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Generalization of Wolstenholme's and Morley's congruences

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Abstract. In this paper, we show that for any prime $p \geq 11$ and any p-integer α, we have $\binom{\alpha p-1}{p-1} \equiv 1 - \alpha(\alpha-1)(\alpha^2-\alpha-1)p\sum_{k=1}^{p-1}\frac{1}{k} + \alpha^2(\alpha-1)^2p^2\sum_{1\leq i<j\leq p-1}\frac{1}{ij}$ (mod p^7). This congruence generalizes the congruences of Wolstenholme, Morley, Glaisher, Carlitz, McIntosh, Tauraso and Meštrović. Furthermore, it allows to rediscover the congruences of Glaisher, Carlitz and Zhao in a simple way.

1. Introduction

As early as in 1819, Babbage [1] showed that for any prime $p \geq 3$, $\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$. In 1862, Wolstenholme ([16], [7, p. 89]) noted that for any prime $p \geq 5$, we have the following congruence:

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{1.1}$$

In 1895, Morley [13] proved that for any prime $p \geq 5$,

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} \pmod{p^3}. \tag{1.2}$$

Five years later, Glaisher ([5, p. 21], [6, p. 323]) generalized Wolstenholme's congruence (1.1) by proving that for any positive integer n and any prime $p \geq 5$, we have

$$\binom{np-1}{p-1} \equiv 1 \pmod{p^3}, \tag{1.3}$$

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$$\binom{np-1}{p-1} \equiv 1 - \frac{1}{3}n(n-1)p^3 B_{p-3} \pmod{p^4}, \tag{1.4}$$

where B_n denotes the n-th Bernoulli number. In the same year, Glaisher stated that for any prime $p \geq 3$,

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^4}.$$

Little more than half a century later, in 1953, CARLITZ ([2], [3]) extended Morley's congruence (1.2) to p^4, by proving that for any prime $p \geq 5$,

$$(-1)^{\frac{p-1}{2}} \binom{p-1}{\frac{p-1}{2}} \equiv 4^{p-1} + \frac{p^3}{12} B_{p-3} \pmod{p^4}. \tag{1.5}$$

A few years before the end of the twentieth century, in 1995, R. J. MCINTOSH [10, p. 385] showed that for any prime $p \geq 7$,

$$\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^5}. \tag{1.6}$$

In 2007, ZHAO [17] gave a result implying that for any prime $p \geq 7$,

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^5}. \tag{1.7}$$

In 2010, TAURASO [15] established, for any prime $p \geq 7$, the congruences

$$\binom{2p-1}{p-1} \equiv 1 + 2p \sum_{k=1}^{p-1} \frac{1}{k} + \frac{2}{3}p^3 \sum_{k=1}^{p-1} \frac{1}{k^3} \pmod{p^6}, \tag{1.8}$$

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} - 2p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \pmod{p^6}. \tag{1.9}$$

Recently, in 2014, MEŠTROVIĆ [12] obtained the following new generalization. For any prime $p \geq 11$,

$$\binom{2p-1}{p-1} \equiv 1 - 2p \sum_{k=1}^{p-1} \frac{1}{k} + 4p^2 \sum_{1 \leq i < j \leq p-1} \frac{1}{ij} \pmod{p^7}. \tag{1.10}$$

In a nice paper [14], ROSEN extended Meštrović's result by studying congruences for the binomial coefficient $\binom{kp-1}{p-1}$ modulo p^n, where k and n are positive integers.

The literature is full of papers about Wolstenholme's congruence, for more details about historical extensions and generalizations, see, for instance, Lehmer [9] and Meštrović's survey [11].

The aim of this paper is to give a congruence for the binomial coefficient $\binom{\alpha p-1}{p-1}$ modulo p^7 involving generalized harmonic numbers, where α is a p-integer. We recall that a rational number α is said to be a p-integer if the denominator b of the irreducible fraction $\frac{a}{b} = \alpha$ is not divisible by p.

2. Preliminaries

For any prime p and any non-negative integer m, we define generalized harmonic numbers H_m by $H_0 = 1$, $H_m = 0$ for $m \geq p$, and

$$H_m = \sum_{1 \leq k_1 < \ldots < k_m \leq p-1} \frac{1}{k_1 \cdots k_m}, \qquad 1 \leq m \leq p-1.$$

Let $P(x)$ be the polynomial defined by

$$P(x) = \binom{x-1}{p-1} = \frac{(x-1)(x-2)\cdots(x-p+1)}{(p-1)!}. \tag{2.1}$$

By writing $P(x)$ in the form $P(x) = \prod_{k=1}^{p-1}\left(1 - \frac{x}{k}\right)$, we obtain

$$P(x) = \sum_{k=0}^{p-1} (-1)^k H_k x^k. \tag{2.2}$$

The proof of the main theorem is based on the following lemma.

Lemma 1. *For any odd prime p and any integer $m \geq 1$, we have the following assertions:*

(1) *If $m \neq p-1$, then $H_m \equiv 0 \pmod{p}$.*

(2) *If $2m-1 \neq p-2$, then $H_{2m-1} \equiv 0 \pmod{p^2}$.*

(3) *If $2m-1 \neq p-4$, then $H_{2m-1} - mpH_{2m} \equiv 0 \pmod{p^4}$.*

Proof. (1) Since $1, \ldots, p-1$ are the roots of $P(x)$, by considering $P(x)$ in $\frac{\mathbb{Z}}{p\mathbb{Z}}[x]$ and Fermat's little theorem, we can write $P(x) = 1 - x^{p-1}$ and deduce from relation (2.2) that $H_{p-1} \equiv -1 \pmod{p}$ and for $m \neq p-1$, $H_m \equiv 0 \pmod{p}$.

(2) Relation (2.1) easily leads to

$$P(x) = P(p-x). \tag{2.3}$$

Using relation (2.2), equality (2.3) can be written as follows:

$$\sum_{k=0}^{p-1}(-1)^k H_k x^k = \sum_{k=0}^{p-1}(-1)^k H_k (p-x)^k.$$

By equating the coefficient of x^{2m-1} on each side of the above relation, we obtain the following identity:

$$H_{2m-1} - mpH_{2m} = \frac{1}{2}p^2 \sum_{k=2m+1}^{p-1} (-1)^k \binom{k}{2m-1} p^{k-2m-1} H_k. \tag{2.4}$$

Thus, according to the first assertion, if $2m \neq p-1$, then $H_{2m-1} \equiv 0 \pmod{p^2}$.

(3) By reducing relation (2.4) modulo p^4, we obtain

$$\begin{aligned} &H_{2m-1} - mpH_{2m} \\ &\quad\equiv -\frac{1}{2}p^2\binom{2m+1}{2}H_{2m+1} + \frac{1}{2}p^3\binom{2m+2}{3}H_{2m+2} \pmod{p^4}. \end{aligned} \tag{2.5}$$

Suppose that $2m-1 \neq p-4$, then $2m+1 \neq p-2$ and $2m+2 \neq p-1$. From the first two assertions, we deduce that $H_{2m-1} - mpH_{2m} \equiv 0 \pmod{p^4}$. □

Lemma 2. *For any integer $n \geq 1$, we have*

$$(-1)^n \binom{2n}{n} = 4^{2n}\binom{n-\frac{1}{2}}{2n}. \tag{2.6}$$

Proof. We have

$$\begin{aligned} 4^{2n}\binom{n-\frac{1}{2}}{2n} &= \frac{2^{2n}}{(2n)!}\prod_{k=1}^{2n}(2(n-k)+1) \\ &= (-1)^n \frac{2^{2n}}{(2n)!}\prod_{k=1}^{n}(2(n+1-k)-1)\prod_{k=n+1}^{2n}(2(k-n)-1). \end{aligned}$$

Using the well-known consecutive odd numbers product formula, we get relation (2.6). □

By taking $n = \frac{p-1}{2}$ in relation (2.6), we obtain

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{\frac{p-1}{2}} = 4^{p-1}\binom{\frac{1}{2}p-1}{p-1}. \tag{2.7}$$

This relation will allow us, in the next section, to generalize Carlitz's congruence (1.5).

Let p be an odd prime and $(S_m)_{m\geq 1}$ the sequence defined by

$$S_m = \sum_{k=1}^{p-1} \frac{1}{k^m}.$$

Some of the following congruences, involving $(S_m)_{m\geq 1}$ are well-known ([6], [4]), but are included here for completeness as the proofs are short.

Lemma 3. *Let p be an odd prime and m a positive integer. If $p-1 \nmid m$, then $S_m \equiv 0 \pmod{p}$. Otherwise, $S_m \equiv -1 \pmod{p}$.*

Proof. Suppose first that $p-1 \mid m$. Then for $1 \leq k \leq p-1$, $\frac{1}{k^m} \equiv 1 \pmod{p}$. Hence $S_m \equiv -1 \pmod{p}$. Suppose now that $p-1 \nmid m$. Let $\overline{g}$ be a generator of the cyclic group $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^*$. Then

$$\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \sum_{j=0}^{p-2} \left(g^j\right)^m \pmod{p}.$$

Since $p-1 \nmid m$, then $g^m \not\equiv 1 \pmod{p}$. Therefore,

$$\sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \sum_{j=0}^{p-2} (g^m)^j \equiv 0 \pmod{p}. \qquad \square$$

Lemma 4. *Let p be an odd prime and m an odd integer.*

(1) *If $p-1 \nmid m+1$, then $S_m \equiv 0 \pmod{p^2}$. Otherwise, $S_m \equiv \frac{1}{2}mp \pmod{p^2}$.*
(2) *If $p-1 \nmid m+3$, then $2S_m + mpS_{m+1} \equiv 0 \pmod{p^4}$. Otherwise, $2S_m + mpS_{m+1} \equiv -\frac{1}{12}m(m+1)(m+2)p^3 \pmod{p^4}$.*
(3) *If $p-1 \nmid m+5$, then $S_m + \frac{1}{2}mpS_{m+1} + \frac{1}{12}m(m+1)p^2S_{m+2} \equiv 0 \pmod{p^6}$.*

Proof. (1) We have

$$S_m = \frac{1}{2}\sum_{k=1}^{p-1} \frac{1}{k^m} + \frac{1}{2}\sum_{k=1}^{p-1} \frac{1}{(p-k)^m} = \frac{1}{2}\sum_{k=1}^{p-1} \frac{(p-k)^m + k^m}{k^m(p-k)^m}.$$

Since m is odd, the binomial theorem yields $(p-k)^m + k^m \equiv mpk^{m-1} \pmod{p^2}$. Then $S_m \equiv -\frac{1}{2}mpS_{m+1} \pmod{p^2}$.

By Lemma 3, if $p-1 \mid m+1$, then $S_{m+1} \equiv -1 \pmod{p}$, thus $S_m \equiv \frac{1}{2}mp \pmod{p^2}$, otherwise $S_m \equiv 0 \pmod{p^2}$.

(2) For $1 \le k \le p-1$, we have

$$\frac{1}{\left(1-\frac{p}{k}\right)^m} \equiv 1 + m\frac{p}{k} + \binom{m+1}{2}\frac{p^2}{k^2} + \binom{m+2}{3}\frac{p^3}{k^3} + \binom{m+3}{4}\frac{p^4}{k^4} + \binom{m+4}{5}\frac{p^5}{k^5} \pmod{p^6}.$$

Then we have

$$\sum_{k=1}^{p-1}\frac{1}{(p-k)^m} \equiv -\sum_{k=1}^{p-1}\frac{1}{k^m} - \sum_{k=1}^{p-1}\frac{mp}{k^{m+1}} - \binom{m+1}{2}\sum_{k=1}^{p-1}\frac{p^2}{k^{m+2}} - \binom{m+2}{3}\sum_{k=1}^{p-1}\frac{p^3}{k^{m+3}} - \binom{m+3}{4}\sum_{k=1}^{p-1}\frac{p^4}{k^{m+4}} - \binom{m+4}{5}\sum_{k=1}^{p-1}\frac{p^5}{k^{m+5}} \pmod{p^6}.$$

It follows that

$$2S_m + mpS_{m+1} \equiv -\tbinom{m+1}{2}p^2S_{m+2} - \tbinom{m+2}{3}p^3S_{m+3} - \tbinom{m+3}{4}p^4S_{m+4} - \tbinom{m+4}{5}p^5S_{m+5} \pmod{p^6}. \tag{2.8}$$

Suppose that $p-1 \nmid m+3$. By Lemma 3, we have $S_{m+3} \equiv 0 \pmod{p}$, and from the first statement, we have $S_{m+2} \equiv 0 \pmod{p^2}$. In this case, relation (2.8) implies that $2S_m + mpS_{m+1} \equiv 0 \pmod{p^4}$. Suppose now that $p-1 \mid m+3$. From the first statement, we have $S_{m+2} \equiv \frac{1}{2}(m+2)p \pmod{p^2}$, and by Lemma 3, $S_{m+3} \equiv -1 \pmod{p}$. In this case, relation (2.8) implies that $2S_m + mpS_{m+1} \equiv -\frac{1}{12}m(m+1)(m+2)p^3 \pmod{p^4}$.

(3) Since $m+4$ and $m+2$ are odd, according to Lemma 3 and the first two statements, if $p-1 \nmid m+5$, then $S_{m+5} \equiv 0 \pmod{p}$, $S_{m+4} \equiv 0 \pmod{p^2}$, and $2S_{m+2} + (m+2)\,pS_{m+3} \equiv 0 \pmod{p^4}$. We deduce from (2.8) that

$$2S_m \equiv -mpS_{m+1} - \frac{m(m+1)}{2}p^2S_{m+2} + \frac{m(m+1)}{3}p^2S_{m+2} \pmod{p^6}.$$

This completes the proof of the lemma. □

Lemma 5. *For any prime $p \ge 5$, the following assertions hold:*

$$\sum_{k=1}^{p-1}\frac{1}{k} \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}, \tag{2.9}$$

$$\sum_{k=1}^{p-1}\frac{1}{k^2} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2}. \tag{2.10}$$

Proof. In [8], the author proved with details relation (2.9), which is Glaisher's result [5]. We deduce from assertion (2) of Lemma 4 that for any odd integer m, $2S_m + mpS_{m+1} \equiv 0 \pmod{p^3}$. By taking $m = 1$ in the last relation and combining with relation (2.9), relation (2.10) holds. □

3. Main results

Our main result is a generalization of Wolstenholme's congruence and Morley's congruence. Moreover, it allows to find again all the generalizations we have listed in the first section.

Theorem 1. *For any odd prime $p \geq 11$ and any p-integer α, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1-\alpha(\alpha-1)(\alpha^2-\alpha-1)p\sum_{k=1}^{p-1}\frac{1}{k}+\alpha^2(\alpha-1)^2p^2\sum_{1\leq i<j\leq p-1}\frac{1}{ij} \pmod{p^7}.$$

Proof. From relation (2.2), we have

$$\binom{\alpha p-1}{p-1} \equiv \sum_{k=0}^{4}(-\alpha)^k H_k p^k - \alpha^5 H_5 p^5 + \alpha^6 H_6 p^6 \pmod{p^7}. \tag{3.1}$$

Since $p \geq 11$, from Lemma 1, we can deduce that $H_6 \equiv 0 \pmod{p}$, $H_5 \equiv 0 \pmod{p^2}$ and $H_3 \equiv 2pH_4 \pmod{p^4}$. Therefore, from relation (3.1), we obtain the following congruence:

$$\binom{\alpha p-1}{p-1} \equiv 1 - \alpha H_1 p + \alpha^2 H_2 p^2 - \alpha^3 (2-\alpha) H_4 p^4 \pmod{p^7}. \tag{3.2}$$

To eliminate H_4 from congruence (3.2), it suffices to take $\alpha = 1$ in this same relation, to get

$$H_4 p^4 \equiv -H_1 p + H_2 p^2 \pmod{p^7}.$$

The substitution of this congruence into congruence (3.2) gives

$$\binom{\alpha p-1}{p-1} \equiv 1 - \alpha(\alpha-1)(\alpha^2-\alpha-1)H_1 p + \alpha^2(\alpha-1)^2 H_2 p^2 \pmod{p^7}. \tag{3.3}$$

The proof of the theorem is complete. □

We notice that Theorem 1 generalizes to p-integers, Meštrović's congruence (1.10). We apply also Theorem 1 to obtain the following generalization to p-integers of Glaisher's congruence (1.3).

Corollary 1. *For any prime $p \geq 5$ and any p-integer α, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1 \pmod{p^3}. \tag{3.4}$$

Proof. We deduce from assertions (1) and (2) of Lemma 1, respectively, for $m = 2$ and $m = 1$, the following congruences.

$$\sum_{1\le i<j\le p-1} \frac{1}{ij} \equiv 0 \pmod{p} \text{ and } \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}, \quad p \ge 5. \tag{3.5}$$

Using these two congruences in Theorem 1, we get relation (3.4) when $p \ge 11$. In the cases $p = 5$ and $p = 7$, the proof can be obtained by a direct calculation. □

We deduce from Theorem 1, the following generalization of the congruences (1.8) and (1.9) of Tauraso.

Corollary 2. *For any odd prime $p \ge 5$ and any p-integer α, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1 - \alpha(\alpha-1)(\alpha^2-\alpha-1)p\sum_{k=1}^{p-1}\frac{1}{k} - \frac{p^2}{2}\alpha^2(\alpha-1)^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^6}, \tag{3.6}$$

$$\binom{\alpha p-1}{p-1} \equiv 1 + \alpha(\alpha-1)p\sum_{k=1}^{p-1}\frac{1}{k} + \frac{1}{6}\alpha^2(\alpha-1)^2p^3\sum_{k=1}^{p-1}\frac{1}{k^3} \pmod{p^6}. \tag{3.7}$$

Proof. The proof in the cases $p = 5$ and $p = 7$ can be obtained by a direct calculation. Suppose now that $p \ge 11$. Using the identity

$$\sum_{1\le i<j\le p-1}\frac{1}{ij} = \frac{1}{2}\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2 - \frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k^2},$$

we deduce from relation (3.5) that

$$p^2\sum_{1\le i<j\le p-1}\frac{1}{ij} \equiv -\frac{1}{2}p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^6}.$$

Given this last congruence, from Theorem 1, we obtain relation (3.6). Taking $m = 1$ in assertion (3) of Lemma 4, the following identity holds for $p \ge 11$,

$$\sum_{k=1}^{p-1}\frac{1}{k} + \frac{1}{2}p\sum_{k=1}^{p-1}\frac{1}{k^2} + \frac{1}{6}p^2\sum_{k=1}^{p-1}\frac{1}{k^3} \equiv 0 \pmod{p^6}.$$

Combining the last congruence with congruence (3.6), we get relation (3.7). □

From relation (3.6) and Lemma 5, we obtain the following corollary, which is a generalization of Glaisher's congruence (1.4).

Corollary 3. *For any prime $p \geq 5$ and any p-integer α, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1 - \frac{1}{3}\alpha(\alpha-1)p^3 B_{p-3} \pmod{p^4}.$$

Note that, by taking $\alpha = \frac{1}{2}$ in Corollary 3 and using relation (2.7), we get Carlitz's congruence (1.5).

For $m = 1$, assertion (2) of Lemma 4 implies that for $p \geq 7$, we have

$$2p\sum_{k=1}^{p-1}\frac{1}{k} + p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 0 \pmod{p^5}. \tag{3.8}$$

From relations (3.6) and (3.8), we obtain the following corollary.

Corollary 4. *For any prime $p \geq 7$ and any p-integer α, we have*

$$\binom{\alpha p-1}{p-1} \equiv 1 + \alpha(\alpha-1)p\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^5}, \tag{3.9}$$

$$\binom{\alpha p-1}{p-1} \equiv 1 - \frac{1}{2}\alpha(\alpha-1)p^2\sum_{k=1}^{p-1}\frac{1}{k^2} \pmod{p^5}. \tag{3.10}$$

By replacing $\alpha = 2$, relation (3.10) allows us to get McIntosh's congruence (1.6), and relation (3.9) allows to get the congruence of Zhao (1.7).

Remark 1. Using the same method as in the proof of Theorem 1, we can obtain a congruence for $\binom{\alpha p-1}{p-1}$ modulo p^9, similar to (3.3), involving H_1, H_2, H_3 and H_4. By exploiting the fact that $\binom{-p-1}{p-1} = \binom{2p-1}{p-1}$, we get $p^4H_4 \equiv 5pH_1 - 5p^2H_2 + 3p^3H_3 \pmod{p^9}$, which allows us to obtain the following congruence:

$$\begin{aligned}\binom{\alpha p-1}{p-1} &\equiv 1 + (\alpha^2-\alpha)(2\alpha^4-4\alpha^3+\alpha^2+\alpha+1)pH_1 \\ &\quad - \alpha^2(\alpha-1)^2(2\alpha^2-2\alpha-1)p^2H_2 + \alpha^3(\alpha-1)^3H_3 \pmod{p^9}.\end{aligned}$$

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