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On characterizations and topology of regular semimetric spaces

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Abstract. We examine semimetric spaces, i.e., spaces fulfilling two of the axioms of a metric space excluding the triangle inequality. We give an overview of many substitutes of the triangle condition which have appeared in the mathematical literature until now and examine the relations between them. We review and compare various ways of introducing a topology in a semimetric space and investigate its metrizability. We also answer some open questions posed by Bessenyei and Páles, Dung and Hang, and Khamsi and Hussain. In particular, we show that a semimetric space is regular in the sense of Bessenyei and Páles if and only if it is uniformly metrizable. This substantially improves a very recent result of Dung and Hang. We also describe a general method of constructing semimetrics satisfying the so-called c-relaxed polygonal inequality and having the property that every open ball is not open and, simultaneously, every closed ball is not closed.

1. Introduction

The concept of a metric space gives way to a myriad of possible generalizations, which have been appearing in the literature for many years, e.g., pseudometric spaces, quasimetric spaces, semimetric spaces and some special types of them. These classes of spaces are obtained by omitting some of the axioms of a metric space. In this paper, we focus our attention on the so-called semimetric spaces, i.e., spaces fulfilling two of the axioms of a metric space excluding the

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triangle condition. Such spaces were first examined by Fréchet [11], and then attracted the attention of mathematicians like Chittenden [4], Wilson [23] and many others. The natural idea regarding this topic is of course finding various substitutes of the classic triangle inequality and examining how they are reflected in the properties of the space. In the papers concerning the subject, many concepts of replacing the classic triangle inequality were given, but the connections between them were not properly established.

The purpose of our paper is to give a thorough overview of some of the aforementioned conditions, as well as examine the relations between them and in many cases, even conclude their equivalence. We also take a closer look at some topological properties of semimetric spaces, among others, at various ways of introducing a topology in a semimetric space. In the final sections we focus our attention on the so-called b-metric spaces and their applications. In this paper, we also answer some open questions posed in [2], [7], [14]. The first of them was asked by Khamsi and Hussain in [14] and concerns the problem of openness of open balls in semimetric spaces. Our Theorem 5.6 gives a general method of constructing semimetrics satisfying the so-called c-relaxed polygonal inequality (see Definition 5.4), for which every open ball is not open and simultaneously, every closed ball is not closed. The proof of Theorem 5.6 is based on Hewitt's [12] resolvability theorem. The second question, posed by Bessenyei and Páles [2], is the following: when is the closure operation defined in a semimetric space via limit operator idempotent? The answer is given in our Theorem 4.3. At last, the following question was asked very recently by Dung and Hang [7, Question 2.4]: is every regular semimetric space having a strong triangle function (see Definitions 2.3 and 2.4) metrizable? In fact, the affirmative answer to this question is already in Niemytzki's paper [19], as noticed by one of the referees. Nevertheless, we obtain here an even stronger result: our Theorem 3.2 gives a list of eight equivalent conditions and shows among others that a semimetric space is regular if and only if it is uniformly metrizable. This is a complete solution of the Dung and Hang [7] problem. Also it is worth noting that the Stone-type theorem for b-metric spaces (see Definition 5.1) obtained recently by An, Dung and Tuyen [1, Theorem 3.15] is an immediate consequence of Theorem 3.2. Indeed, every b-metric space is regular, so by Theorem 3.2, is metrizable and hence paracompact in view of the classical Stone theorem (see, e.g., [8, Theorem 4.1.1]). Finally, let us notice that recently, semimetric spaces have also attracted the attention of many mathematicians working in fixed point theory: see, e.g., the book of Kirk and Shahzad [22], and the references therein, as well as the paper [13].

2. Overview of various triangle-like conditions

In this section, we recall some basic notions and conditions which have been appearing in the mathematical literature since the beginning of the 20th century. Let us begin with introducing the basic notion of a semimetric space.

Definition 2.1. Let $X \neq \emptyset$, $d : X \times X \to [0, \infty)$. We call the pair (X, d) a *semimetric space* if for every $x, y \in X$:

(M1) $d(x, y) = 0 \iff x = y$;

(M2) $d(x, y) = d(y, x)$.

Remark 2.2. One of the referees pointed out to us that in the literature, e.g., in [17, Chapter 10], a different terminology is used. Namely, if we let d generate a topology τ_d (for the definition, see Section 4), then the topological space (X, τ_d) is called a *symmetric* space, whereas a semimetric space means a symmetric space in which all open balls are neighbourhoods. However, in our paper we adopted the terminology from Wilson's [23] pioneering paper, similarly as done in recent papers, e.g., [2], [7] and [16], and the monograph [15].

In a semimetric space one defines notions such as *open ball*, *convergence*, *Cauchy sequence*, *completeness*, etc. in the same way as in metric spaces. Spaces fulfilling just (M1) and (M2) can be quite far from metric spaces. On the other hand, we can consider semimetric spaces with an additional axiom, which in a way substitutes the classical metric triangle inequality, but is actually weaker. This results in obtaining various properties of a semimetric space to a smaller or greater extent reflecting those of a metric space. It was already mentioned in Section 1 that semimetric spaces were first examined by Fréchet, Chittenden and Wilson. Chittenden [4] gave the following substitute of the triangle inequality:

> *"There exists a positive function ψ such that $\lim_{t\to 0}\psi(t)=0$ and if $d(x,y)\le\varepsilon$ and $d(y,z)\le\varepsilon$, then $d(x,z)\le\psi(\varepsilon)$, for any $\varepsilon>0$."*

However, he did not specify whether the function ψ can attain infinite values. This leads to two possible interpretations of his idea, which we will refer to as conditions (Chi) and (Chi$^\infty$):

(Chi) there exists a function $\psi : [0, \infty) \to [0, \infty)$ such that: $\psi(0) = 0$, $\lim_{t\to 0^+} \psi(t) = 0$ and

$$\forall_{\varepsilon>0}\, \forall_{x,y,z\in X}\, [(d(x, y) \le \varepsilon \wedge d(y, z) \le \varepsilon) \Longrightarrow d(x, z) \le \psi(\varepsilon)]\,; \tag{1}$$

(Chi$^\infty$) there exists a function $\psi : [0, \infty) \to [0, \infty]$ such that: $\psi(0) = 0$, $\lim_{t\to 0^+} \psi(t) = 0$ and (1) holds.

In particular, in the monograph of ENGELKING [8, p. 417] there appears condition (Chi). In 1931, WILSON [23] considered the following conditions as possible substitutes of the triangle inequality:

(W3) $\forall_{x,y\in X}\ \forall_{(x_n)\subseteq X}\ [(d(x_n,x)\to 0 \wedge d(x_n,y)\to 0) \Longrightarrow x=y]$;

(W4) $\forall_{x\in X}\ \forall_{(x_n),(y_n)\subseteq X}\ [(d(x_n,x)\to 0 \wedge d(x_n,y_n)\to 0) \Longrightarrow d(y_n,x)\to 0]$;

(W5) $\forall_{(x_n),(y_n),(z_n)\subseteq X}\ [(d(x_n,y_n)\to 0 \wedge d(y_n,z_n)\to 0) \Longrightarrow d(x_n,z_n)\to 0]$.

(In fact, as pointed out by one of the referees, conditions (W4) and (W5) are attributed to Pitcher and Chittenden; see conditions (2) and (4) in [21], respectively.) WILSON [23] concluded that in a semimetric space (W5) is equivalent to the condition of Chittenden. However, this is true only under the assumption that the function ψ may attain infinite values, i.e., ψ is as in (Chi$^\infty$), which we illustrate in the following example.

Example 2.1. Let $X=\mathbb{R}$ and define a semimetric d as follows:

$$d(x,y) := \begin{cases} |x-y|, & \text{if } |x-y|\le 1,\\ |x|+|y|, & \text{if } |x-y|>1. \end{cases}$$

It is easily seen that if $|x-y|>1$, then $d(x,y)>1$, so $d(x,y)\le 1$ implies that $d(x,y)=|x-y|$. Hence we may infer that if $d(x_n,y_n)\to 0$ and $d(y_n,z_n)\to 0$, then $|x_n-y_n|\to 0$ and $|y_n-z_n|\to 0$, so $|x_n-z_n|\to 0$. Again, by the definition of d, we get that $d(x_n,z_n)\to 0$, so (W5) is satisfied. On the other hand, condition (Chi) is not fulfilled. Indeed, let $(x_n),(y_n),(z_n)$ be defined as follows:

$$x_n := n, \quad y_n := n+1 \quad \text{and} \quad z_n := n+2 \qquad \text{for } n\in\mathbb{N}.$$

Now suppose, on the contrary, that condition (Chi) holds. Then $d(x_n,y_n) = d(y_n,z_n)=1$ implies that $d(x_n,z_n)\le\psi(1)$, which yields a contradiction, since $d(x_n,z_n)=2n+2\to\infty$.

A more up-to-date substitute of the triangle inequality comes from the work of BESSENYEI and PÁLES [2], who proposed the following notion of *regularity* of a semimetric space.

Definition 2.3. We call a semimetric space (X,d) *regular* if there exists a function $\Phi:[0,\infty)^2\to[0,\infty]$ such that $\Phi(0,0)=0$, Φ is symmetric, continuous at the origin, nondecreasing in each of its variables and for all $x,y,z\in X$,

$$d(x,z)\le\Phi(d(x,y),d(y,z)).$$

Then Φ is said to be a *triangle function* for d. This definition has recently been strengthened by Kirk and Shahzad [16] in the following way.

Definition 2.4. A triangle function Φ for semimetric d is called a *strong triangle function* if for all $x, y, z, w \in X$,

$$|d(x,y) - d(z,w)| \leq \Phi\left(d(x,z), d(y,w)\right).$$

Very recently, Dung and Hang [7, Theorem 2.1] have shown that if (X, d) is a regular semimetric space having the strong triangle function, which is continuous in each of its variables at $(0,0)$, then (X, d) is metrizable with a metric ρ such that for any $(x_n) \subseteq X$, (x_n) is a Cauchy sequence in (X, d) if and only if (x_n) is a Cauchy sequence in (X, ρ). Consequently, (X, d) is complete if and only if (X, ρ) is complete. As already mentioned in the Introduction, we obtain a much stronger result in Section 3: see the equivalence (iv) $\iff$ (viii) in Theorem 3.2, which also immediately implies that the classes of Cauchy sequences in (X, d) and (X, ρ) are identical.

3. Equivalent conditions for the regularity of a semimetric space

To understand the nature of the conditions discussed in the previous section and also to show the connections between them, we have collected some of them and shown that they are in fact equivalent. Note that among them there are conditions (W5), (Chi$^\infty$) and the regularity in the Bessenyei–Páles sense. Moreover, the equivalence (iv) $\iff$ (viii) shows that the answer to [7, Question 2.4] is positive. In its proof we use [8, Theorem 8.1.10], which is attributed to Tukey (see [8, p. 535]).

Theorem 3.1 (Tukey). *For any sequence $(V_n)_{n\in\mathbb{N}}$ of members of a uniformity on a set X, where $V_0 := X \times X$ and $3V_{n+1} \subseteq V_n$ for $n \in \mathbb{N}$, there exists a pseudometric ρ on X such that for any $n \in \mathbb{N}$,*

$$V_n \subseteq \left\{(x,y) \in X \times X : \rho(x,y) \leq \frac{1}{2^n}\right\} \subseteq V_{n-1}.$$

Now let (X, d) be a semimetric space. For $x \in X$ and $r > 0$, denote

$$B(x,r) := \{y \in X : d(y,x) < r\}.$$

Theorem 3.2. *Let (X, d) be a semimetric space. The following conditions are equivalent:*

(i) $\forall_{(x_n),(y_n),(z_n)\subseteq X}\ [(d(x_n, y_n) \to 0 \wedge d(y_n, z_n) \to 0) \Longrightarrow d(x_n, z_n) \to 0]$;

(ii) $\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,y,z\in X}\ (\max\{d(x, y), d(y, z)\} < \delta \Longrightarrow d(x, z) < \varepsilon)$;

(iii) $\lim\limits_{\varepsilon\to 0^+} \sup\limits_{y\in X} \operatorname{diam} B(y, \varepsilon) = 0$;

(iv) *(X, d) is regular;*

(v) *there exists a function $\varphi : [0,\infty) \to [0,\infty]$ such that $\varphi(0) = 0$, φ is continuous at the origin, nondecreasing, and for all $x, y, z \in X$,*

$$d(x, z) \le \varphi(d(x, y) + d(y, z));$$

(vi) *there exists a function $\psi : [0,\infty) \to [0,\infty]$ such that $\psi(0)=0$, $\lim\limits_{t\to 0^+} \psi(t)=0$, and*

$$\forall_{\varepsilon>0}\ \forall_{x,y,z\in X}\ (\max\{d(x, y), d(y, z)\} \le \varepsilon \Longrightarrow d(x, z) \le \psi(\varepsilon))\,;$$

(vii) *the family $\mathcal{B} := \{V_n : n \in \mathbb{N}\}$, where for $n \in \mathbb{N}$,*

$$V_n := \left\{(x, y) \in X \times X : d(x, y) < \frac{1}{n}\right\},$$

is a base of a uniform structure on X;

(viii) *(X, d) is uniformly metrizable, i.e., there exists a metric ρ on X such that d and ρ are uniformly equivalent, that is*

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall x, y \in X \quad (d(x, y) < \delta \implies \rho(x, y) < \varepsilon)$$

and

$$\forall\varepsilon > 0\ \exists\delta > 0\ \forall x, y \in X \quad (\rho(x, y) < \delta \implies d(x, y) < \varepsilon)\,.$$

Proof.

(i) $\Longrightarrow$ (ii). Suppose to the contrary that there exists $\varepsilon > 0$ such that for every $\delta > 0$, there exist $x, y, z \in X$ such that $\max\{d(x, y), d(y, z)\} < \delta$ and $d(x, z) \ge \varepsilon$. Then in particular, for every $n \in \mathbb{N}$, there exist $x_n, y_n, z_n \in X$ such that

$$\max\{d(x_n, y_n), d(y_n, z_n)\} < \frac{1}{n} \text{ and } d(x_n, z_n) \ge \varepsilon.$$

Consider the sequences $(x_n), (y_n), (z_n)$. Clearly, $d(x_n, y_n) \to 0$, $d(y_n, z_n) \to 0$, but $d(x_n, z_n) \ge \varepsilon$ for $n \in \mathbb{N}$, so $d(x_n, z_n) \nrightarrow 0$, a contradiction.

(ii) $\Longrightarrow$ (iii). Condition (ii) is equivalent to

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{x,y,z\in X}\ \ (x,z\in B(y,\delta)\Longrightarrow d(x,z)<\varepsilon)\,,$$

and implies further that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \sup_{y\in X}\operatorname{diam}B(y,\delta)\le\varepsilon.$$

Obviously, if $0<\delta_1\le\delta_2$, then $\sup_{y\in X}\operatorname{diam}B(y,\delta_1)\le\sup_{y\in X}\operatorname{diam}B(y,\delta_2)$, so we obtain that

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{0<\delta'\le\delta}\ \sup_{y\in X}\operatorname{diam}B(y,\delta')\le\varepsilon,$$

which means that (iii) is fulfilled.

(iii) $\Longrightarrow$ (iv). It was proved by Bessenyei and Páles [2].

(iv) $\Longrightarrow$ (v). For $t\in[0,\infty)$, define $\varphi(t):=\Phi(t,t)$. Then $\varphi:[0,\infty)\to[0,\infty]$, $\varphi(0)=0$, φ is nondecreasing and continuous at the origin. Moreover, for arbitrary $x,y,z\in X$, we have

$$\begin{aligned}d(x,z)&\le\Phi(d(x,y),d(y,z))\le\Phi(d(x,y)+d(y,z),d(x,y)+d(y,z))\\&=\varphi(d(x,y)+d(y,z)),\end{aligned}$$

so the implication is proved.

(v) $\Longrightarrow$ (vi). Put $\psi(t):=\varphi(2t)$. Then of course $\psi:[0,\infty)\to[0,\infty]$, $\psi(0)=0$, ψ is continuous at the origin. Now let $\varepsilon>0$ and $x,y,z\in X$, and assume that $\max\{d(x,y),d(y,z)\}\le\varepsilon$. From (v), we obtain that

$$d(x,z)\le\varphi(d(x,y)+d(y,z))\le\varphi(\varepsilon+\varepsilon)=\psi(\varepsilon).$$

(vi) $\Longrightarrow$ (i). Let $(x_n),(y_n),(z_n)\subseteq X$ be such that $d(x_n,y_n)\to0$, $d(y_n,z_n)\to0$. Fix $\varepsilon>0$. Since $\lim_{t\to0^+}\psi(t)=0$, there is $\delta>0$ be such that $\psi(\delta)<\varepsilon$. Then there exists $n_0\in\mathbb{N}$ with $d(x_n,y_n),d(y_n,z_n)\le\delta$ for $n\ge n_0$. For such n, from (vi), we have that $d(x_n,z_n)\le\psi(\delta)<\varepsilon$.

We have shown that conditions (i)-(vi) are equivalent.

(v) $\Longrightarrow$ (vii). Notice that for every $i,j\in\mathbb{N}$, we have that $V_i\cap V_j=V_{\max\{i,j\}}$. By [8, Proposition 8.1.14] it suffices to show that the following conditions are satisfied:

(BU1) $\forall_{V_1,V_2\in\mathcal{B}}\ \exists_{V\in\mathcal{B}}\ V\subseteq V_1\cap V_2$;

(BU2) $\forall_{V\in\mathcal{B}}\ \exists_{W\in\mathcal{B}}\ 2W := \{(x,z) : \exists_{y\in X}\ ((x,y)\in W \wedge (y,z)\in W)\} \subseteq V$;
(BU3) $\bigcap \mathcal{B} = \triangle := \{(x,x) :\ x\in X\}$.

Hence (BU1) is satisfied in a trivial way. Now we have to show that

$$\forall_{n\in\mathbb{N}}\ \exists_{k\in\mathbb{N}}\ 2V_k = \left\{(x,z) : \exists_{y\in X}\ \left(d(x,y) < \frac{1}{k}\ \wedge d(y,z) < \frac{1}{k}\right)\right\} \subseteq V_n.$$

Fix $n\in\mathbb{N}$ and take $k\in\mathbb{N}$ such that $\varphi\left(\frac{2}{k}\right) < \frac{1}{n}$. For any pair $(x,z)\in 2V_k$, we have

$$d(x,z) \le \varphi(d(x,y)+d(y,z)) \le \varphi\left(\frac{1}{k}+\frac{1}{k}\right) < \frac{1}{n}.$$

Thus $(x,z)\in V_n$, and (BU2) is fulfilled as well. It remains to show that (BU3) holds, i.e., $\bigcap\limits_{n\in\mathbb{N}} V_n = \triangle$. Obviously, $\triangle \subseteq \bigcap\limits_{n\in\mathbb{N}} V_n$, and if we take $(x,y)\in \bigcap\limits_{n\in\mathbb{N}} V_n$, then $d(x,y) < \frac{1}{n}$ for $n\in\mathbb{N}$, so $d(x,y)=0$, and hence $(x,y)\in\triangle$.

(vii) $\Rightarrow$ (i) Let $\varepsilon > 0$. Let $m\in\mathbb{N}$ be such that $\frac{1}{m} < \varepsilon$. By (vii), there exists $k\in\mathbb{N}$ with $2V_k \subseteq V_m$. Let $(x_n), (y_n), (z_n) \subseteq X$ be such that $d(x_n,y_n)\to 0$ and $d(y_n,z_n)\to 0$. There exists $n_0\in\mathbb{N}$ such that for $n\ge n_0$,

$$\max\{d(x_n,y_n), d(y_n,z_n)\} < \frac{1}{k}.$$

Then $(x_n,z_n)\in 2V_k\subseteq V_m$, so $d(x_n,z_n) < \frac{1}{m} < \varepsilon$ for $n\ge n_0$. Thus $d(x_n,z_n)\to 0$.

(v) $\Longrightarrow$ (viii) For $n\in\mathbb{N}$, set $U_n := \left\{(x,y)\in X\times X : d(x,y) < \frac{1}{n}\right\}$. We will define an increasing sequence (k_n) of positive integers such that

$$3U_{k_{n+1}} \subseteq U_{k_n} \quad \text{for } n\in\mathbb{N}.$$

Set $k_1 := 1$. Now let $n\in\mathbb{N}$, and assume that k_n is already defined. By hypothesis, there exists $\delta > 0$ such that $\varphi(\delta) < \frac{1}{k_n}$. Again, since $\lim\limits_{t\to 0^+}\varphi(t) = 0$, there exists $k_{n+1}\in\mathbb{N}$ such that

$$\varphi\left(\frac{2}{k_{n+1}}\right) < \frac{\delta}{2} \quad \text{and} \quad k_{n+1} > \max\left\{\frac{2}{\delta}, k_n\right\}.$$

Let $(x,y)\in 3U_{k_{n+1}}$. Then there exists $u,v\in X$ such that $(x,u),(u,v),(v,y)\in U_{k_{n+1}}$, i.e.,

$$\max\{d(x,u), d(u,v), d(v,y)\} < \frac{1}{k_{n+1}}.$$

Then, using the monotonicity of φ, we get that

$$d(x,y) \le \varphi(d(x,u)+d(u,y)) \le \varphi(d(x,u)+\varphi(d(u,v)+d(v,y))). \tag{2}$$

Since $d(u,v)+d(v,y) < \frac{2}{k_{n+1}}$, we obtain that

$$\varphi(d(u,v)+d(v,y)) \le \varphi\left(\frac{2}{k_{n+1}}\right) < \frac{\delta}{2}.$$

Furthermore, we have that

$$d(x,u) < \frac{1}{k_{n+1}} < \frac{\delta}{2}.$$

Consequently, we get that

$$d(x,u)+\varphi(d(u,v)+d(v,y)) < \delta.$$

Hence, and by (2), we obtain that $d(x,y) \le \varphi(\delta) < \frac{1}{k_n}$, so $(x,y) \in U_{k_n}$. Now set $V_0 := X \times X$ and $V_n := U_{k_n}$ for $n \in \mathbb{N}$. By Theorem 3.1, there exists a pseudometric ρ on X such that for any $n \in \mathbb{N}$,

$$V_n \subseteq \left\{(x,y) \in X \times X : \rho(x,y) \le \frac{1}{2^n}\right\} \subseteq V_{n-1}. \tag{3}$$

Hence, if $\rho(x,y)=0$, then $(x,y) \in \bigcap_{n\in\mathbb{N}} V_{n-1} = \Delta$, so $x=y$. Thus ρ is a metric on X, and moreover, by (3), for any $n \in \mathbb{N}$ and $x,y \in X$, the following implications hold:

$$\text{if } \rho(x,y) \le \frac{1}{2^{n+1}}, \text{ then } d(x,y) < \frac{1}{k_n}; \quad \text{if } d(x,y) < \frac{1}{k_n}, \text{ then } \rho(x,y) \le \frac{1}{2^n}.$$

This easily yields that d and ρ are uniformly equivalent.

(viii) $\Longrightarrow$ (i) Let $x_n, y_n, z_n \in X$ for $n \in \mathbb{N}$, and assume that $d(x_n,y_n) \to 0$ and $d(y_n,z_n) \to 0$. By hypothesis, d and ρ are uniformly equivalent, so we may infer that $\rho(x_n,y_n) \to 0$ and $\rho(y_n,z_n) \to 0$, which implies that $\rho(x_n,z_n) \to 0$ since ρ is a metric. Again, by hypothesis, we obtain that $d(x_n,z_n) \to 0$, which completes the proof. $\square$

4. Topologies in a semimetric space

It is natural to consider in what ways one can introduce a topology in a semimetric space. In [1], the authors considered b-metric spaces (i.e., a special case of semimetric spaces – see Definition 5.1) with various topologies. We would like to present these considerations in a more general setting. Let us consider the following topologies in a semimetric space (X, d).

(1) $\tau_d := \{U \subseteq X : \forall_{x\in U}\ \exists_{r>0}\ \ B(x, r) \subseteq U\}$.

(2) Let τ^d be a topology on X, generated by the following base:

$$\mathcal{B} := \{B(x_1, r_1) \cap \cdots \cap B(x_n, r_n) : n \in \mathbb{N}, x_1, ..., x_n \in X, r_1, ..., r_n > 0\}.$$

(3) Now let us introduce a topology via a limit operator. First, let us recall the following definition (see, e.g., [8, Problem 1.7.18]).

Definition 4.1. Let X be a nonempty set. A mapping $\lambda : D_\lambda \to X$, where $\emptyset \neq D_\lambda \subseteq X^{\mathbb{N}}$, is called a *limit operator* on X, if it satisfies the following conditions:

(L1) if $x_n = x$ for all $n \in \mathbb{N}$, then $\lambda(x_n) = x$;

(L2) if $\lambda(x_n) = x$, then $\lambda(x_{k_n}) = x$ for every subsequence (x_{k_n}) of (x_n);

(L3) if (x_n) does not converge to x, then it contains a subsequence (x_{k_n}) such that no subsequence of (x_{k_n}) converges to x.

Note that the only technical restriction imposed on the set D_λ in the above definition is for it to include all constant sequences. Using a limit operator λ, one can define the operation of closure by letting

$$x \in \overline{A} \text{ if and only if } A \text{ contains a sequence } (x_n) \text{ such that } \lambda(x_n) = x.$$

Of course, in a semimetric space satisfying condition (W3), one can define the limit operator in a natural way, namely, we put

$$D_\lambda := \{(x_n) \subseteq X : \exists_{x\in X}\ d(x_n, x) \to 0\},$$

and for $(x_n) \in D_\lambda$, $\lambda(x_n) := x$.

Then, the topology obtained via the limit operator is of the following form:

$$\tau := \{U \subseteq X : X \setminus U \text{ is sequentially closed}\},$$

where a set is called *sequentially closed* if it contains all limits of its convergent sequences.

Now we present a theorem which shows some connections between the above topologies. It is a generalization of [1, Proposition 3.3].

Theorem 4.2. *Let* (X, d) *be a semimetric space. Then* $\tau_d = \tau \subseteq \tau^d$.

Proof. We will first show that $\tau_d \subseteq \tau$. Let $U \in \tau_d$. Let $x_n \in X \setminus U$ for $n \in \mathbb{N}$, $x \in X$ and $d(x_n, x) \to 0$. Now suppose that $x \in U$. Then there exists $r > 0$ such that $B(x, r) \subseteq U$. Also, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $d(x_n, x) < r$, hence $x_n \in B(x, r) \subseteq U$, which yields a contradiction. Now we will show the reverse inclusion. Let $U \in \tau$. Suppose that $U \notin \tau_d$. Then there exists $x_0 \in U$ such that $B(x_0, \frac{1}{n}) \nsubseteq U$, for any $n \in \mathbb{N}$. Hence, for $n \in \mathbb{N}$, there exists $x_n \in X \setminus U$ such that $d(x_n, x_0) < \frac{1}{n}$, so $x_n \to x_0$. Since $X \setminus U$ is sequentially closed, we get that $x_0 \in X \setminus U$, which yields a contradiction. The inclusion $\tau_d \subseteq \tau^d$ is obvious. □

It is worth to note here that, in general, $\tau \not\supseteq \tau^d$, as there exist semimetrics for which some open balls are not open. The examples can be found in [1], [20] and [22]. We will also present other examples in the next section.

Now we will examine the connection between a classic condition concerning limit operators and Wilson's axioms. Note that for the closure operation defined in (3) to be idempotent, it is necessary and sufficient for the limit operator to fulfil an additional condition [8, Problem 1.7.18]:

(L4) If $\lambda(x_n) = x$ and $\lambda(y_n(k)) = x_n$ for $n \in \mathbb{N}$, then there exist sequences of positive integers $k_1, k_2, \ldots$ and $m_1, m_2, \ldots$ such that $\lambda(y_{m_n}(k_n)) = x$.

Moreover, it turns out that the following theorem holds, which binds together one of the Wilson's axioms and the above condition. This answers the question posed by Bessenyei and Páles [2] at the end of their paper. One of the referees pointed out to us that this answer is rather well known and can be found, e.g., in [18], however, we hope that our Theorem 4.3 and Example 4.1 shed more light on relations between conditions (W4) and (L4).

Theorem 4.3. *For any semimetric space* (X, d), (W4) *implies* (L4). *In particular, if* (X, d) *is regular, then the closure operation is idempotent.*

Proof. We will show that (L4) is satisfied. Assume that $d(x_n, x) \overset{n\to\infty}{\longrightarrow} 0$, and for $n \in \mathbb{N}$, $d(y_n(k), x_n) \overset{k\to\infty}{\longrightarrow} 0$. For every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $d(x_n, y_n(k_n)) < \frac{1}{n}$. As (W4) is fulfilled, we obtain that $d(y_n(k_n), x) \overset{n\to\infty}{\longrightarrow} 0$. □

However, the following example shows that the above implication cannot be reversed.

Example 4.1. Let $X := \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, and define a semimetric $d : X \times X \to [0, \infty)$ as follows:

$$d(x,y) := \begin{cases} 1, & \text{for } x = \frac{1}{2n-1},\ y = 0 \text{ or } x = 0,\ y = \frac{1}{2n-1},\ n \in \mathbb{N}, \\ |x-y|, & \text{otherwise.} \end{cases}$$

Notice that (X, d) does not fulfil (W4) (consider the sequences $x_n := \frac{1}{2n}$ and $y_n := \frac{1}{2n-1}$, and $x := 0$). Now we will show that (L4) holds for (X, d). Take $(x_n)_{n\in\mathbb{N}} \subseteq X$ convergent to $x \in X$, and for each $n \in \mathbb{N}$ let $y_n(k) \overset{k\to\infty}{\longrightarrow} x_n$. It suffices to show that there exists a sequence (z_n) such that $z_n \to x$ and $z_n \in \{y_m(k) : k, m \in \mathbb{N}\}$. Consider the following two cases:

(I) $x \neq 0$.

Then there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for $n \geq n_0$. Analogously, every sequence $(y_n(k))_{k\in\mathbb{N}}$ with $n \geq n_0$ is constant for sufficiently large k. Precisely:

$$\forall_{n\geq n_0}\ \exists_{k_n\in\mathbb{N}}\ \forall_{j\geq k_n} y_n(j) = x.$$

Consider $(z_n)_{n\in\mathbb{N}}$ defined as follows:

$$z_n := \begin{cases} y_n(n), & \text{if } n < n_0, \\ y_n(k_n), & \text{if } n \geq n_0. \end{cases}$$

Clearly, $z_n = x$ for $n \geq n_0$, so $z_n \to x$.

(II) $x = 0$.

Note that if $x_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $x_n \in \left\{\frac{1}{2k} : k \in \mathbb{N}\right\} \cup \{0\}$. Without loss of generality, assume that $x_n \in \left\{\frac{1}{2k} : k \in \mathbb{N}\right\} \cup \{0\}$ for $n \in \mathbb{N}$. Let us consider the following partition of $\mathbb{N}$:

$$\mathcal{N}_1 := \{n \in \mathbb{N} :\ x_n \neq 0\} \quad \text{and} \quad \mathcal{N}_2 := \{n \in \mathbb{N} :\ x_n = 0\}.$$

Let $n \in \mathbb{N}$. If $n \in \mathcal{N}_1$, then there exists $k_n \in \mathbb{N}$ such that $y_n(k_n) = x_n$. If $n \in \mathcal{N}_2$, then $d(y_n(k), 0) \to 0$, so there exists k_n for which $d(y_n(k_n), 0) < \frac{1}{n}$. Consider the sequence $(z_n)_{n\in\mathbb{N}}$, where $z_n := y_n(k_n)$ for $n \in \mathbb{N}$. Then $z_n \to 0$, since

$$0 \leq d(z_n, 0) = d(y_n(k_n), 0) \leq \max\left\{\frac{1}{n}, d(x_n, 0)\right\} \to 0.$$

At the end of this section, it is natural to ask a question concerning metrizability of a semimetric space. We have already considered the case of regular semimetric spaces, which turn out to be uniformly metrizable according to Theorem 3.2. In [1], the authors proved that every b-metric space (whose definition will be recalled in the next section) is semimetrizable (see [1, Theorem 3.4]). However, Chittenden [4] had proved a stronger result much earlier, showing even metrizability for a wider class of semimetric spaces, i.e., those fulfilling the condition (Chi). The connections between those spaces and b-metric spaces will be discussed in the next section. Subsequently, Chittenden's result was extended by Niemytzki [19] and Wilson [23], who had proved that (W4) is in fact a sufficient condition for metrizability of a semimetric space.

5. *b*-metric spaces and *c*-relaxed polygonal inequality

In this section, we will focus our attention on the so-called b-metric spaces, which were already mentioned in the previous section. It may be good to point out here that this notion originates from the book of Bourbaki [3], who used the term quasiultrametric. However, in many papers its origin is associated with Czerwik's paper [6], who introduced the term b-metric.

Definition 5.1. Let (X, d) be a semimetric space, $K \geq 1$. (X, d) is called a *b-metric space* if d fulfils the following condition:

(B3) $d(x, z) \leq K(d(x, y) + d(y, z))$.

However, we note that the conditions (B3) and (Chi) are in fact connected.

Proposition 5.2. *For any semimetric space* (X, d), (B3) *implies* (Chi).

Proof. Assume that (X, d) fulfils (B3). It is easily seen that (Chi) holds with a function ψ defined by $\psi(t) := 2Kt$ for $t \in [0, \infty)$. □

On the other hand, the following theorem describes a large class of semimetrics satisfying condition (Chi) which are not b-metrics.

Theorem 5.3. *Let (X, ρ) be an unbounded metric space. Then there exists a semimetric d such that $\tau_d = \tau_\rho$, d satisfies Chittenden's condition* (Chi) *and d is not a b-metric.*

Proof. By hypothesis, there exists a sequence $(x_n)_{n=0}^{\infty}$ such that

$$\rho(x_n, x_0) \to \infty, \quad r := \rho(x_0, x_1) \geq 1, \quad \rho(x_0, x_n) > r, \quad \text{for } n \geq 2.$$

Denote by $\overline{B}_\rho(x_0, r)$ the closed ball with center x_0 and radius r in (X, ρ). For $x, y \in X \setminus \{x_0\}$, set $d(x, y) := \rho(x, y)$, and for $x \in X$, define

$$d(x, x_0) := \begin{cases} \rho(x, x_0), & \text{if } x \in \overline{B}_\rho(x_0, r), \\ \rho^2(x, x_0)), & \text{otherwise,} \end{cases}$$

and $d(x_0, x) := d(x, x_0)$. Observe that $d(x, y) = \rho(x, y)$ if $\rho(x, y) \leq r$, so if (y_n) is such that $\rho(y_n, y) \to 0$ for some $y \in X$, then $d(y_n, y) \to 0$. On the other hand, it follows from the definition of d that $d(x, y) \geq \rho(x, y)$ for any $x, y \in X$, so if $d(y_n, y) \to 0$, then $\rho(y_n, y) \to 0$. Consequently, by Theorem 4.2, we may infer that $\tau_d = \tau_\rho$.

Now suppose, on the contrary, that d is a b-metric. Then there exists $K \geq 1$ such that for any $n \geq 2$,

$$d(x_0, x_n) \leq K\left(d(x_0, x_1) + d(x_1, x_n)\right), \text{ i.e.,}$$

$$\rho^2(x_0, x_n) \leq K\left(\rho(x_0, x_1) + \rho(x_1, x_n)\right) \leq K\left(2\rho(x_0, x_1) + \rho(x_0, x_n)\right).$$

Hence $\rho^2(x_0, x_n) - K\rho(x_0, x_n) - 2K\rho(x_0, x_1) \leq 0$, so letting n tend to ∞, we obtain that $\infty \leq 0$, a contradiction.

Finally, we show that d satisfies (Chi). Let $\varepsilon > 0$, $x, y, z \in X$, and $d(x, y) \leq \varepsilon$ and $d(y, z) \leq \varepsilon$. If $x, y, z \in X \setminus \{x_0\}$, then $d(x, z) \leq 2\varepsilon$. If $x = y$ or $x = y$ or $y = z$, then $d(x, z) \leq \varepsilon$. So assume that x, y, z are pairwise distinct and $x_0 \in \{x, y, z\}$. The following two cases are possible:

(1) $x = x_0$ or $z = x_0$. Then either $d(x, z) = \rho(x, z)$ or $d(x, z) = \rho^2(x, z)$.
In the first case, since $\rho \leq d$, we get

$$d(x, z) \leq \rho(x, y) + \rho(y, z) \leq d(x, y) + d(y, z) \leq 2\varepsilon.$$

In the second case, we have

$$d(x, z) \leq \left(\rho(x, y) + \rho(y, z)\right)^2 \leq \left(d(x, y) + d(y, z)\right)^2 \leq 4\varepsilon^2.$$

(2) $y = x_0$. Then we have

$$d(x, z) = \rho(x, z) \leq \rho(x, y) + \rho(y, z) \leq d(x, y) + d(y, z) \leq 2\varepsilon.$$

Consequently, condition (Chi) is satisfied with ψ defined by

$$\psi(\varepsilon) := \begin{cases} 2\varepsilon, & \text{for } 0 < \varepsilon \leq \frac{1}{2}, \\ 4\varepsilon^2, & \text{for } \varepsilon > \frac{1}{2}. \end{cases}$$

□

FAGIN *et al.* [9] proposed a stronger concept than the one of a b-metric, which they actually call "distance measures". Instead of simply requiring the function d to fulfil a relaxed form of the triangle inequality, they impose a condition which allows to carry even more properties of metric spaces to a semimetric space (X, d).

Definition 5.4. Let (X, d) be a semimetric space and $c \geq 1$. We say that the function d satisfies a *c-relaxed polygonal inequality* if, for every $n \in \mathbb{N}$ and for every finite sequence $x_0, x_1, \ldots, x_n$ of elements of X, the following inequality holds:

$$d(x_0, x_n) \leqslant c\left(d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)\right).$$

It is easy to see that every semimetric d which satisfies the c-relaxed polygonal inequality is a b-metric. The reverse implication, however, does not hold. A proper counterexample is given in [9]. Another obvious fact is that a semimetric d is in fact a metric if and only if it satisfies the 1-relaxed polygonal inequality. The authors of [9] also gave an interesting characterization of semimetrics which satisfy the condition from Definition 5.4.

Theorem 5.5 (Fagin et al.)**.** *Let (X, d) be a semimetric space and $c \geq 1$. Then d satisfies the c-relaxed polygonal inequality if and only if there exists a metric ρ on X such that*

$$\forall_{x,y \in X}\, \rho(x, y) \leqslant d(x, y) \leqslant c\rho(x, y).$$

Now we will come back to the discussion on the possible lack of openness of open balls in semimetric spaces. As we have already mentioned, the question about openness of an open ball, posed by KHAMSI and HUSSAIN in [14], had already been answered in [22], [20] and [1]. All of those examples were build on sets with isolated points, and moreover, they indicated a fixed open ball which was not open. Here we describe a large class of connected semimetric spaces with semimetrics satisfying the c-relaxed polygonal inequality for which even every open ball is not open, and simultaneously, every closed ball is not closed. In our proof we use a fact shown first by HEWITT [12] (see also [5, Theorem 3.7]) that every metric space (X, ρ) without isolated points is *resolvable*, i.e., there exists a subset D of X such that both D and $X \setminus D$ are dense in X. Inspired by remarks of one of the referees, we simplified our original proof of the following:

Theorem 5.6. *Let (X, ρ) be an unbounded connected metric space. Then for every $c > 1$, there exists a semimetric d on X satisfying the c-relaxed polygonal inequality such that $\tau_d = \tau_\rho$, and for any $x \in X$ and $r > 0$, the open ball $B_d(x, r)$ is not open and the closed ball $\overline{B}_d(x, r)$ is not closed.*

Proof. Since (X, ρ) is connected, it has no isolated points, so by the above-mentioned Hewitt's theorem, there exists a dense subset D of X such that $X \setminus D$ is also dense in X. For $x, y \in X$, set

$$d(x,y) := \begin{cases} \rho(x,y), & \text{if } x, y \in D \text{ or } x, y \notin D, \\ c\rho(x,y), & \text{otherwise.} \end{cases} \tag{4}$$

Clearly, d is a semimetric, and for any $x, y \in X$,

$$\rho(x,y) \leq d(x,y) \leq c\rho(x,y),$$

which implies that $\tau_d = \tau_\rho$, and moreover, by Theorem 5.5, d satisfies the c-relaxed polygonal inequality. Now fix $x \in D$ and $r > 0$. Clearly, the band

$$B := \left\{ y \in X : \frac{r}{c} < \rho(x,y) < r \right\}$$

is a nonempty open set, so it contains some $z \in D$. Since

$$B_d(x,r) = (B_\rho(x,r) \cap D) \cup B_\rho\left(x, \frac{r}{c}\right), \tag{5}$$

we may infer that $z \in B_d(x,r)$. We show that $z \notin \operatorname{Int} B_d(x,r)$. Consider any open ball $B_\rho(z, r')$. Since $z \in B$ and B is open, we may assume without loss of generality that $B_\rho(z,r') \subseteq B$. Since $X \setminus D$ is dense in X, there exists w in $(X \setminus D) \cap B_\rho(z,r')$. In particular, $w \in B$, so $\rho(x,w) > r/c$. Thus by (5), we get that $w \notin B_d(x,r)$, which shows that $B_\rho(z,r') \not\subseteq B_d(x,r)$. Hence $B_d(x,r)$ is not open if $x \in D$. Now, if $x \in X \setminus D$, then we may repeat the above argument interchanging the roles between D and $X \setminus D$. Consequently, in every case, the open ball $B_d(x,r)$ is not open.

Finally, we show that the closed ball $\overline{B}_d(x,r)$ is not closed. As explained above, it suffices to consider only the case when $x \in D$. Then

$$\overline{B}_d(x,r) = \left(\overline{B}_\rho(x,r) \cap D\right) \cup \overline{B}_\rho\left(x, \frac{r}{c}\right),$$

so $\overline{B}_d(x,r) \supseteq B \cap D$, and hence $\operatorname{cl} \overline{B}_d(x,r) \supseteq \operatorname{cl}(B \cap D)$. Since B is open and D is dense in X, we conclude that $\operatorname{cl}(B \cap D) = \operatorname{cl} B$. Consequently, we get that

$$\operatorname{cl} \overline{B}_d(x,r) \supseteq \operatorname{cl} B \supseteq B \setminus D.$$

Clearly, $B \setminus D$ is nonempty and $(B \setminus D) \cap \overline{B}_d(x,r) = \emptyset$, so $\overline{B}_d(x,r)$ is not closed. □

Remark 5.7. The above proof shows in fact that the assumptions on a metric space (X, ρ) in Theorem 5.6 can be weakened: it suffices that X has no isolated points and (X, ρ) is such that for any $x \in X$, the set $\{\rho(x, y) : y \in X\}$ is dense in $[0, \infty)$. For example, the metric space $(\mathbb{Q}, \rho_e)$, where $\mathbb{Q}$ is the set of all rationals and ρ_e is the Euclidean metric, has such a property, whereas Theorem 5.6 is not applicable to it. Moreover, the extension of Theorem 5.6 can be reversed in the following way. Let (X, ρ) be a metric space without isolated points. If for any $c > 1$, the semimetric d defined by (4) is such that every open ball in (X, d) is not open, then for any $x \in X$, the set $\{\rho(x, y) : y \in X\}$ is dense in $[0, \infty)$. We omit the proof of this somewhat technical result.

6. Applications

As far as the applications are concerned, semimetrics turned out to be surprisingly useful in several IT problems. Taking pattern matching, for example, distance functions used in most common techniques of comparing shapes fail to satisfy the triangle inequality. However, in some applications, we need at least some weaker form of the triangle inequality. This is where b-metrics and c-relaxed polygonal inequality become useful. If we use some b-metric d to calculate the distance $d(A, B)$ between shapes A and B, then (with a certain imprecision) we can approximate the distance $d(A, C)$ between shape A and any shape C, which is relatively close to B. Using this fact, we can basically cluster giant database of shapes into classes, effectively reducing the amount of comparisons needed to find the picture we are looking for. Various types of semimetrics also appear in measuring distance between top k lists, as a method of measuring how the two sets of "k best matching objects", obtained by various comparison methods differ. More on this topic can be read in [9] and [10].

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