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An extension of the sine addition formula on groups and semigroups

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Abstract. The functional equation $f(xy) = f(x)g(y) + g(x)f(y)$ is called the sine addition formula, and in a very general setting it is known that g must be the average of two multiplicative functions. Here we consider the case in which the two multiplicative functions coincide, but we generalize that case to a functional equation with four unknown functions. That is, assuming that M is a nonzero multiplicative function, we solve $f(xy) = k(x)M(y)+g(x)h(y)$ for the four unknown functions f, g, h, k on groups and certain semigroups under the additional assumption that the unknown functions are at least central.

1. Introduction

The functional equation

$$f(xy) = f(x)g(y) + g(x)f(y)$$

on a semigroup $(S, \cdot)$ is known as the *sine addition formula*, since it has as one of its solutions on $(\mathbb{R}, +)$ the pair $f = \sin$, $g = \cos$ for $f, g : \mathbb{R} \to \mathbb{R}$. It also has other solutions on $(\mathbb{R}, +)$ such as $f(x) = cxe^{kx}$, $g(x) = e^{kx}$ for constants c and k. In the more abstract setting of semigroups, the role of exponential functions is played by multiplicative functions. A function M from a semigroup S to a field K is said to be *multiplicative* if it satisfies the Cauchy functional equation

$$M(xy) = M(x)M(y), \quad x, y \in S.$$

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In other words, M is a semigroup homomorphism of S into $(K, \cdot)$. In this abstract setting the role of cosine is played by the average of two multiplicative functions.

General results circa 2013 about the sine addition formula on groups and semigroups are summarized in [4, Chapter 4]. In particular, we cite Theorem 4.1 which states among other things that on a topological semigroup S, if the pair $f, g : S \to \mathbb{C}$ is a continuous solution of the sine addition formula with $f \neq 0$, then g must be of the form

$$g = \frac{1}{2}(M_1 + M_2)$$

for continuous multiplicative functions M_1, M_2. Also f and g must be abelian (see below for definition). Furthermore, if $M_1 \neq M_2$, then $f = a(M_1 - M_2)$ for some (nonzero) complex constant a. The form of f is also determined in the case $M_1 = M_2$ provided that S is a group. Later an improvement was made in [2] for the case $M_1 = M_2$ under the assumption that S is a semigroup generated by its squares or a regular semigroup.

Here we solve a variant of the sine addition formula containing four unknown functions, in the case where one instance of g in the sine addition formula is replaced by a multiplicative function M (so we are in the case $M_1 = M_2$). Specifically, we consider the functional equation

$$f(xy) = k(x)M(y) + g(x)h(y), \quad x, y \in S, \tag{1}$$

for four unknown functions $g, h, k : S \to K$ and $f : S \cdot S \to K$. Here $M : S \to K$ is a given multiplicative function, K is a (commutative) field, S may be a group or a semigroup, and $S \cdot S = \{xy : x, y \in S\}$. In the case that S is a semigroup, then we assume as in [2] that either S is generated by its squares or S is regular.

Throughout this paper we will assume that M is not the trivial multiplicative function which is identically 0. When we state that a function is *nonzero*, we mean that it is not the zero function.

Both the sine addition formula and equation (1) belong to a class of functional equations known as Levi–Civita equations (see [5, Section 10] or [4, Chapter 5]). For this type of equation, the structure of solutions (under certain assumptions concerning the linear independence of functions on the right hand side of the equation) can be described with the help of representations of S. In order to apply this theory, one must first take into account the possible linear dependence of functions on the right hand side. Even after doing that, explicit formulas for the representations may be difficult to obtain. If S is a topological abelian group and one is looking for continuous solutions, then things are a bit easier.

In that context it is known that (again under linear independence assumptions) the continuous solutions are normal exponential polynomials.

In the present article, we take a more direct approach, avoiding any assumptions about linear independence and any use of representations of S. It is interesting that it is possible to use elementary methods to find the forms of the other four functions f, g, h, k in (1) just by assuming that the single function M is multiplicative and the other functions are at least central (see below for definition).

The organization of the paper is as follows. In the next section, we begin with some definitions and prove a couple of lemmas we need in order to solve equation (1). Solutions of that equation are given in the third section. The final section concerns continuous solutions of (1) on topological groups and semigroups, followed by some examples and results in special cases.

2. Preliminaries

We begin with some definitions.

Let S be a semigroup and X a set. A function $\phi : S \to X$ is *central* provided that $\phi(xy) = \phi(yx)$ for all $x, y \in S$. Function $\phi : S \to X$ is *abelian* if $\phi(x_1x_2 \cdots x_n) = \phi(x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(n)})$ for all $x_1, \ldots, x_n \in S$, all permutations π of $\{1, \ldots, n\}$ and all $n = 2, 3, \ldots$. Clearly, every abelian function is central.

Note that any multiplicative function M taking values in a field is automatically abelian.

On a semigroup S there is a natural ideal associated with any nonzero multiplicative function that takes the value 0 somewhere. Let K be a field. If $M : S \to K$ is a nonzero multiplicative function, then we define

$$I_M = \{x \in S : M(x) = 0\},$$

which we can think of as the *null ideal* of M. (Note that in general we allow for the possibility that I_M may be empty, even though we are thinking of it as an ideal.) It is easy to see that I_M is a two-sided ideal if not empty, since if $x \in I_M$ and $y \in S$, then $M(xy) = M(x)M(y) = 0$, and the same is true if $y \in I_M$ and $x \in S$. It follows that $T := S \setminus I_M = \{t \in S : M(t) \neq 0\}$ is a subsemigroup of S. The null ideal I_M (if nonempty) is proper since $M \neq 0$, thus T is nonempty. It may also happen that $T = S$ if I_M is empty.

If S is a group and $M : S \to K$ is a nonzero multiplicative function, then M is nowhere zero. Indeed, if there is $x_0 \in S$ such that $M(x_0) = 0$, then

$M(x) = M(x_0)M(x_0^{-1}x) = 0$ for every $x \in S$. Thus $I_M = \emptyset$ when S is a group. A nonzero multiplicative function from a group S into $\mathbb{C}$ is called a *character* on S.

If S is a topological semigroup, we let $C(S)$ denote the algebra of continuous functions from S into $\mathbb{C}$.

For any field K, let $K^* = K \setminus \{0\}$ denote the subset of nonzero elements of K.

In tandem with multiplicative functions from a semigroup S to a field K, we also have the notion of additive function, which is another type of homomorphism. A function $A : S \to K$ is said to be *additive* if

$$A(xy) = A(x) + A(y), \quad x, y \in S.$$

In other words, A is a semigroup homomorphism of S into $(K, +)$.

We continue with two lemmas needed for our main results.

Lemma 1. *Let S be a semigroup, let K be a field, and suppose $f, g : S \to K$ satisfy the functional equation*

$$f(xy) = f(x)g(y), \quad x, y \in S. \tag{2}$$

If $f \neq 0$, then g is multiplicative, say $g = M$. Furthermore,

(i) *if $M = 0$, then $f = 0$ on $S \cdot S$;*

(ii) *if $M \neq 0$ and f is central, then $f = \mu M$ for some constant $\mu \in K$.*

Proof. Using associativity in S, we have

$$f(x)g(yz) = f(x \cdot yz) = f(xy \cdot z) = f(xy)g(z) = f(x)g(y)g(z), \quad x, y, z \in S.$$

Thus g is multiplicative if there is any $x_0 \in S$ for which $f(x_0) \neq 0$.

Now with $g = M$, equation (2) shows that $f = 0$ on $S \cdot S$ in the case $M = 0$. If $M \neq 0$, then there exists $y_0 \in S$ such that $M(y_0) \neq 0$. Using centrality of f, we find that

$$f(x)M(y) = f(xy) = f(yx) = f(y)M(x),$$

and putting $y = y_0$ here, we arrive at $f = \mu M$ where $\mu = f(y_0)/M(y_0)$. □

Lemma 2. *Let S be a semigroup, let K be a field, let $\sigma \in K^*$, let $M : S \to K$ be a nonzero multiplicative function, and let $A : S \to K$. Then $k : S \to K$ is a central solution of*

$$k(xy) = [k(x) + \sigma M(x)A(y)]M(y), \quad x, y \in S, \tag{3}$$

if and only if $k = (\rho + \sigma A)M$ *for some* $\rho \in K$*, and the restriction of* A *to the subsemigroup* $T = S \setminus I_M$ *is additive. (The values* $A(y)$ *for* $y \in I_M$ *are arbitrary because they are always multiplied by* $M(y) = 0$*, both in the functional equation* (3) *and in the solution formula for* k*.)*

Proof. Interchanging x and y in (3), we get (again since k is central)

$$k(x)M(y) + \sigma M(x)A(y)M(y) = k(xy) = k(yx) = k(y)M(x) + \sigma M(y)A(x)M(x),$$

or

$$[k(x) - \sigma A(x)M(x)]M(y) = [k(y) - \sigma A(y)M(y)]M(x).$$

Using $y = y_0 \in S$ such that $M(y_0) \neq 0$, we obtain

$$k(x) - \sigma A(x)M(x) = \rho M(x)$$

for some constant $\rho \in K$. Thus $k = (\rho + \sigma A)M$.

Substituting the form of k into (3) and simplifying, we arrive at

$$A(xy)M(xy) = [A(x) + A(y)]M(x)M(y), \quad x, y \in S,$$

since $\sigma \neq 0$. Since $M(t) \neq 0$ for all $t \in T$, we have $A(xy) = A(x) + A(y)$ for all $x, y \in T$ as claimed.

Conversely, it is easy to see that such a function k satisfies equation (3). □

3. Solutions of (1)

Initially we do not consider topological aspects, but they are easily incorporated later and will be dealt with in the final section.

Recall our main functional equation (1):

$$f(xy) = k(x)M(y) + g(x)h(y), \quad x, y \in S.$$

Since we will assume that $f \neq 0$ in our main results, let us first dispense with the simple situation $f = 0$.

Proposition 3. *Let* S *be a semigroup, let* K *be a field, and let* $M : S \to K$ *be a nonzero multiplicative function. Suppose* $f = 0$ *in equation* (1)*, so that* $g, h, k : S \to K$ *satisfy the functional equation*

$$0 = k(x)M(y) + g(x)h(y), \quad x, y \in S. \tag{4}$$

The solutions fall into two categories:

(i) *$g = k = 0$ and h arbitrary; or*

(ii) *$k = -cg$, $h = cM$ for some constant $c \in K$, and $g(\neq 0)$ is arbitrary.*

Proof. Putting $y = y_0$ such that $M(y_0) \neq 0$ in (4), we find that $k = -cg$ where $c := h(y_0)/M(y_0)$. With this, equation (4) can be written in the form

$$0 = g(x)[h(y) - cM(y)], \quad x, y \in S,$$

showing that either $g = 0$ or $h = cM$. □

Now we state our two main theorems, one for groups and the other for semigroups. A joint proof is given for both cases.

Theorem 4. *Let G be a group, let K be a field, and let $M : G \to K^*$ be a nonzero multiplicative function. Suppose $f, g, h, k : G \to K$ satisfy equation (1), where f, g, k are central, h is abelian, and $f \neq 0$.*

The solutions fall into four categories:

(i) *$f = aM$, $g = 0$, $k = aM$, and h is arbitrary;*

(ii) *$f = aM$, $h = c_1M$, $k = aM - c_1g$, where g is arbitrary (nonzero);*

(iii) *$f = c_1M + bdM'$, $g = bM'$, $h = c_2M + dM'$, $k = c_1M - bc_2M'$; or*

(iv) *$f = (c_2 + bc_1 + bA)M$, $g = bM$, $h = (c_1 + A)M$, $k = (c_2 + bA)M$;*

for some constants $a, b, d \in K^*$ and $c_1, c_2 \in K$, where $M' : G \to K^*$ is a nonzero multiplicative function and $A : G \to K$ is a nonzero additive function.

Conversely, each of these combinations of functions is a solution of (1).

There are two differences between the result on groups (above) and the result on semigroups (below). First, solution category (iv) becomes a bit more complicated on semigroups because I_M may be a nonempty proper subset of S in the semigroup case, but I_M is empty in the group case. Second, a fifth category of solution may be possible in the semigroup case if $S \cdot S \neq S$ (which cannot occur if S is a group).

Theorem 5. *Let S be a semigroup which is generated by its squares, let K be a field, and let $M : S \to K$ be a nonzero multiplicative function. Suppose $g, h, k : S \to K$ and $f : S \cdot S \to K$ satisfy equation (1), where f, g, k are central, h is abelian, and $f \neq 0$.*

The solutions fall into four categories if $S \cdot S = S$:

(i) *$f = aM$, $g = 0$, $k = aM$, and h is arbitrary;*

(ii) *$f = aM$, $h = c_1M$, $k = aM - c_1g$, where g is arbitrary (nonzero);*

(iii) *$f = c_1M + bdM'$, $g = bM'$, $h = c_2M + dM'$, $k = c_1M - bc_2M'$; or*

(iv) $f = (c_2 + bc_1 + bA)M$, $g = bM$, $h = (c_1 + A)M$, $k = (c_2 + bA)M$;

for some constants $a, b, d \in K^*$ *and* $c_1, c_2 \in K$, *where* $M' : S \to K$ *is a nonzero multiplicative function, and* $A : S \to K$ *is a function which is additive and nonzero on the nonempty subsemigroup* $T = \{t \in S : M(t) \neq 0\}$ *and has arbitrary values on* $S \setminus T = I_M$.

If $S \cdot S \neq S$, *then the above formulas for* f *must be restricted to* $S \cdot S$ *(for example,* $f = aM|_{S \cdot S}$ *in* (i)*). In addition to the first four, there is also a fifth category of solution:*

(v) $f(xy) = c_1 M(xy) + bg(x)g(y)$, $h = c_2 M + bg$, $k = c_1 M - c_2 g$;

for some constants $b \in K^*$ *and* $c_1, c_2 \in K$, *where* $g|_{S \cdot S} = 0$ *and* $g(x_1)g(y_1) = g(x_2)g(y_2)$ *whenever* $x_1 y_1 = x_2 y_2$.

Conversely, each of these combinations of functions is a solution of (1).

Proof. First we handle the special case $g = 0$ separately. In this case, (1) reduces to

$$f(xy) = k(x)M(y), \quad x, y \in S.$$

Using associativity, here we calculate that

$$k(x)M(y)M(z) = k(x)M(yz) = f(x \cdot yz) = f(xy \cdot z) = k(xy)M(z), \quad x, y, z \in S.$$

Since M is nonzero, we conclude that

$$k(x)M(y) = k(xy), \quad x, y \in S.$$

Since k is central, we get from Lemma 1 that $k = aM$ for some constant $a \in K$. Thus it follows that $f(xy) = aM(xy)$, that is $f = aM|_{S \cdot S}$. (If $S \cdot S = S$, this is just $f = aM$.) Noting that $f \neq 0$, we must specify that $a \in K^*$, and this constitutes solution (i).

Henceforth we assume that $g \neq 0$. Again using associativity with equation (1), we calculate that

$$k(xy)M(z) + g(xy)h(z) = f(xy \cdot z) = f(x \cdot yz) = k(x)M(yz) + g(x)h(yz),$$

for all $x, y, z \in S$. This rearranges to

$$[k(xy) - k(x)M(y)]M(z) = g(x)h(yz) - g(xy)h(z), \quad x, y, z \in S. \tag{5}$$

Taking $z = z_0$ here such that $M(z_0) \neq 0$, we deduce that

$$k(xy) + cg(xy) = k(x)M(y) + g(x)h'(y), \quad x, y \in S, \tag{6}$$

for some constant $c \in K$ and function $h' : S \to K$. For future reference, we observe that h' is defined by

$$h'(y) = \frac{h(yz_0)}{M(z_0)}, \quad y \in S,$$

and thus h' is central since h is abelian.

Now apply the same process to (6) as we did to (1), with $k + cg$ in place of f and h' in place of h. On one hand, we have

$$\begin{aligned} k(xy \cdot z) + cg(xy \cdot z) &= k(xy)M(z) + g(xy)h'(z) \\ &= [k(x)M(y) + g(x)h'(y) - cg(xy)]M(z) + g(xy)h'(z) \\ &= [k(x)M(y) + g(x)h'(y)]M(z) + g(xy)[h'(z) - cM(z)]. \end{aligned}$$

And on the other hand,

$$k(x \cdot yz) + cg(x \cdot yz) = k(x)M(yz) + g(x)h'(yz) = k(x)M(y)M(z) + g(x)h'(yz).$$

Therefore, we conclude that

$$g(xy)[h'(z) - cM(z)] = g(x)[h'(yz) - h'(y)M(z)], \quad x, y, z \in S. \tag{7}$$

There are two cases to consider.

Case 1. Suppose $h' = cM$. Then (6) reduces to

$$k(xy) + cg(xy) = [k(x) + cg(x)]M(y), \quad x, y \in S.$$

Applying Lemma 1 to this equation, we conclude that (since $k + cg$ is central)

$$k + cg = aM$$

for some constant $a \in K$.

Returning this information to (1), we have now

$$f(xy) = [aM(x) - cg(x)]M(y) + g(x)h(y) = aM(xy) + g(x)[h(y) - cM(y)].$$

Since f is central, we see that

$$g(x)[h(y) - cM(y)] = g(y)[h(x) - cM(x)],$$

and since $g \neq 0$, this means that

$$h = cM + \mu g$$

for some constant $\mu \in K$. Now equation (5) reduces to

$$0 = \mu[g(x)g(yz) - g(xy)g(z)], \quad x, y, z \in S. \tag{8}$$

We have two subcases to consider.

Subcase 1(a). Suppose $\mu = 0$. Here we have $h = cM$ and $k = aM - cg$. Inserting these into (1), we find that

$$f(xy) = aM(x)M(y) = aM(xy).$$

Thus $f = aM|_{S \cdot S}$, and this is solution (ii) (noting that $a \neq 0$ since $f \neq 0$), where we have put $c_1 := c$.

Subcase 1(b). Suppose $\mu \neq 0$. Then from (8) we get

$$g(x)g(yz) = g(xy)g(z), \quad x, y, z \in S.$$

Choosing $z = z_0$ such that $g(z_0) \neq 0$, we arrive at

$$g(x)g'(y) = g(xy), \tag{9}$$

for some function $g' : S \to K$. It now follows from Lemma 1 that g' is multiplicative, with either $g' = 0$ and $g = 0$ on $S \cdot S$, or $g' \neq 0$ and g is a constant times the multiplicative function g'.

For the moment, let us suppose that $S \cdot S = S$. Then since $g \neq 0$ it follows that also $g' \neq 0$, hence

$$g = bM'$$

for some multiplicative map $M' : S \to K$ and $b \in K^*$ (both nonzero since $g \neq 0$). Thus we have $h = cM + \mu bM'$, $k = aM - cbM'$, and (1) becomes

$$\begin{aligned} f(xy) &= [aM(x) - cbM'(x)]M(y) + bM'(x)[cM(y) + \mu bM'(y)] \\ &= aM(xy) + \mu b^2 M'(xy). \end{aligned}$$

Thus we have $f = aM + \mu b^2 M'$, and this is solution (iii) with $d := \mu b$ ($\neq 0$), $c_1 := a$, and $c_2 := c$.

If S is a group, then we are finished with this subcase. Now suppose S is a semigroup and $S \cdot S \neq S$. If $g' \neq 0$, then we proceed as we did for $S \cdot S = S$, and get situation (iii) again. The remaining possibility is $g' = 0$ and $g|_{S \cdot S} = 0$. In this case, (1) becomes

$$f(xy) = aM(x)M(y) + \mu g(x)g(y) = aM(xy) + \mu g(x)g(y).$$

This formula gives consistent values for f only if the value of $g(x)g(y)$ is uniquely determined by the value of xy. This constitutes solution (v) with $b := \mu$ ($\neq 0$), $c_1 := a$, and $c_2 := c$.

That concludes Case 1.

Case 2. Suppose $h' \neq cM$. Using $z = z_0$ so that $h'(z_0) - cM(z_0) \neq 0$ in (7), we find that

$$g(xy) = g(x)h''(y), \quad x, y \in S, \tag{10}$$

for some function $h'' : S \to K$. Since $g \neq 0$, we again apply Lemma 1 and conclude that h'' is multiplicative: either $h'' = 0$ and $g|_{S \cdot S} = 0$, or $h'' \neq 0$ and g is a constant times the multiplicative function h''.

First let us suppose that $h'' = 0$ and $g|_{S \cdot S} = 0$ (which can occur only if $S \cdot S \neq S$). Now (7) tells us that

$$0 = h'(yz) - h'(y)M(z), \quad y, z \in S,$$

since $g \neq 0$. Applying Lemma 1 once more, we see that $h' = \tau M$ for some $\tau \in K$ with $\tau \neq c$ (since $h' \neq cM$). Then equation (6) yields

$$k(xy) = [k(x) + \tau g(x)]M(y), \quad x, y \in S,$$

and since k is central, we have

$$[k(x) + \tau g(x)]M(y) = [k(y) + \tau g(y)]M(x), \quad x, y \in S,$$

so $k + \tau g = c_1 M$ for some $c_1 \in K$. Hence $k = c_1 M - \tau g$. With this, (1) becomes

$$f(xy) = [c_1 M(x) - c\tau g(x)]M(y) + g(x)h(y) = c_1 M(xy) + g(x)[h(y) - \tau M(y)]$$

for all $x, y \in S$. Since f is central, this gives

$$g(x)[h(y) - \tau M(y)] = g(y)[h(x) - \tau M(x)],$$

and since $g \neq 0$, we have $h(y) - \tau M(y) = bg(y)$ for some $b \in K$. Thus $h = \tau M + bg$, and we are again in situation (v), where $c_2 := \tau$. (Note that here $b \neq 0$, else we are back to solution (ii).)

The other possible outcome of (10) is that $M' := h'' \neq 0$ and

$$g = bM'$$

for some constant $b \neq 0$ and nonzero multiplicative map $M' : S \to K$. Now (7) takes the form

$$bM'(xy)[h'(z) - \tau M(z)] = bM'(x)[h'(yz) - h'(y)M(z)], \quad x, y, z \in S,$$

which reduces to

$$h'(yz) = M'(y)h'(z) + [h'(y) - \tau M'(y)]M(z). \tag{11}$$

Interchanging y, z, we get

$$h'(zy) = M'(z)h'(y) + [h'(z) - \tau M'(z)]M(y).$$

Recalling that h' is central, the last two equations yield

$$[h'(y) - \tau M'(y)][M(z) - M'(z)] = [h'(z) - \tau M'(z)][M(y) - M'(y)].$$

There are two subcases to consider.

Subcase 2(a). Suppose $M \neq M'$. Then taking $z = t_0$ such that $M(t_0) - M'(t_0) \neq 0$, we deduce that $h'(y) - \tau M'(y) = c[M(y) - M'(y)]$ for some constant $c \in K$. Thus,

$$h' = cM + (\tau - c)M'.$$

Returning the forms of g and h' to (7) and simplifying, the equation simplifies to

$$0 = (\tau - c)M'(x)M'(y)M(z), \quad x, y, z \in S.$$

But since $\tau \neq c$ and both M' and M are nonzero, this is impossible. Therefore, this subcase cannot occur.

Subcase 2(b). Suppose $M = M'$. Now (11) takes the form

$$h'(yz) = M(y)h'(z) + [h'(y) - \tau M(y)]M(z).$$

Let $T = \{t \in S : M(t) \neq 0\} = S \setminus I_M$. On this subsemigroup we can divide the previous equation by $M(yz)$ and write

$$\frac{h'(yz)}{M(yz)} = \frac{h'(z)}{M(z)} + \frac{h'(y)}{M(y)} - \tau, \quad y, z \in T.$$

This means that the function $A' : T \to K$ defined by

$$A'(t) = \frac{h'(t)}{M(t)} - \tau, \quad t \in T,$$

is additive, and we have

$$h'(t) = (\tau + A'(t))M(t), \quad t \in T. \tag{12}$$

If $I_M = \emptyset$ (which is the case if S is a group), then we have $T = S$ and (dropping the prime mark from A')

$$h' = (\tau + A)M,$$

where $A : S \to K$ is additive. We show that h' has the same form when $I_M \neq \emptyset$.

Indeed, suppose S is a semigroup generated by its squares and $y \in I_M = S \setminus T$. Then we have from (11)

$$h'(yz) = h'(y)M(z), \quad y \in I_M, z \in S.$$

In particular, this means that $h'(yz) = 0$ for $y, z \in I_M$. We want to show that h' vanishes on I_M. Let $x \in I_M$. Since S is generated by its squares, we have $x = x_1^2 x_2^2 \cdots x_n^2$ for some $x_i \in S$ ($i = 1, \ldots, n$). Now $0 = M(x) = M(x_1)^2 M(x_2)^2 \cdots M(x_n)^2$, so $M(x_i) = 0$ for some i, thus $x_i \in I_M$. Hence $x = (x_1^2 x_2^2 \cdots x_i) \cdot (x_i x_{i+1}^2 \cdots x_n^2) = y_1 y_2$ with $y_1, y_2 \in I_M$, since I_M is an ideal. Therefore, we have $h'(x) = h'(y_1 y_2) = 0$ for all $x \in I_M$. Now extend A' to a function $A : S \to K$ by defining $A(x) := A'(x)$ for $x \in T$, and defining $A(x)$ arbitrarily for $x \in S \setminus T = I_M$. Merging this with (12), we arrive again at $h' = (\tau + A)M$.

Recall also that $g = bM' = bM \neq 0$. Returning this information to (6) results in

$$k(xy) = k(x)M(y) + bM(x)A(y)M(y), \quad x, y \in S.$$

By Lemma 2, the solution of this equation is

$$k = (c_2 + bA)M$$

for some constant $c_2 \in K$.

Next, equation (5) reveals that

$$h(yz) = [h(z) + A(y)M(z)]M(y), \quad y, z \in S.$$

Since h is central, this equation also is governed by Lemma 2. The outcome is that

$$h = (c_1 + A)M$$

for some constant $c_1 \in K$. With this, equation (1) finally becomes

$$f(xy) = (c_2 + bA(x))M(x)M(y) + bM(x)(c_1 + A(y))M(y),$$

which simplifies to

$$f(xy) = (c_2 + bc_1 + bA(xy))M(xy).$$

This is solution (iv), and Case 2 is finished.

Conversely, it is easily verified that the combinations of functions in (i), (ii), (iii), (iv) (and (v) in the semigroup case) are solutions of (1), and the proof is complete. □

Remarks.

1. Note that $S \cdot S = S$ for any semigroup S that contains an identity element, that is if S is a monoid. So in Theorem 5, we can dispense with solution (v) if S is a monoid.
2. The only place in the proof of Theorem 5 where we used the assumption that the semigroup S is generated by its squares was in proving that $h' = 0$ on I_M in Subcase 2(b). The proof works under other conditions. For example, it works if S is regular, which by definition means that for every $x \in S$ there exists $y \in S$ such that $x = xyx$.

4. Topological considerations, some examples and special cases

Topological versions of Theorems 4 and 5 are easy to obtain. Again we state the results separately but combine their proofs into one.

Corollary 6. *Let G be a topological group, and let $M \in C(G)$ be a character on G. Suppose $f, g, h, k \in C(G)$ satisfy equation (1), where f, g, k are central, h is abelian, and $f \neq 0$.*

The solutions fall into four categories:

(i) *$f = aM$, $g = 0$, $k = aM$, and h is arbitrary;*

(ii) *$f = aM$, $h = c_1 M$, $k = aM - c_1 g$, and g is arbitrary (nonzero);*

(iii) *$f = c_1 M + bdM'$, $g = bM'$, $h = c_2 M + dM'$, $k = c_1 M - bc_2 M'$;* or

(iv) *$f = (c_2 + bc_1 + bA)M$, $g = bM$, $h = (c_1 + A)M$, $k = (c_2 + bA)M$;*

for some constants $a, b, d \in \mathbb{C}^*$ and $c_1, c_2 \in \mathbb{C}$, where $M' \in C(G)$ is a character and $A \in C(G)$ is a nonzero additive function.

Conversely, each of these combinations of functions is a continuous solution of (1).

Corollary 7. *Let S be a topological semigroup which is generated by its squares, and let $M \in C(S)$ be a nonzero multiplicative function. Suppose $g, h, k \in C(S)$ and $f \in C(S \cdot S)$ satisfy equation* (1)*, where f, g, k are central, h is abelian, and $f \neq 0$.*

The solutions fall into four categories if $S \cdot S = S$:

(i) $f = aM$, $g = 0$, $k = aM$, *and h is arbitrary;*

(ii) $f = aM$, $h = c_1 M$, $k = aM - c_1 g$, *and g is arbitrary (nonzero);*

(iii) $f = c_1 M + bdM'$, $g = bM'$, $h = c_2 M + dM'$, $k = c_1 M - bc_2 M'$; *or*

(iv) $f = (c_2 + bc_1 + bA)M$, $g = bM$, $h = (c_1 + A)M$, $k = (c_2 + bA)M$;

for some constants $a, b, d \in \mathbb{C}^*$ and $c_1, c_2 \in \mathbb{C}$, where $M' \in C(S)$ is a nonzero multiplicative function, and $A : S \to \mathbb{C}$ is a function which is additive, nonzero, and continuous on the nonempty subsemigroup $T = \{t \in S : M(t) \neq 0\}$ and arbitrary on $S \setminus T = I_M$.

If $S \cdot S \neq S$, then the above formulas for f must be restricted to $S \cdot S$ (for example, $f = aM|_{S \cdot S}$ in (i)*). In addition to the first four, there is also a fifth category of solution:*

(v) $f(xy) = c_1 M(xy) + bg(x)g(y)$, $h = c_2 M + bg$, $k = c_1 M - c_2 g$;

for some constants $b \in \mathbb{C}^*$ and $c_1, c_2 \in \mathbb{C}$, where $g|_{S \cdot S} = 0$ and $g(x_1)g(y_1) = g(x_2)g(y_2)$ whenever $x_1 y_1 = x_2 y_2$.

Proof. By Theorems 4 and 5, the only things needing proof are the continuity statements regarding M' and A. For M', the continuity follows from $M' = g/b$. For A, continuity on $T (= G$ in the group case) follows from $A(t) = h(t)/M(t) - c_1$ for $t \in T$. □

We close the paper with some examples, a remark, and two specialized results.

In the first two examples S is commutative, so all functions on S are abelian (and thus central).

The first example is an application of Corollary 7. Let S be the open interval $(0, 1) \subset \mathbb{R}$ under multiplication, so S is generated by its squares and $S \cdot S = S$. Recall that the nonzero multiplicative functions in $C(0, 1)$ are of the form $M(x) = x^\lambda$ where λ runs through the complex numbers. Hence $T = S$ in Corollary 7 above.

Also note that the nonzero additive functions in $C(0,1)$ are of the form $A(x) = a \log x$ for some constant $a \in \mathbb{C}^*$.

Example 8. Assuming $f \neq 0$, the continuous solutions of (1) are the following (where in each case $M(x) = x^\lambda$):

(i) $f(x) = k(x) = ax^\lambda$, $g = 0$, where $h \in C(0,1)$ is arbitrary;

(ii) $f(x) = ax^\lambda$, $h(x) = c_1 x^\lambda$, $k(x) = ax^\lambda - c_1 g(x)$, with $g \in C(0,1)$ arbitrary (nonzero);

(iii) $f(x) = c_1 x^\lambda + bdx^\mu$, $g(x) = bx^\mu$, $h(x) = c_2 x^\lambda + dx^\mu$, $k(x) = c_1 x^\lambda - bc_2 x^\mu$; and

(iv) $f(x) = (c_2 + bc_1 + ba \log x)x^\lambda$, $g(x) = bx^\lambda$, $h(x) = (c_1 + a \log x)x^\lambda$, $k(x) = (c_2 + ba \log x)x^\lambda$;

for some constants $a, b, d \in \mathbb{C}^*$ and $c_1, c_2, \mu, \lambda \in \mathbb{C}$.

The next example is an application of Theorem 5. Note that a nonzero multiplicative function on $(0,1)$ can never take the value 0, so I_M is empty and $T = S$ in the notation of Theorem 5. Here we have taken $k = h = f$ so that our functional equation more closely resembles the sine addition formula.

Example 9. Let S again be the open interval $(0,1) \subset \mathbb{R}$ under multiplication, so $S \cdot S = S$, let K be a field, and let $M : (0,1) \to K$ be a nonzero multiplicative function. Suppose $f, g : (0,1) \to K$ satisfy

$$f(xy) = f(x)M(y) + g(x)f(y), \quad x, y \in (0,1),$$

with $f \neq 0$, and consider the four types of solutions provided by Theorem 5. Solution type (i) holds with $f = aM$ and $g = 0$. Solution type (ii) cannot occur, because $k = f$ forces $c_1 = 0$ which makes $h = f$ impossible. In solution type (iii), $k = f$ forces $c_2 = -d$, so $h = -d(M - M')$; then $h = f$ requires f to have the same form, so $c_1 = -d$ and $b = 1$; thus the solutions in this category are $f = d(M' - M)$ and $g = M'$, where $M' \neq M$ since $f \neq 0$. Finally, in category (iv) we see that $k = f$ forces $bc_1 = 0$, which is possible only if $c_1 = 0$; then $h = AM = f$, so $c_2 = 0$, again $b = 1$, and the solutions are $f = AM$ and $g = M$. Summarizing, the solutions are:

(a) $f = aM$ and $g = 0$ for some constant $a \in K^*$;

(b) $f = a(M - M')$ and $g = M' \neq M$ for some nonzero multiplicative function $M' : (0,1) \to K$ and constant $a \in K^*$;

(c) $f = AM$ and $g = M$ for some nonzero additive function $A : (0,1) \to K$.

Our next example is non-abelian. Let $S = M(2,\mathbb{C})$ be the semigroup of complex 2×2 matrices under the operation of matrix multiplication. This semigroup is generated by its squares (see, for example, page 192 in [2]), so we may apply Corollary 7 to find the continuous solutions of (1). Also S is a monoid (as it contains the identity matrix), so we need not concern ourselves with solution category (v).

The nonzero multiplicative functions in $C(S)$ are given by [2, Lemma 5.4(ii)]. One of them is $M = 1$; all the others are of the form

$$M(X) = |\det(X)|^{\lambda-n}(\det(X))^n, \quad X \in GL(2,\mathbb{C}),$$

for some $\lambda \in \mathbb{C}$ with positive real part and some $n \in \mathbb{Z}$, and $M(X) = 0$ when $\det(X) = 0$.

In the case $M = 1$, we have $I_M = \emptyset$ and $T = S \setminus I_M = S$. For all other continuous and nonzero M, we find that I_M is the ideal $\{X \in S|\det(X) = 0\}$ of singular matrices. In this case, $T = S \setminus I_M$ is the subsemigroup $GL(2,\mathbb{C}) = \{X \in S|\det(X) \neq 0\}$ consisting of the invertible matrices.

The additive functions in $C(GL(2,\mathbb{C}))$ are provided in [2, Lemma 5.4(iv)] and are given by

$$A(X) = \gamma \log(\det(X)), \quad X \in GL(2,\mathbb{C}),$$

where γ runs through $\mathbb{C}$. If $M \neq 1$, then $T = S \setminus I_M = GL(2,\mathbb{C})$, so this will be the form of A in the next example for that case.

If, on the other hand, we suppose $M = 1$, then $I_M = \emptyset$ and $T = S \setminus I_M = S$. If $A \in C(S)$ is additive, then its restriction to the subsemigroup $GL(2,\mathbb{C})$ must be of the form in the previous paragraph. Now in solution category (iv), we have $h = c_1 + A$ and $h \in C(S)$. Thus the continuity of h as $\det(X)$ tends to 0 requires that $\gamma = 0$, so A vanishes on $GL(2,\mathbb{C})$. Now we show that A also vanishes on the subsemigroup $S \setminus GL(2,\mathbb{C})$ of singular matrices. From [3] and [1], we learn that every element of this subsemigroup is a product of at most two idempotents. If X is an idempotent ($X^2 = X$), then $A(X) = A(X^2) = 2A(X)$, so $A(X) = 0$. If $V \in S \setminus GL(2,\mathbb{C})$ is not idempotent, then there exist idempotents X, Y such that $V = XY$, so $A(V) = A(XY) = A(X) + A(Y) = 0$. Therefore, $A = 0$ in solution category (iv) when $M = 1$.

As a final preparation for the next example, we note that the forms of continuous central functions $g : S \to \mathbb{C}$ can be found in [2, Proposition 5.5(c)], where it is shown that

$$g(X) = \Psi(\mathrm{tr}(X), \det(X)), \quad X \in S,$$

where $\Psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous and $\mathrm{tr}(X)$ is the trace of X.

Example 10. Let $S = M(2, \mathbb{C})$ be the multiplicative semigroup of complex 2×2 matrices. We get the solutions $f, g, h, k \in C(S)$ of equation (1) with f, g, k central, h abelian, and $f \neq 0$ by plugging the relevant forms of M, M', A (and g in solution (ii)) from the preceding discussion into the solution formulas (i)-(iv) of Corollary 7.

We note here the necessity of f being central for our results. In a private communication, Henrik Stetkær has shared with us an example of a non-central continuous solution of the functional equation

$$f(xy) = f(x)M(y) + \check{M}(x)f(y), \quad x, y \in G, \tag{13}$$

where G is the $(ax + b)$-group, M is a continuous character on G, and $\check{M}(x) := M(x^{-1})$. This equation is clearly of the form (1). Our group G consists of elements of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, for $a, b \in \mathbb{R}$ with $a > 0$, under the operation of matrix multiplication. For brevity, we write

$$(a, b) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

According to [4, Example 3.13], the continuous characters on G are of the form

$$M_t(a, b) = a^t, \quad (a, b) \in G,$$

where $t \in \mathbb{C}$. In Example 2.10 of the same reference we find that the continuous additive functions on G into $\mathbb{C}$ have the form

$$A_c(a, b) = c \log a, \quad (a, b) \in G,$$

for some constant $c \in \mathbb{C}$.

It is easily checked that f, M defined by

$$f(a, b) = b/\sqrt{a}, \quad M(a, b) = 1/\sqrt{a}, \quad (a, b) \in G,$$

are continuous, nonzero, and satisfy equation (13). (Note that $M = M_{-1/2}$ and $\check{M} = M_{1/2}$.) It is not difficult to check that this solution does not fit any of the forms in Corollary 6. The explanation is that this f is not central. For example,

$$f((1, 2)(2, 1)) = f(2, 3) = 3/\sqrt{2} \neq 5/\sqrt{2} = f(2, 5) = f((2, 1)(1, 2)).$$

We end the paper with two specialized results in which we prescribe that g in equation (1) is a multiplicative function M'. The first is a consequence of our main theorems.

Corollary 11. *Let S be either a semigroup which is generated by its squares or a group, let K be a field, and let $M, M' : S \to K$ be nonzero multiplicative functions. Suppose $h, k : S \to K$ and $f : S \cdot S \to K$ satisfy the functional equation*

$$f(xy) = k(x)M(y) + M'(x)h(y), \quad x, y \in S, \tag{14}$$

where f, k are central, h is abelian, and $f \neq 0$. The solutions fall into three categories:

(a) $f = aM$, $M' = 0$, $k = aM$, *and h is arbitrary;*

(b) $f = c_1M + c_3M'$, $h = c_2M + c_3M'$, $k = c_1M - c_2M'$; *or*

(c) $f = (c_2 + c_1 + A)M$, $h = (c_1 + A)M$, $k = (c_2 + A)M$, $M' = M$;

for some constants $a \in K^*$ and $c_1, c_2, c_3 \in K$, where $A : S \to K$ is a function which is additive and nonzero on the (nonempty) subsemigroup $T = \{t \in S : M(t) \neq 0\}$. In the group case $T = S$ holds, and in the semigroup case the solution formulas for f must be restricted to $S \cdot S$. Moreover, in solution category (b) we must choose (c_1, c_3) so that $f \neq 0$.

Conversely, each of these combinations of functions is a solution of (14).

Proof. In either the group or semigroup case, we have at least four categories of potential solutions as listed in Theorems 5 and 4. In category (i) of those two theorems, we are in category (a) of the present corollary. In category (ii) of those two theorems, the fact that $g = M'$ means that $k = aM - c_2M'$; this solution is contained in category (b) here (with $c_3 = 0$ and $c_1 = a \in K^*$). In category (iii) of Theorems 5 and 4, we must take $b = 1$ to get $g = M'$; thus we are in category (b) again. In category (iv), we must again take $b = 1$ so that g is a nonzero multiplicative function ($g = M' = M$ this time); thus we are in category (c) here. In the semigroup case, we see there is a possible fifth category if $S \cdot S \neq S$, but since $g = M'$ is specified, we fall back into category (b) again and the proof is finished. □

Finally, we note that in the special case of (14) in which $f = h = k$, we can get by with fewer assumptions.

Proposition 12. *Let S be either a semigroup which is generated by its squares or a group, let K be a field, and let $M, M' : S \to K$ be multiplicative functions which are not both zero. The central solutions $f : S \to K$ of the functional equation*

$$f(xy) = f(x)M(y) + M'(x)f(y), \quad x, y \in S, \tag{15}$$

fall into two categories:

(a) $f = a(M - M')$, $M \neq M'$; *or*

(b) $f = AM$, $M = M' =: M$;

for some constant $a \in K$ *and a function* $A : S \to K$ *which is additive on* $T = S \setminus I_M = \{t \in S : M(t) \neq 0\}$ *and arbitrary on* I_M*. In the group case,* $T = S$*.*

Conversely, each of these combinations of functions is a solution of (15)*.*

Proof. First, suppose $M = M'(\neq 0)$. Then [2, Lemma 3.4(ii)] tells us that f has the form given in (b). If, on the other hand, $M \neq M'$, then we use the fact that f is central to get

$$f(x)M(y) + M'(x)f(y) = f(xy) = f(yx) = f(y)M(x) + M'(y)f(x),$$

which yields

$$f(x)[M(y) - M'(y)] = f(y)[M(x) - M'(x)].$$

Putting $y = y_0$ such that $M(y_0) \neq M'(y_0)$, we see that f has the form given in (a). □

References

[1] R. J. H. Dawlings, Product of idempotents in the semigroup of singular endomorphisms of a finite dimensional vector space, *Proc. Roy. Soc. Edinburgh Sect. A* **91** (1981–82), 123–133.

[2] B. Ebanks and H. Stetkær, d'Alembert's other functional equation on monoids with an involution, *Aequationes Mathematicae* **89** (2015), 187–206.

[3] J. A. Erdös, On products of idempotent matrices, *Glasgow Math. J.* **8** (1967), 118–122.

[4] H. Stetkær, Functional Equations on Groups, *World Scientific Publishing*, *Singapore*, 2013.

[5] L. Székelyhidi, Convolution Type Functional Equations on Topological Abelian Groups, *World Scientific Publishing*, *Singapore*, 1991.

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