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# Convergence rates in the strong law of large numbers for negatively orthant dependent random variables with general moment conditions

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**Abstract.** Let $\{a_n, n \geq 1\}$ be a sequence of real numbers with $0 < a_n/n^{1/p} \uparrow$ for some $1 \leq p < 2$, and let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed negatively orthant dependent random variables. In this paper, it is shown that $\sum_{n=1}^{\infty} n^{r-1} \times P\{|X| > a_n\} < \infty$ is equivalent to $\sum_{n=1}^{\infty} n^{r-2} P\{\max_{1\leq m\leq n} |S_m - mEXI(|X| \leq a_n)| > \varepsilon a_n\} < \infty$, $\forall\, \varepsilon > 0$, where $r \geq 1$ and $S_n = \sum_{k=1}^{n} X_k$.

## 1. Introduction

A sequence $\{U_n, n \geq 1\}$ of random variables is said to converge completely to the constant $\theta$ if

$$\sum_{n=1}^{\infty} P\{|U_n - \theta| > \varepsilon\} < \infty, \quad \forall\, \varepsilon > 0.$$

This concept of complete convergence was introduced by Hsu and Robbins [12], and they proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected

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value if the variance of the summands is finite. Their result has been generalized and extended by many authors. Let $r \geq 1$, $0 < p < 2$, $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$. Then the following three statements are equivalent:

$$E|X|^{rp} < \infty, \quad \text{where } EX = 0 \text{ if } rp \geq 1, \tag{1.1}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon n^{1/p}\} < \infty, \quad \forall\, \varepsilon > 0, \tag{1.2}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\{\max_{1\leq m\leq n} |S_m| > \varepsilon n^{1/p}\} < \infty, \quad \forall\, \varepsilon > 0. \tag{1.3}$$

Baum and Katz [3] proved the equivalence of (1.1) and (1.2). In the case of $p = 1$, the equivalence was already proved by Katz [14]. Chow [8] proved the equivalence of (1.1) and (1.3).

The property (1.3) represents information regarding the convergence rate in the Marcinkiewicz–Zygmund strong law of large numbers. In the case $r > 1$, (1.3) implies

$$\sum_{n=1}^{\infty} n^{r-2} P\{\sup_{m\geq n} |S_m|/m^{1/p} > \varepsilon\} < \infty, \quad \forall\, \varepsilon > 0,$$

which is equivalent to

$$\sum_{n=1}^{\infty} 2^{n(r-1)} P\{\sup_{m\geq 2^n} |S_m|/m^{1/p} > \varepsilon\} < \infty, \quad \forall\, \varepsilon > 0,$$

and hence we have

$$P\{\sup_{m\geq n} |S_m|/m^{1/p} > \varepsilon\} = o\left(\frac{1}{n^{r-1}}\right), \quad \forall\, \varepsilon > 0.$$

Since the convergence of $S_n/n^{1/p} \to 0$ a.s. is equivalent to $P\{\sup_{m\geq n} |S_m|/m^{1/p} > \varepsilon\} \to 0, \forall\, \varepsilon > 0$, (1.3) describes the convergence rate in the Marcinkiewicz–Zygmund strong law of large numbers.

For a sequence of not necessarily independent random variables satisfying a Hájek–Rényi type maximal inequality, the convergence rate in the strong law of large numbers was established by Fazekas [9], Fazekas and Klesov [10], Tómács [23], and Tómács and Líbor [24].

Obviously, (1.3) implies (1.2). But, the converse is not true in the non-independent case (see, for example, Bai *et al.* [2]). So it is interesting to obtain the complete convergence for the maximum partial sums of non-independent random variables.

It is also interesting to find more generalized moment conditions for the complete convergence. In fact, LANZINGER [17], GUT and STADTMÜLLER [11], and CHEN and SUNG [6] extended the Baum–Katz theorem under higher order moment conditions, SUNG [22] obtained the complete convergence for pairwise i.i.d. random variables under some generalized moment conditions, and LI *et al.* [19] characterized the complete convergence for heavy-tailed random variables via integral test. In particular, CHEN *et al.* [7] obtained the following result. Let $r \geq 1$, $1 \leq p < 2$, $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_n, n \geq 1\}$ a sequence of constants with $0 < a_n/n^{1/p} \uparrow$. Then the following three statements are equivalent:

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X| > a_n\} < \infty, \tag{1.4}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n - nb_n| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0, \tag{1.5}$$

$$\sum_{n=1}^{\infty} n^{r-2} P\{\max_{1\leq m\leq n} |S_m - mEXI(|X| \leq a_n)| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0. \tag{1.6}$$

They proved the above equivalent statements by using the Rosenthal inequality for the maximum partial sums of independent random variables.

In this paper, we prove the equivalence of (1.4) and (1.6) for a sequence of negatively orthant dependent (NOD) random variables. Although the Rosenthal inequality holds for the partial sums of NOD random variables (see Lemma 2.1), it is not known whether the Rosenthal inequality holds for the maximum partial sums. For the proof of (1.4) $\Rightarrow$ (1.6), we use a Rosenthal type inequality (Lemma 2.2) instead of the Rosenthal inequality and a delicate truncation of random variables. For the proof of the converse, we use a new method which is not the standard one.

We first recall the concept of negatively quadrant dependent random variables. The concept of negative quadrant dependence was introduced by LEHMANN [18]. Two random variables $X$ and $Y$ are said to be negatively quadrant dependent if

$$P(X \leq x, Y \leq y) \leq P(X \leq x)P(Y \leq y)$$

for all real numbers $x$ and $y$. A sequence of random variables is said to be pairwise negatively quadrant dependent if every two random variables in the sequence are negatively quadrant dependent.

A finite family of random variables $\{X_1, \ldots, X_n\}$ is said to be NOD if the following two inequalities hold:

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

and

$$P(X_1 > x_1, \cdots, X_n > x_n) \leq \prod_{i=1}^{n} P(X_i > x_i)$$

for all real numbers $x_1, \ldots, x_n$. An infinite family of random variables is NOD if every finite subfamily is NOD.

For more properties about NOD, we refer to Joag-Dev and Proschan [13], Bozorgnia *et al.* [4], and Asadian *et al.* [1]. For limiting results about NOD, we refer to Bai *et al.* [2], Qiu *et al.* [21], and Chen and Sung [5].

Now we state the main results. Some preliminary lemmas will be given in Section 2, and the proofs of the main results will be detailed in Section 3. An application of our main results will be given in Section 4.

**Theorem 1.1.** *Let $r > 1$ and $1 \leq p < 2$. Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, $\{a_n, n \geq 1\}$ a sequence of real numbers with $0 < a_n/n^{1/p} \uparrow$. Then (1.4) and (1.6) are equivalent.*

When $r = 1$ and $1 < p < 2$, it is an open problem whether (1.6) holds or not even under the moment condition $E|X|^{rp} < \infty$ and $a_n = n^{1/p}$. However, when $r = 1$ and $p = 1$, we have the following result for pairwise negatively quadrant dependent random variables. Since the pairwise negative quadrant dependence is weaker than NOD, the following theorem satisfies for NOD random variables.

**Theorem 1.2.** *Let $\{X, X_n, n \geq 1\}$ be a sequence of identically distributed pairwise negatively quadrant dependent random variables with partial sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$, and $\{a_n, n \geq 1\}$ a sequence of real numbers with $0 < a_n/n \uparrow$. Then the following statements are equivalent:*

$$\sum_{n=1}^{\infty} P\{|X| > a_n\} < \infty, \tag{1.7}$$

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1 \leq m \leq n} |S_m - mEXI(|X| \leq a_n)| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0, \tag{1.8}$$

$$a_n^{-1} \sum_{k=1}^{n} (X_k - EX_kI(|X_k| \leq a_k)) \to 0 \quad \text{a.s.} \tag{1.9}$$

*Remark 1.1.* When $0 < a_n/n \uparrow \infty$, the random variable with the general moment condition (1.7) may have infinite mean. Kruglov [16] and Sung [22] obtained strong laws of large numbers for pairwise i.i.d. random variables with infinite means.

*Remark 1.2.* For a sequence of pairwise i.i.d. random variables $\{X, X_n, n \geq 1\}$, many authors obtained complete convergence results. Sung [22] proved that (1.7) implies that

$$\sum_{n=1}^{\infty} n^{-1} P\{|S_n - nEXI(|X| \leq a_n)| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0,$$

which is weaker than (1.8). When $a_n = n$ for $n \geq 1$, (1.7) is equivalent to $E|X| < \infty$. Bai *et al.* [2] proved that $EX = 0$ is equivalent to

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\leq m\leq n} |S_m| > \varepsilon n\} < \infty, \quad \forall\, \varepsilon > 0.$$

Kruglov [15] proved that, for a sequence of pairwise i.i.d. and non-negative random variables, the pair of expressions $E|X| < \infty$ and $EX = \mu$ is equivalent to $\sum_{n=1}^{\infty} n^{-1} P\{|S_n - n\mu| > \varepsilon n\} < \infty$ for all $\varepsilon > 0$.

Throughout this paper, the symbol $C$ denotes a positive constant which is not necessarily the same one in each appearance, and $I(A)$ denotes the indicator function of the event $A$. It proves convenient to define $n(t) = [n^t]$ for any $t > 0$, where $[x]$ denotes the integer part of $x$.

## 2. Preliminary lemmas

In this section, we present some preliminary lemmas. Lemma 2.1 is the Rosenthal inequality for the sum of NOD random variables.

**Lemma 2.1** (Asadian *et al.* [1]). *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^s < \infty$, for some $s \geq 2$ and all $n \geq 1$. Then there exists a positive constant $C$ depending only on $s$ such that for all $n \geq 1$,*

$$E\Big|\sum_{k=1}^{n} X_k\Big|^s \leq C\left\{\sum_{k=1}^{n} E|X_k|^s + \left(\sum_{k=1}^{n} EX_k^2\right)^{s/2}\right\}.$$

Using Theorem 3 in MÓRICZ [20], Lemma 2.1 gives a Rosenthal type inequality for the maximum of partial sums of NOD random variables. Since the term $(\log 2n)^s$ is added, Lemma 2.2 is not the Rosenthal inequality for the maximum of partial sums.

**Lemma 2.2.** *Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables with $EX_n = 0$ and $E|X_n|^s < \infty$, for some $s \geq 2$ and all $n \geq 1$. Then there exists a positive constant $C$ depending only on $s$ such that for all $n \geq 1$,*

$$E \max_{1 \leq m \leq n} \Big| \sum_{k=1}^{m} X_k \Big|^s \leq C \left( \frac{\log 2n}{\log 2} \right)^s \left\{ \sum_{k=1}^{n} E|X_k|^s + \left( \sum_{k=1}^{n} EX_k^2 \right)^{s/2} \right\}.$$

**Lemma 2.3.** *Let $r \geq 1$, $0 < p < 2$, and $X$ be a random variable. Suppose that $\{a_n, n \geq 1\}$ is a sequence of positive numbers with $a_n \uparrow$. Then (1.4) implies that*

$$n^r P\{|X| > a_n\} \to 0 \quad \text{as } n \to \infty.$$

PROOF. By (1.4),

$$\sum_{k=n+1}^{2n} k^{r-1} P\{|X| > a_k\} \leq \sum_{k=n+1}^{\infty} k^{r-1} P\{|X| > a_k\} \to 0$$

as $n \to \infty$. Note that by $a_n \uparrow$,

$$\sum_{k=n+1}^{2n} k^{r-1} P\{|X| > a_k\} \geq \sum_{k=n+1}^{2n} n^{r-1} P\{|X| > a_{2n}\} = n^r P\{|X| > a_{2n}\}.$$

Hence,

$$n^r P\{|X| > a_{2n}\} \to 0$$

as $n \to \infty$. It follows that for $n \geq 2$,

$$n^r P\{|X| > a_n\} \leq n^r P\{|X| > a_{2[n/2]}\} \leq 3^r [n/2]^r P\{|X| > a_{2[n/2]}\} \to 0$$

as $n \to \infty$. So we arrive at the desired result. □

**Lemma 2.4.** *Under the assumptions of Theorem 1.1, if $rp \geq 2$ and (1.4) holds, then for any $t \in (0, 1]$,*

$$a_n^{-2} \cdot n a_{n(t)}^2 P\{|X| > a_{n(t)}\} \leq C n^{1-2/p}$$

*and*

$$a_n^{-2} \cdot n E|X|^2 I(|X| \leq a_{n(t)}) \leq C n^{1-2/p}.$$

Proof. By $a_n/n^{1/p} \uparrow$ and Lemma 2.3,

$$a_n^{-2} \cdot na_{n(t)}^2 P\{|X| > a_{n(t)}\} \leq n^{1-2/p} n(t)^{2/p} P\{|X| > a_{n(t)}\}$$
$$\leq Cn^{2(t-1)/p+1} \cdot n^{-rt} = Cn^{1-2/p+(2-rp)t/p} \leq Cn^{1-2/p}.$$

By [7, Lemma 2.3] in Chen *et al.*,

$$a_n^{-2} \cdot nE|X|^2 I(|X| \leq a_{n(t)}) \leq a_n^{-2} \cdot nE|X|^2 I(|X| \leq a_n) \leq Cn^{1-2/p}. \qquad \square$$

**Lemma 2.5.** *Under the assumptions of Theorem 1.1, if* (1.4) *holds, then, for any* $s > rp$ *and any* $t \in (0, 1)$,

$$\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} (\log n)^s \cdot na_{n(t)}^s P\{|X| > a_{n(t)}\} < \infty$$

*and*

$$\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} (\log n)^s \cdot nE|X|^s I(|X| \leq a_{n(t)}) < \infty.$$

Proof. By $a_n/n^{1/p} \uparrow$ and Lemma 2.3,

$$\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} (\log n)^s \cdot na_{n(t)}^s P\{|X| > a_{n(t)}\}$$
$$\leq \sum_{n=1}^{\infty} n^{r-1-s/p} n(t)^{s/p} (\log n)^s P\{|X| > a_{n(t)}\}$$
$$\leq C \sum_{n=1}^{\infty} n^{r-1-s/p} n(t)^{s/p-r} (\log n)^s \leq C \sum_{n=1}^{\infty} n^{-1-(s-rp)(1-t)/p} (\log n)^s < \infty.$$

By $a_n/n^{1/p} \uparrow$ again,

$$\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} (\log n)^s \cdot nE|X|^s I(|X| \leq a_{n(t)})$$
$$= \sum_{n=1}^{\infty} n^{r-1} \cdot a_n^{-s} (\log n)^s \sum_{k=1}^{n(t)} E|X|^s I(a_{k-1} < |X| \leq a_k) \ (a_0 = 0)$$
$$\leq \sum_{n=1}^{\infty} n^{r-1} \cdot a_n^{-s} (\log n)^s \sum_{k=1}^{n(t)} a_k^s P\{a_{k-1} < |X| \leq a_k\} \leq$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} n^{r-1}\cdot a_n^{-s}(\log n)^s a_{n(t)}^s n(t)^{-s/p}\sum_{k=1}^{n(t)} k^{s/p}P\{a_{k-1}<|X|\le a_k\}\\
&\le \sum_{n=1}^{\infty} n^{r-1-s/p}\cdot(\log n)^s\sum_{k=1}^{n(t)} k^{s/p}P\{a_{k-1}<|X|\le a_k\}\\
&\le \sum_{k=1}^{\infty} k^{s/p}P\{a_{k-1}<|X|\le a_k\}\sum_{n=[k^{1/t}]}^{\infty} n^{r-1-s/p}(\log n)^s\\
&\le C\sum_{k=1}^{\infty} k^{s/p}P\{a_{k-1}<|X|\le a_k\}\cdot k^{(r-s/p)/t}(\log k)^s\\
&= \sum_{k=1}^{\infty} k^{r}P\{a_{k-1}<|X|\le a_k\}\cdot k^{-(s-rp)(1-t)/(pt)}(\log k)^s\\
&\le C\sum_{k=1}^{\infty} k^{r}P\{a_{k-1}<|X|\le a_k\}<\infty.
\end{aligned}
$$

So we complete the proof. □

**Lemma 2.6.** *Under the assumptions of Theorem 1.1, if* (1.4) *holds, then, for any* $t\in(1/r,1)$,

$$a_n^{-1}\cdot nE\left((X-a_{n(t)})I(a_{n(t)}<X\le a_n)+(a_n-a_{n(t)})I(X>a_n)\right)\to 0,\quad \text{as } n\to\infty.$$

Proof. Note that $0\le (X-a_{n(t)})I(a_{n(t)}<X\le a_n)+(a_n-a_{n(t)})I(X>a_n)\le XI(a_{n(t)}<X\le a_n)+a_nI(X>a_n)$. Hence by Lemma 2.3,

$$
\begin{aligned}
0&\le a_n^{-1}\cdot nE\left((X-a_{n(t)})I(a_{n(t)}<X\le a_n)+(a_n-a_{n(t)})I(X>a_n)\right)\\
&\le a_n^{-1}\cdot nE\left(XI(a_{n(t)}<X\le a_n)+a_nI(X>a_n)\right)\\
&\le nP\{|X|>a_{n(t)}\}\le Cn^{1-rt}\to 0,\quad \text{as } n\to\infty.
\end{aligned}
$$

□

**Lemma 2.7.** *Let* $\{X_n, n\ge 1\}$ *be a sequence of random variables, and* $\{a_n, n\ge 1\}$ *a sequence of real numbers with* $0<a_n\uparrow$. *Assume that*

$$\sum_{n=1}^{\infty} n^{-1}P\{\max_{1\le k\le n}|X_k|>Ma_n\}<\infty$$

*for some* $M>0$. *Then*

$$P\{\max_{1\le k\le n}|X_k|>Ma_{2n}\}\to 0\quad \text{as } n\to\infty.$$

Proof. Noting that

$$P\{\max_{1\le k\le n}|X_k|>Ma_{2n}\}\le 2\sum_{m=n+1}^{2n}m^{-1}P\{\max_{1\le k\le m}|X_k|>Ma_m\},$$

we have the desired result. □

**Lemma 2.8.** *Let $r\ge 1$ and $0<p<2$. Let $X$ be a random variable, and $\{a_n,n\ge 1\}$ a sequence of real numbers with $0<a_n\uparrow$. Then (1.4) is equivalent to*

$$\sum_{n=1}^{\infty}n^{r-1}P\{|X|>a_{mn}\}<\infty$$

*for some integer $m\ge 1$.*

Proof. It is enough to prove the necessity. Assume that $\sum_{n=1}^{\infty}n^{r-1}P\{|X|>a_{mn}\}<\infty$ for some integer $m\ge 1$. Note that for any $1\le i\le m$,

$$(mn+i)^{r-1}P\{|X|>a_{mn+i}\}\le(2m)^{r-1}\cdot n^{r-1}P\{|X|>a_{mn}\}.$$

Hence, by the assumption,

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{r-1}P\{|X|>a_n\}\\&\quad=\sum_{n=1}^{m}n^{r-1}P\{|X|>a_n\}+\sum_{n=1}^{\infty}\sum_{i=1}^{m}(mn+i)^{r-1}P\{|X|>a_{mn+i}\}\\&\quad\le\sum_{n=1}^{m}n^{r-1}P\{|X|>a_n\}+(2m)^{r-1}m\sum_{n=1}^{\infty}n^{r-1}P\{|X|>a_{mn}\}<\infty.\end{aligned}$$

The proof is completed. □

**Lemma 2.9.** *Under the conditions of Lemma 2.8, we further assume that $0<a_n/n^{1/p}\uparrow$. Then (1.4) is equivalent to*

$$\sum_{n=1}^{\infty}n^{r-1}P\{|X|>Ma_n\}<\infty$$

*for some $M>0$.*

PROOF. It is enough to prove the necessity. Assume that $\sum_{n=1}^{\infty} n^{r-1}P\{|X| > Ma_n\} < \infty$ for some $M > 0$. If $0 < M \leq 1$, then (1.4) holds trivially. Now we consider the case of $M > 1$. Then there exists a positive integer $i$ such that $2^{i/p} \geq M$. By $a_n/n^{1/p} \uparrow$, we have $2^{-1/p}a_{2n} \geq a_n$. Then

$$\sum_{n=1}^{\infty} n^{r-1}P\{M^{-1}2^{1/p}|X| > a_{2n}\} = \sum_{n=1}^{\infty} n^{r-1}P\{|X| > M2^{-1/p}a_{2n}\} \leq \sum_{n=1}^{\infty} n^{r-1}P\{|X| > Ma_n\} < \infty.$$

By Lemma 2.8,

$$\sum_{n=1}^{\infty} n^{r-1}P\{2^{1/p}|X| > Ma_n\} < \infty.$$

If we use this method $i-1$ times more, then

$$\sum_{n=1}^{\infty} n^{r-1}P\{2^{i/p}|X| > Ma_n\} < \infty.$$

So (1.4) holds. □

## 3. Proofs of the main results

In this section, we provide the proofs of the main results.

PROOF OF THEOREM 1.1. (1.4) ⇒ (1.6). For $1 \leq k \leq n$ and $n \geq 1$, set

$$X_{nk} = -a_nI(X_k < -a_n) + X_kI(|X_k| \leq a_n) + a_nI(X_k > a_n).$$

Then $EX_{nk} = -a_nP\{X < -a_n\} + a_nP\{X > a_n\} + EXI(|X| \leq a_n)$. By Lemma 2.3, $a_n^{-1} \cdot n\,|-a_nP\{X < -a_n\} + a_nP\{X > a_n\}| \leq a_n^{-1} \cdot na_nP\{|X| > a_n\} \leq Cn^{1-r} \to 0$ as $n \to \infty$. Therefore, to prove (1.6), it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2}P\left\{\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m}(X_k - EX_{nk})\Big| > \varepsilon a_n\right\} < \infty, \quad \forall\, \varepsilon > 0. \tag{3.1}$$

Note that

$$\left\{\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m}(X_k - EX_{nk})\Big| > \varepsilon a_n\right\} \subset \cup_{k=1}^{n}\{|X_k| > a_n\} \cup \left\{\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m}(X_{nk} - EX_{nk})\Big| > \varepsilon a_n\right\}.$$

Hence, to prove (3.1), it is enough by (1.4) to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le m\le n}\Big|\sum_{k=1}^{m}(X_{nk}-EX_{nk})\Big| > \varepsilon a_n\right\} < \infty. \tag{3.2}$$

For any $t \in (1/r, 1)$, set

$$\begin{aligned}
X_{nk}^{(1)} &= -a_{n(t)}I(X_k < -a_{n(t)}) + X_k I(|X_k| \le a_{n(t)}) + a_{n(t)}I(X_k > a_{n(t)}),\\
X_{nk}^{(2)} &= (X_k - a_{n(t)})I(a_{n(t)} < X_k \le a_n) + (a_n - a_{n(t)})I(X_k > a_n),\\
X_{nk}^{(3)} &= (X_k + a_{n(t)})I(-a_n \le X_k < -a_{n(t)}) - (a_n - a_{n(t)})I(X_k < -a_n).
\end{aligned}$$

Then $X_{nk} = X_{nk}^{(1)} + X_{nk}^{(2)} + X_{nk}^{(3)}$, and $\{X_{nk}^{(i)}, 1 \le k \le n\}$, $1 \le i \le 3$, are all NOD by Bozorgnia *et al.* [4]. Hence, to prove (3.2), it is enough to show that for all $\varepsilon > 0$ and $i = 1, 2, 3$,

$$I_i = \sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le m\le n}\Big|\sum_{k=1}^{m}(X_{nk}^{(i)}-EX_{nk}^{(i)})\Big| > \varepsilon a_n\right\} < \infty.$$

By the Markov inequality and Lemma 2.2, we get that for any $s \ge 2$,

$$\begin{aligned}
I_1 &\le C\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s} E\max_{1\le m\le n}\Big|\sum_{k=1}^{m}(X_{nk}^{(1)}-EX_{nk}^{(1)})\Big|^s\\
&\le C\left(\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s}(\log n)^s\left(\sum_{k=1}^{n}E|X_{nk}^{(1)}|^2\right)^{s/2} + \sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s}(\log n)^s\sum_{k=1}^{n}E|X_{nk}^{(1)}|^s\right)\\
&= C(I_{11}+I_{12}).
\end{aligned}$$

If $rp \ge 2$, we take $s$ large enough such that $r - 2 - s/p + s/2 < -1$. Note that $s > rp$. Then by Lemma 2.4,

$$I_{11} \le C\sum_{n=1}^{\infty} n^{r-2-s/p+s/2}(\log n)^s < \infty.$$

Since $s > rp$, $I_{12} < \infty$, by Lemma 2.5. If $rp < 2$, we take $s = 2$ (in this case, $I_{11} = I_{12}$). Then $I_{11} = I_{12} < \infty$, by Lemma 2.5 again.

Since $X_{nk}^{(2)} \geq 0$, we have

$$\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m} X_{nk}^{(2)}\Big| = \sum_{k=1}^{n} X_{nk}^{(2)},$$

and by Lemma 2.6,

$$\begin{aligned} & a_n^{-1}\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m} EX_{nk}^{(2)}\Big| \\ & \quad = a_n^{-1}\cdot nE\big((X-a_{n(t)})I(a_{n(t)}<X\leq a_n)+(a_n-a_{n(t)})I(X>a_n)\big)\to 0. \end{aligned}$$

Therefore, to prove $I_2<\infty$, it is enough to show that for all $\varepsilon>0$,

$$I_2' = \sum_{n=1}^{\infty} n^{r-2} P\left\{\Big|\sum_{k=1}^{n}(X_{nk}^{(2)}-EX_{nk}^{(2)})\Big| > \varepsilon a_n\right\} < \infty.$$

By the Markov inequality and Lemma 2.1, we get that for any $s\geq 2$,

$$\begin{aligned} I_2' &\leq C\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s} E\Big|\sum_{k=1}^{n}(X_{nk}^{(2)}-EX_{nk}^{(2)})\Big|^s \\ &\leq C\left(\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s}\left(\sum_{k=1}^{n} E|X_{nk}^{(2)}|^2\right)^{s/2} + \sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s}\sum_{k=1}^{n} E|X_{nk}^{(2)}|^s\right) \\ &= C(I_{21}+I_{22}). \end{aligned}$$

If $rp\geq 2$, we have by [7, Lemma 2.3] in Chen *et al.* and Lemma 2.3 that

$$\begin{aligned} a_n^{-2}\sum_{k=1}^{n} E|X_{nk}^{(2)}|^2 &\leq a_n^{-2}\cdot n\left(EX^2 I(a_{n(t)}<X\leq a_n)+a_n^2 P\{X>a_n\}\right) \\ &\leq a_n^{-2}\cdot nE|X|^2 I(|X|\leq a_n) + nP\{|X|>a_n\} \\ &\leq Cn^{1-2/p}+Cn^{1-r}\leq Cn^{1-2/p}. \end{aligned}$$

Taking $s$ large enough such that $r-2-s/p+s/2<-1$, we have

$$I_{21} = \sum_{n=1}^{\infty} n^{r-2}\left(a_n^{-2}\sum_{k=1}^{n} E|X_{nk}^{(2)}|^2\right)^{s/2} \leq C\sum_{n=1}^{\infty} n^{r-2-s/p+s/2} < \infty.$$

By Lemma 2.4 in [7] and (1.4),

$$\begin{aligned} I_{22} &\le C\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s} n\left(E|X|^s I(a_{n(t)} < X \le a_n) + a_n^s P\{X > a_n\}\right) \\ &\le C\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s} nE|X|^s I(|X| \le a_n) + C\sum_{n=1}^{\infty} n^{r-1} P\{|X| > a_n\} \\ &\le C\sum_{n=1}^{\infty} n^{r-1} P\{|X| > a_n\} < \infty. \end{aligned}$$

If $rp < 2$, we take $s = 2$ (in this case, $I_{21} = I_{22}$). Then

$$I_{21} = I_{22} \le \sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-2} n\left(EX^2 I(a_{n(t)} < X \le a_n) + a_n^2 P\{X > a_n\}\right) < \infty$$

by [7, Lemma 2.4] and (1.4) again.
By the same argument, as $I_2 < \infty$, we have $I_3 < \infty$.

(1.6) ⇒ (1.4). Noting that

$$\max_{1\le k\le n} |X_k| - a_n \le \max_{1\le k\le n} |X_k - EXI(|X|\le a_n)| \le 2\max_{1\le m\le n} |S_m - mEXI(|X|\le a_n)|,$$

we have by (1.6) that

$$\sum_{n=1}^{\infty} n^{r-2} P\{\max_{1\le k\le n} |X_k| > (1+\varepsilon)a_n\} < \infty, \quad \forall\, \varepsilon > 0,$$

which, together with Lemma 2.7, implies that

$$P\{\max_{1\le k\le n} |X_k| > (1+\varepsilon)a_{2n}\} \to 0$$

as $n \to \infty$. Hence, we have by Lemma A.6 in Zhang and Wen [25] that

$$\begin{aligned} nP\{|X| > (1+\varepsilon)a_{2n}\} &= \sum_{k=1}^{n} P\{|X_k| > (1+\varepsilon)a_{2n}\} \le CP\{\max_{1\le k\le n} |X_k| > (1+\varepsilon)a_{2n}\} \\ &\le CP\{\max_{1\le k\le n} |X_k| > (1+\varepsilon)a_n\}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X| > (1+\varepsilon)a_{2n}\} < \infty, \quad \forall\, \varepsilon > 0,$$

and hence (1.4) holds by Lemmas 2.8 and 2.9. So we complete the proof. □

PROOF OF THEOREM 1.2. The proof of (1.7) $\Leftrightarrow$ (1.9) can be done by the same method as in Theorem 2.3 of SUNG [22]. We proceed by proving that (1.7) $\Rightarrow$ (1.8) and (1.8) $\Rightarrow$ (1.7). We first prove that (1.7) $\Rightarrow$ (1.8). When $a_n/n \uparrow \infty$, by Lemma 2.4 in [22], we have

$$a_n^{-1} \cdot nEXI(|X| \le a_n) \to 0$$

as $n \to \infty$. Hence to prove (1.8), it is enough to prove that

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} |S_m| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0. \tag{3.3}$$

Note that $\max_{1\le m\le n} |S_m| \le \sum_{k=1}^n |X_k| = \sum_{k=1}^n X_k^+ + \sum_{k=1}^n X_k^-$, and $\{X_n^+, n\ge 1\}$ and $\{X_n^-, n \ge 1\}$ are sequences of pairwise negatively quadrant dependent random variables with

$$\sum_{n=1}^{\infty} P\{|X^{\pm}| > a_n\} \le \sum_{n=1}^{\infty} P\{|X| > a_n\} < \infty,$$

where $x^+ = \max\{0, x\}$ and $x^- = (-x)^+$. By the same method as in Theorem 2.2 of [22],

$$\sum_{n=1}^{\infty} n^{-1} P\left\{\sum_{k=1}^{n} X_k^{\pm} > \varepsilon a_n\right\} < \infty, \quad \forall\, \varepsilon > 0,$$

which implies (3.3).

When $a_n/n$ converges to a real positive number, $\sum_{n=1}^{\infty} P\{|X| > a_n\} < \infty$ is equivalent to $E|X| < \infty$, and hence

$$a_n^{-1} \cdot n|EXI(|X| \le a_n) - EX| = a_n^{-1} \cdot n|EXI(|X| > a_n)| \le a_n^{-1} \cdot nE|X|I(|X| > a_n) \to 0.$$

Therefore, to prove (1.8), it is enough to prove that

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} |S_m - mEX| > \varepsilon a_n\} < \infty, \quad \forall\, \varepsilon > 0. \tag{3.4}$$

But, (3.4) holds by the same method as in Theorem 1.2 of BAI *et al.* [2].

The proof of (1.8) $\Rightarrow$ (1.7) is similar to that in Theorem 1.1, and so we omit it. □

## 4. An application

In this section, we give the convergence rate in the strong law of large numbers for identically distributed pairwise negatively quadrant dependent random variables with the standard Cauchy distribution. Since the standard Cauchy distribution does not have finite moments of order greater than or equal to one, the classical law of large numbers cannot be applied to this case.

**Theorem 4.1.** *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of identically distributed pairwise negatively quadrant dependent random variables with the standard Cauchy distribution. Let* $\{a_n, n \geq 1\}$ *be a sequence of positive real numbers satisfying* $0 < a_n/n \uparrow$. *Then the following statements are equivalent:*

$$\sum_{n=1}^{\infty} a_n^{-1} < \infty, \tag{4.1}$$

$$\sum_{n=1}^{\infty} n^{-1} P\left\{\max_{1\leq m\leq n}\Big|\sum_{k=1}^{m} X_k\Big| > \varepsilon a_n\right\} < \infty, \quad \forall\, \varepsilon > 0, \tag{4.2}$$

$$a_n^{-1}\sum_{k=1}^{n} X_k \to 0 \quad a.s. \tag{4.3}$$

Proof. Since $X$ has the standard Cauchy distribution, we have that $EXI(|X| \leq a_n) = 0$. By Theorem 1.2, it remains to prove that (1.7) is equivalent to (4.1). Since $X$ has the standard Cauchy distribution, we obtain that

$$\sum_{n=1}^{\infty} P\{|X| > a_n\} = \frac{2}{\pi}\sum_{n=1}^{\infty}\int_{a_n}^{\infty}\frac{1}{(1+x^2)}\,dx.$$

Noting that $1/(2x^2) \leq 1/(1+x^2) \leq 1/x^2$ for $x \geq 1$, we have that $\sum_{n=1}^{\infty}\int_{a_n}^{\infty}\frac{1}{(1+x^2)}\times dx < \infty$ is equivalent to $\sum_{n=1}^{\infty} a_n^{-1} < \infty$. It follows that (1.7) is equivalent to (4.1). □

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## References

[1] N. Asadian, V. Fakoor and A. Bozorgnia, Rosenthal's type inequalities for negatively orthant dependent random variables, *J. Iranian Stat. Soc. ((JIRSS)* **5** (2006), 69–75.

[2] P. Bai, P. Chen and S.H. Sung, On complete convergence and the strong law of large numbers for pairwise independent random variables, *Acta Math. Hungar.* **142** (2014), 502–518.

[3] L. E. Baum and M. Katz, Convergence rates in the law of large numbers, *Trans. Amer. Math. Soc.* **120** (1965), 108–123.

[4] A. Bozorgnia, R. F. Patterson and R. L. Taylor, Limit theorems for dependent random variables, In: Proceedings of the First World Congress of Nonlinear Analysts, *de Gruyter, Berlin*, 1996, 1639–1650.

[5] P. Chen and S. H. Sung, Complete convergence and strong laws of large numbers for weighted sums of negatively orthant dependent random variables, *Acta Math. Hungar.* **148** (2016), 83–95.

[6] P. Chen and S. H. Sung, A Baum–Katz theorem for i.i.d. random variables with higher order moments, *Statist. Probab. Lett.* **94** (2014), 63–68.

[7] P. Chen, J. Yi and S. H. Sung, An extension of the Baum–Katz theorem to i.i.d. random variables with general moment conditions, *J. Inequal. Appl.* **2015** (2015), 9 pp.

[8] Y. S. Chow, Delayed sums and Borel summability of independent, identically distributed random variables, *Bull. Inst. Math. Acad. Sinica* **1** (1973), 207-220.

[9] I. Fazekas, On a general approach to the strong laws of large numbers, *J. Math. Sci. (N.Y.)* **200** (2014), 411–423.

[10] I. Fazekas and O. Klesov, A general approach to the strong law of large numbers, *Theory Probab. Appl.* **45** (2001), 436–449.

[11] A. Gut and U. Stadtmüller, An intermediate Baum–Katz theorem, *Statist Probab. Lett.* **81** (2011), 1486–1492.

[12] P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Nat. Acad. Sci. U.S.A.* **33** (1947), 25–31.

[13] K. Joag-Dev and F. Proschan, Negative association of random variables with applications, *Ann. Statist.* **11** (1983), 286–295.

[14] M. Katz, The probability in the tail of a distribution, *Ann. Math. Statist.* **34** (1963), 312–318.

[15] V. M. Kruglov, Strong law of large numbers for weighted sums, In: Stability Problems for Stochastic Models, Part II, *TVP/VSP, Moscow/Utrecht*, 1994, 139–150.

[16] V. M. Kruglov, A strong law of large numbers for pairwise independent identically distributed random variables with infinite means, *Statist. Probab. Lett.* **78** (2008), 890–895.

[17] H. Lanzinger, A Baum–Katz theorem for random variables under exponential moment conditions, *Statist. Probab. Lett.* **39** (1998), 89–95.

[18] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.* **37** (1966), 1137–1153.

[19] W. Li, P. Chen and T.-C. Hu, Complete convergence for moving average processes associated to heavy-tailed distributions and applications, *J. Math. Anal. Appl.* **420** (2014), 66–76.

[20] F. Móricz, Moment inequalities and the strong laws of large numbers, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **35** (1976), 299–314.

[21] D. Qiu, Q. Wu and P. Chen, Complete convergence for negatively orthant dependent random variables, *J. Ineq. Appl.* **2014** (2014), 12 pp.

[22] S. H. Sung, On the strong law of large numbers for pairwise i.i.d. random variables with general moment conditions, *Statist. Probab. Lett.* **83** (2013), 1963–1968.

[23] T. Tómács, A general method to obtain the rate of convergence in the strong law of large numbers, *Ann. Math. Inform.* **34** (2007), 97–102.

[24] T. Tómács and Z. Líbor, A Hájek–Rényi type inequality and its applications, *Ann. Math. Inform.* **33** (2006), 141–149.

[25] L. Zhang and J. Wen, Strong laws for sums of B-valued mixing random fields, *Chinese Ann. Math.* **20** (2001), 205–216 (in *Chinese*).

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