Publ. Math. Debrecen
In-print:: Ref. no.: 8075 (2018), 1–17

Certain almost Kenmotsu metrics satisfying the Miao–Tam equation

By AMALENDU GHOSH (Hooghly) and DHRITI SUNDAR PATRA (Ranchi)

Abstract. In this paper, we consider the Miao–Tam equation within the framework of Kenmotsu and almost Kenmotsu manifolds. These classes have a nice connection with the warped product of real line and almost Hermitian (Kähler) manifolds. Here, we prove that a Kenmotsu metric satisfying the Miao–Tam equation is Einstein. Next, we study the Miao–Tam equation on almost Kenmotsu manifolds satisfying some nullity conditions. To this end, we construct some examples of almost Kenmotsu manifolds that satisfy the Miao–Tam equation.

1. Introduction

A classical problem in differential geometry is to find Riemannian metrics on a given compact manifold M^n that provide constant scalar curvature. In this sense, it is crucial to study the critical points of the total scalar curvature functional through variational approach. Einstein and Hilbert proved that the critical points of the total scalar curvature functional $\mathcal{S} : \mathcal{M} \longrightarrow R$ defined by

$$\mathcal{S}(g) = \int_M r_g dv_g,$$

on a compact orientable Riemannian manifold (M^n, g) restricted to the set of all Riemannian metrics $\mathcal{M}$ (where $\mathcal{M}$ denotes the set of all Riemannian metrics

---

Mathematics Subject Classification: 53C25, 53C20, 53C15.
Key words and phrases: the Miao–Tam equation, almost Kenmotsu manifold, Kenmotsu manifold, Einstein manifold, nullity distributions.
The author A. Ghosh has been supported by the UGC (India) under the scheme Minor Research Project in Science, Sanction No. PSW- 018/15-16, Dated 15-11-2016.

on (M^n, g) of unit volume, r_g the scalar curvature, and dv_g the volume form of g), are Einstein (see [3]). This stimulated many interesting research. In [13], the authors studied the variational properties of the volume functional over the space of constant scalar curvature on a given compact Riemannian manifold with boundary. This leads to the following definition:

Definition 1. Let (M^n, g), $n > 2$ be a compact Riemannian manifold with a smooth boundary metric ∂M. Then g is said to be a critical metric if there exists a smooth function $\lambda : M^n \to \mathbb{R}$ such that

$$-(\Delta_g \lambda)g + \nabla_g^2 \lambda - \lambda S = g, \tag{1.1}$$

on M and $\lambda = 0$ on ∂M, where Δ_g, $\nabla_g^2 \lambda$ are the Laplacian, Hessian operator with respect to the metric g, and S is the $(0, 2)$ Ricci curvature of g. The function λ is known as the potential function.

For brevity, the metrics which satisfy (1.1) are known as Miao–Tam critical metrics and we refer to equation (1.1) as the Miao–Tam equation. In [13], Miao–Tam proved that any Riemannian metric g satisfying equation (1.1) must have constant scalar curvature, from which it follows that a critical metric g always has constant scalar curvature. The existence of such metrics was proved on some certain classes of warped product spaces which include the usual spatial Schwarzschild metrics and Ads–Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres (cf. Corollaries 3.1 and 3.2 of [12]).

In [13], the authors classified the Einstein and conformally flat Riemannian manifolds satisfying (1.1). In fact, they proved that any connected, compact, Einstein manifold with smooth boundary satisfying Miao–Tam critical condition is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$. Similar characterization was obtained when g is a conformally flat metric on a simply connected manifold M such that the boundary of (M, g) is isometric to a round sphere. We also point out that the last result has been generalized in dimension 4 under the Bach flat assumption by Barros *et al.* [2]. Moreover, Miao–Tam also exhibited a general method to construct the metric on the warped product that satisfies equation (1.1). Recently, the authors studied equation (1.1) on a certain class of odd dimensional Riemannian manifolds (namely, contact metric manifolds (see [16])), and proved that a complete K-contact metric satisfying the Miao–Tam critical condition is isometric to a unit sphere $\mathbb{S}^{2n+1}$. In [11], Kenmotsu proved that a warped product of a line and a Kählerian manifold satisfies equations (2.3), (2.4), and, an almost contact metric manifold satisfying these equations is known as a Kenmotsu manifold. Conversely, a Kenmotsu manifold M is locally a warped product $I \times_f N^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c, and N^{2n} is a Kählerian manifold. Further, we also highlight that there is a nice connection between almost Kenmotsu manifolds and warped products of real line with almost Hermitian manifolds, because most known examples of almost Kenmotsu manifolds arise from the warped products of a Kaehler manifold and the real line. Thus in the one hand, there exist warped product spaces that satisfy the Miao–Tam equation, and on the other hand, there exist warped spaces which carry Kenmotsu and almost Kenmotsu structures. For this, we are motivated to study the Miao–Tam equation on Kenmotsu and almost Kenmotsu manifolds.

The organization of this paper is as follows. After some rudiments of Kenmotsu and almost Kenmotsu manifolds in Section 2, we study the Miao–Tam equation within the farmework of Kenmotsu manifolds in Section 3, where we prove that a Kenmotsu metric satisfying the Miao–Tam equation is Einstein. Next, we consider the Miao–Tam equation on certain classes of almost Kenmotsu manifolds satisfying some nullity conditions. Finally, we present some examples of Kenmotsu and almost Kenmotsu metrics on the warped product of real line and Kaehler manifolds that satisfies the Miao–Tam equation.

2. Preliminaries

A contact manifold is a Riemannian manifold M of dimension $(2n+1)$ which carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. The form η is usually known as the contact form on M. It is well known that a contact manifold admits an almost contact metric structure on (φ, ξ, η, g), where φ is a tensor field of type $(1,1)$, ξ a global vector field known as the characteristic vector field (or the Reeb vector field), and g is Riemannian metric, such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y on M. It follows from equation (2.1) that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$ (see [4, p. 43]). A Riemannian manifold M together with the almost contact metric structure (φ, ξ, η, g) is said to be an almost contact metric manifold. On almost contact metric manifolds one can always define a fundamental 2-form Φ by $\Phi(X,Y) = g(X, \varphi Y)$ for all vector fields X, Y on M. An almost contact metric structure of M is said to be contact metric if $\Phi = d\eta$, and is said to be almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Further, a condition for an almost contact metric structure being normal is equivalent to the vanishing of the $(1,2)$-type torsion tensor N_φ, defined by $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold, and the normality condition is given by

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.3}$$

for all vector fields X, Y on M. On a Kenmotsu manifold [11] it holds:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.5}$$

$$Q\xi = -2n\xi, \tag{2.6}$$

for all vector fields X, Y on M, where ∇ denotes the operator of covariant differentiation of g, R the curvature tensor of g, and Q the Ricci operator associated with the $(0,2)$ Ricci tensor S given by $S(Y,Z) = g(QY,Z)$ for all vector fields Y, Z on M. A more general class of manifolds containing the class of Kenmotsu manifolds is the class of $(0,\beta)$-Kenmotsu (or simply β-Kenmotsu) manifolds, which are a special kind of trans-Sasakian manifold, as we now recall. An almost contact metric manifold M is said to be trans-Sasakian if there exist two functions α and β on M such that

$$(\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{2.7}$$

for any vector fields X, Y on M. If $\alpha = 0$, then M is said to be a β-Kenmotsu manifold. Kenmotsu manifolds appear to be a particular case of β-Kenmotsu manifolds, for $\beta = 1$. On an almost Kenmotsu manifold we define the operator h by $h = \frac{1}{2}\pounds_\xi \varphi$ on M, where $\pounds_\xi$ is the Lie differentiation with respect to ξ. For an almost Kenmotsu manifold the following formulas are valid (see [18], [8]):

$$h\xi = 0, \quad \operatorname{Tr} h = \operatorname{Tr}(h\varphi) = 0, \quad h\varphi = -\varphi h, \tag{2.8}$$

$$\nabla_X \xi = X - \eta(X)\xi - \varphi h X, \tag{2.9}$$

for any vector field X on M.

An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized (κ, μ)-almost Kenmotsu manifold if ξ belongs to the generalized (κ, μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.10}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M. An almost Kenmotsu manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ is said to be a generalized $(\kappa,\mu)'$-almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa,\mu)'$-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\}, \tag{2.11}$$

for all vector fields X, Y on M, where $h' = ho\varphi$ and κ, μ are smooth functions on M. Moreover, if both κ and μ are constants in equation (2.11), then M is called a $(\kappa,\mu)'$-almost Kenmotsu manifold. Classifications of almost Kenmotsu manifolds with ξ belong to $(\kappa,\ \mu)$ or $(\kappa,\mu)'$-nullity distribution were worked out by several authors. For more details, we refer the reader to [18], [8], [15]. The following formulas are valid on a generalized (κ,μ) or $(\kappa,\mu)'$-almost Kenmotsu manifold (e.g., [8]):

$$h'^2 = (\kappa+1)\varphi^2, \text{ respectively, } h^2 = (\kappa+1)\varphi^2, \tag{2.12}$$

$$Q\xi = 2n\kappa\xi. \tag{2.13}$$

In this connection, we mention that in [8] Dileo and Pastore (see Proposition 4.1) proved that on a $(\kappa,\mu)'$-almost Kenmotsu manifold, $\mu = -2$. Consider $X \in \mathcal{D}$ an eigenvector of h' with eigenvalue σ, where $\mathcal{D}$ is the distribution such that $\mathcal{D} = \ker(\eta)$. It follows from (2.12) that $\sigma^2 = -(\kappa+1)$, and therefore $\kappa \leq -1$ and $\sigma = \pm\sqrt{-\kappa-1}$. The equality holds if and only if $h = 0$ (equivalently, $h' = 0$). Thus, $h' \neq 0$ if and only if $\kappa < -1$.

We now recall the notion of warped product manifolds for our later use. Let $(N, J, \tilde{g})$ be an almost Hermitian manifold and consider the warped product $M = \mathbb{R} \times_f N$ with the metric $g = g_0 + f^2\tilde{g}$, where f is a positive function on $\mathbb{R}$, and g_0 is the standard metric on $\mathbb{R}$. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$, and define the tensor field φ on $\mathbb{R} \times_f N$ such that $\varphi X = JX$ if X is a vector field on N, and $\varphi X = 0$ if X is tangent on $\mathbb{R}$. Then it is easy to verify that M admits an almost contact metric structure. An interesting characterization of an almost Kenmotsu manifold through the warped product of a real line and an almost Hermitian manifold is given by the following (see [1]).

Lemma 2.1. *Let N be an almost Hermitian manifold. Then the warped product $\mathbb{R} \times_f N$ is a $(0,\beta)$-trans Sasakian manifold, with $\beta = \frac{f'}{f}$ if and only if N is Kählerian.*

3. Kenmotsu manifolds satisfying the Miao–Tam equation

First, we deduce the expression of the curvature tensor that satisfies the Miao–Tam equation.

Lemma 3.1. *If a Riemannian manifold (M^n, g) satisfies the Miao–Tam equation, then its curvature tensor R can be expressed as*

$$R(X,Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda\{(\nabla_X Q)Y - (\nabla_Y Q)X\} + (Xf)Y - (Yf)X, \quad (3.1)$$

for any vector fields X, Y on M and $f = -\frac{r\lambda+1}{n-1}$.

PROOF. Equation (1.1) can be exhibited as

$$\nabla_X D\lambda = \lambda QX + (1 + \triangle_g \lambda)X, \quad (3.2)$$

for any vector field X on M. Now, tracing (1.1), we obtain $\triangle_g \lambda = -\frac{r\lambda+n}{n-1}$. Then equation (3.2) transforms into

$$\nabla_X D\lambda = \lambda QX + fX, \quad (3.3)$$

for any vector field X on M. Taking the covariant derivative of (3.3) along an arbitrary vector field Y on M, we obtain

$$\nabla_Y(\nabla_X D\lambda) = (Y\lambda)QX + \lambda\{(\nabla_Y Q)X + Q(\nabla_Y X)\} + (Yf)X + f\nabla_Y X,$$

for any vector field X on M. Applying the preceding equation and (3.3) in the well-known expression of the curvature tensor $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we obtain the required result. □

Here we characterize a Kenmotsu metric that satisfies the Miao–Tam equation. First, we recall the following formula. The detailed proof of this can be found in [9]. However, for completion, we present a sketch of the proof.

Lemma 3.2. *On any Kenmotsu manifold of dimension $(2n+1)$, the following formula is valid:*

$$(\nabla_\xi Q)Y = -2QY - 4nY, \quad (3.4)$$

for any vector field Y.

PROOF. By virtue of (2.4), we have

$$(\pounds_\xi g)(Y,Z) = g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) = 2\{g(Y,Z) - \eta(Y)\eta(Z)\}. \quad (3.5)$$

Differentiating (3.5) covariantly along an arbitrary vector field, X gives

$$(\nabla_X \pounds_\xi g)(Y,Z) = 2\{2\eta(X)\eta(Y)\eta(Z) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}. \tag{3.6}$$

Making use of this in the following formula [19, p. 23]:

$$(\nabla_X \pounds_V g)(Y,Z)) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y),$$

we have

$$\begin{aligned} &g((\pounds_\xi \nabla)(X,Y),Z) + g((\pounds_\xi \nabla)(X,Z),Y) \\ &\quad = 2\{2\eta(X)\eta(Y)\eta(Z) - g(X,Y)\eta(Z) - g(X,Z)\eta(Y)\}. \end{aligned}$$

By a straightforward combinatorial computation, the foregoing equation yields

$$(\pounds_\xi \nabla)(Y,Z) = 2\{\eta(Y)\eta(Z) - g(Y,Z)\xi\}. \tag{3.7}$$

Taking covariant differentiation of (3.7) along X and using (2.4), we find

$$\begin{aligned} (\nabla_X \pounds_\xi \nabla)(Y,Z) = 2\{&g(X,Y)\eta(Z)\xi + g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi \\ &- g(Y,Z)X + \eta(Y)\eta(Z)X - 3\eta(X)\eta(Y)\eta(Z)\xi\}. \end{aligned}$$

Utilizing this in the following formula [19, p. 23]:

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

we obtain

$$(\pounds_\xi R)(X,Y)Z = 2\{g(X,Z)Y - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \tag{3.8}$$

Taking g-trace of this over X, we deduce

$$(\pounds_\xi S)(Y,Z) = 4n\{\eta(Y)\eta(Z) - g(Y,Z)\}. \tag{3.9}$$

Moreover, Lie differentiating the identity $S(Y,Z) = g(QY,Z)$, we have

$$(\pounds_\xi S)(Y,Z) = (\pounds_\xi g)(QY,Z) + g((\pounds_\xi Q)Y,Z). \tag{3.10}$$

On the other hand, replacing Y by QY in (3.5) and using (2.9), we deduce

$$(\pounds_\xi g)(QY,Z) = 2\{g(QY,Z) + 2n\eta(Y)\eta(Z)\}.$$

Using this and (3.10) in equation (3.11), we obtain $(\pounds_\xi Q)Y = -2QY - 4nY$. By a straightforward computation and taking into account of (2.4), we complete the proof. □

We now characterize the solution of the Miao–Tam equation within the framwork of Kenmotsu manifold. Precisely, we prove

Theorem 3.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a Kenmotsu manifold. If (g, λ) satisfies the Miao–Tam equation, then g is Einstein and locally $\lambda = A\cosh t + B\sinh t + \frac{1}{2n}$.*

Proof. Taking g-trace of (3.4) yields $\xi r = -2(r+2n(2n+1))$. Since (g, λ) is a solution of the the Miao–Tam equation, the scalar curvature r of g is constant, and hence $r = -2n(2n+1)$. Making use of this in $f = -\frac{r\lambda+1}{2n}$, we deduce $f = (2n+1)\lambda - \frac{1}{2n}$, and hence

$$Xf = (2n+1)(X\lambda). \tag{3.11}$$

On the other hand, since $\nabla_\xi \xi = 0$ (follows from (2.6)) and $\xi\lambda = g(\xi, D\lambda)$, taking into account (2.6) and (3.3), we deduce

$$\xi(\xi\lambda) = g(\nabla_\xi D\lambda, \xi) = \lambda - \frac{1}{2n}. \tag{3.12}$$

Now, taking covariant derivative of (2.5) over an arbitrary vector field X on M and using (2.4), we obtain

$$(\nabla_X Q)\xi = -QX - 2nX. \tag{3.13}$$

Setting $Y = \xi$ in (3.1) and using (2.6), (3.4), (3.13), we have

$$R(X,\xi)D\lambda = -2n(X\lambda)\xi - (\xi\lambda)QX + \lambda(QX + 2nX) + (Xf)\xi - (\xi f)X, \tag{3.14}$$

for any vector field X on M. Further, from (2.5), we deduce $R(X, \xi)Z = g(X, Z)\xi - \eta(Z)X$. By virtue of this, the foregoing equation reduces to

$$(\lambda - \xi\lambda)QX + (2n\lambda - \xi f + \xi\lambda)X - (2n+1)(X\lambda)\xi + (Xf)\xi = 0. \tag{3.15}$$

Making use of (3.11), the last equation reduces to

$$(\lambda - \xi\lambda)\{QX + 2nX\} = 0, \tag{3.16}$$

for all vector fields X on M. Now, suppose that $\lambda = \xi\lambda$ in some open set $\mathcal{O}$ in M. Then we have $\xi\lambda = \lambda - \frac{1}{2n} = \lambda$ on $\mathcal{O}$, which is clearly a contradiction. Hence $QX = -2nX$ for all vector fields X on M, and M is thus Einstein. So, the scalar curvature becomes $r = -2n(2n+1)$. Thus equation (3.3) can be exhibited as

$$\nabla_X D\lambda = \left(\lambda - \frac{1}{2n}\right)X, \tag{3.17}$$

for all vector fields X on M. We know that a Kenmotsu manifold is locally a warped product $I \times_f N^{2n}$, where I is an open interval of the real line, N^{2n} is a Kähler manifold, $f^2 = ce^{2t}$, and t is the coordinate of I. Since $\xi = \frac{\partial}{\partial t}$, it follows from (3.17) that

$$\frac{d^2\lambda}{dt^2} = \lambda - \frac{1}{2n}.$$

Its solution can be exhibited as $\lambda = A\cosh t + B\sinh t + \frac{1}{2n}$, where A, B are constants on M. □

4. Almost Kenmotsu manifolds satisfying the Miao–Tam equation

Before entering into our main results, we now recall the following:

Lemma 4.1 ([15, Proposition 3.2]). *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold with $h \neq 0$. Then*

$$\xi(\lambda) = -\lambda(\mu + 2), \quad \xi(\kappa) = -2(\kappa + 1)(\mu + 2). \tag{4.1}$$

Recently, Wang–Liu [18] obtained some expression of the Ricci operator on generalized (κ, μ) or $(\kappa, \mu)'$-almost Kenmotsu manifolds.

Lemma 4.2 ([18, Lemma 3.4]). *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a generalized (κ, μ)-almost Kenmotsu manifold with $h' \neq 0$. For $n > 1$, the Ricci operator Q of M^{2n+1} can be expressed as*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2(n - 1)h'X + \mu hX, \tag{4.2}$$

for any vector field X on M. Also, the scalar curvature of M is $2n(\kappa - 2n)$.

Lemma 4.3 ([18, Lemma 3.3]). *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$. For $n > 1$, the Ricci operator Q of M can be expressed as*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - [\mu - 2(n - 1)h']X, \tag{4.3}$$

for any vector field X on M. Further, if κ and μ are constants and $n \geq 1$, then $\mu = -2$, and hence

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2nh'X, \tag{4.4}$$

for any vector field X on M. In both cases, the scalar curvature of M is $2n(\kappa–2n)$.

Remark 4.1. We note that there is no Einstein almost Kenmotsu manifold satisfying the hypothesis in Lemma 4.2 or Lemma 4.3. Indeed, if M is Einstein, then $S = \frac{r}{2n+1}g$. Since $Q\xi = 2n\kappa\xi$ and the scalar curvature is $2n(\kappa - 2n)$, we see that $\kappa = -1$, and hence $h' = 0$. This contradicts the hypothesis $h' \neq 0$. Thus an Einstein almost Kenmotsu manifold satisfying any one of the nullity conditions, that is either generalized (κ, μ), or generalized $(\kappa, \mu)'$ with $h' \neq 0$, does not exist.

We now consider the Miao–Tam equation on $(\kappa, -2)'$-almost Kenmotsu manifold with $h' \neq 0$, and prove the following.

Theorem 4.1. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, -2)'$-almost Kenmotsu manifold with $h' \neq 0$. If there is a non-constant function λ on M satisfying the Miao–Tam equation, then M^3 is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$, and for $n > 1$, M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$, or, $B^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$; where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2}{n} - \frac{1}{n^2}$, $B^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2}{n} - \frac{1}{n^2}$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1+\frac{1}{n})t}$, with c, c' positive constants.*

Proof. Since the scalar curvature of a $(\kappa, -2)'$-almost Kenmotsu manifold with $h' \neq 0$ is $2n(\kappa - 2n)$ (follows from Lemma 4.3), equation (3.1) can be written as

$$\begin{aligned} R(X,Y)D\lambda = {} & (X\lambda)QY - (Y\lambda)QX + \lambda\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ & + (2n - \kappa)\{(X\lambda)Y - (Y\lambda)X\}, \end{aligned} \tag{4.5}$$

for any vector fields X, Y on M. Therefore, substituting X by ξ in (4.5), then taking its inner product with ξ and using (2.13), we get

$$\begin{aligned} g(R(\xi,Y)D\lambda, \xi) = {} & \{2n(\kappa+1) - \kappa\}\{(\xi\lambda)\eta(Y) - (Y\lambda)\} \\ & + \lambda\{g((\nabla_\xi Q)Y, \xi) - g((\nabla_Y Q)\xi, \xi)\}, \end{aligned} \tag{4.6}$$

for all vector fields Y on M. Taking covariant derivative of (2.13) along an arbitrary vector field Y on M, we have $(\nabla_Y Q)\xi + Q(\nabla_Y \xi) = 2n\kappa\nabla_Y\xi$. By virtue of (2.9), the last equation reduces to

$$(\nabla_Y Q)\xi = 2n\kappa(Y - \varphi hY) - Q(Y - \varphi hY), \tag{4.7}$$

for any vector field Y on M. Now, using (4.7) and (2.8) in (4.6) provides

$$g(R(\xi,Y)D\lambda, \xi) = \{2n(\kappa+1) - \kappa\}\{(\xi\lambda)\eta(Y) - (Y\lambda)\}, \tag{4.8}$$

for all vector fields Y on M. Further, taking the scalar product of (2.11) with $D\lambda$, and then replacing X by ξ in the resulting equation and using $g(Y, D\lambda) = Y\lambda$ gives

$$g(R(\xi, Y)D\lambda, \xi) = \kappa g(D\lambda - (\xi\lambda)\xi, Y) + 2g(D\lambda, h'Y),$$

for any vector field Y on M, where we have also used $\mu = -2$. Combining the last two equations and using the self-adjointness of $h\varphi$, we get

$$n(\kappa + 1)\{D\lambda - (\xi\lambda)\xi\} - h\varphi D\lambda = 0. \tag{4.9}$$

Now, operating the foregoing equation by $h\varphi$ and using $h\xi = 0$ yields $n(\kappa + 1)h\varphi D\lambda + h^2\varphi^2 D\lambda = 0$. By virtue of (4.9), (2.12), $\varphi\xi = 0$ and the first equation in (2.1), the preceding equation yields

$$n^2(\kappa + 1)^2\{D\lambda - (\xi\lambda)\xi\} - (\kappa + 1)\varphi^2 D\lambda = 0.$$

Moreover, making use of (2.1) the last equation reduces to $(\kappa + 1)\{n^2(\kappa + 1) + 1\}\{D\lambda - (\xi\lambda)\xi\} = 0$. Since $\kappa < -1$, the foregoing equation gives

$$\{n^2(\kappa + 1) + 1\}\{D\lambda - (\xi\lambda)\xi\} = 0. \tag{4.10}$$

Thus, we have either $n^2(\kappa + 1) + 1 = 0$, or $n^2(\kappa + 1) + 1 \neq 0$.

Case 1. In this case, we have $\kappa = -1 - \frac{1}{n^2}$. For $n = 1$, $\kappa = \mu = -2$, and therefore from Theorem 4.2 of DILEO and PASTORE [8], we deduce that M^3 is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$, and for $n > 1$, M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$, or, $B^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$; where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2}{n} - \frac{1}{n^2}$, $B^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2}{n} - \frac{1}{n^2}$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1+\frac{1}{n})t}$, with c, c' positive constants.

Case 2. In this case, it follows from (4.10) that $D\lambda = (\xi\lambda)\xi$. Taking covariant derivative of $D\lambda = (\xi\lambda)\xi$ along an arbitrary vector field X on M and using (2.1), (2.9), we deduce

$$\nabla_X D\lambda = X(\xi\lambda)\xi + (\xi\lambda)(X - \eta(X)\xi - \varphi hX). \tag{4.11}$$

Since the scalar curvature (from Lemma 3.3) is $2n(\kappa - 2n)$, equation (3.3) gives

$$\nabla_X D\lambda = \lambda\{QX + (2n - \kappa)X\} - \frac{1}{2n}X, \tag{4.12}$$

for any vector field X on M. Making use of (4.11) in (4.12), it follows that

$$\lambda QX = \left\{(\kappa - 2n)\lambda + (\xi\lambda) + \frac{1}{2n}\right\} X + X(\xi\lambda)\xi - (\xi\lambda)\{\eta(X)\xi + \varphi hX\},$$

for any vector field X on M. Comparing this with (4.4), we deduce that

$$\left\{\kappa\lambda + (\xi\lambda) + \frac{1}{2n}\right\} X + X(\xi\lambda)\xi + \{(\xi\lambda) + 2n\lambda\}h'X \\ - \{2n(\kappa+1)\lambda + (\xi\lambda)\}\eta(X)\xi = 0, \tag{4.13}$$

for any vector field X on M. Now, tracing (4.13) over X and noting that $\operatorname{Tr} h' = 0$, we have

$$(2n+1)\left\{\kappa\lambda + (\xi\lambda) + +\frac{1}{2n}\right\} + \xi(\xi\lambda) - \{2n(\kappa+1)\lambda + (\xi\lambda)\} = 0. \tag{4.14}$$

Next, substituting X by ξ in equation (3.3) and then taking its scalar product with ξ yields $\xi(\xi\lambda) = \lambda\{2n(\kappa+1) - \kappa\} - \frac{1}{2n}$. By virtue of this, equation (4.14) takes the form

$$\kappa\lambda + (\xi\lambda) + \frac{1}{2n} = 0. \tag{4.15}$$

Therefore, applying φ^2 to (4.13) and taking into account (4.15), we infer that $\{(\xi\lambda) + 2n\lambda\}\varphi^2 h'X = 0$ for any vector field X on M. Further, making use of the first equation of (2.1), $h' = h \circ \varphi$, $h\varphi = -\varphi h$ and $\varphi\xi = 0$, the last equation reduces to

$$((\xi\lambda) + 2n\lambda)h'X = 0, \tag{4.16}$$

for any vector field X on M. By virtue of (4.15), equation (4.16) gives $\{(2n - \kappa)\lambda - \frac{1}{2n}\}h'X = 0$ for any vector field X on M. Since h' is non-vanishing, $\kappa < -1$. Thus, it follows that $\lambda = \frac{1}{2n(2n-\kappa)}$, which is constant. This completes the proof. □

Next, we extend the last result for a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold. Note that the metric satisfying the Miao–Tam equation has constant scalar curvature (see [13]). Further, the scalar curvature of a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$ is $2n(\kappa - 2n)$ (follows from Lemma 4.3). Hence, it follows that κ is constant. Therefore, from Lemma 4.1 we have $(\kappa + 1)(\mu + 2) = 0$. Since $\kappa < -1$, we must have $\mu = -2$. Thus from the last theorem we have the following.

Theorem 4.2. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$, $n>1$, be a generalized $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$. If there is a non-constant function λ on M satisfying the Miao–Tam equation, then g is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha)\times_f\mathbb{R}^n$, or, $B^{n+1}(\alpha)\times_{f'}\mathbb{R}^n$; where $f=ce^{(1-\frac{1}{n})t}$ and $f'=c'e^{(1+\frac{1}{n})t}$, with c, c' positive constants.*

Finally, we examine the existence of the solution of the Miao–Tam equation on a generalized (κ,μ)-almost Kenmotsu manifold with $h\neq 0$.

Theorem 4.3. *There does not exist any solution of the Miao–Tam equation on a generalized (κ,μ)-almost Kenmotsu manifold $M^{2n+1}(\varphi,\xi,\eta,g)$, $n>1$, with $h\neq 0$.*

Proof. Suppose there exists a non-trivial smooth function λ such that (g,λ) is a solution of the Miao–Tam equation (1.1). Then it satisfies the curvature equation (3.1). Therefore, taking the scalar product of (3.1) with ξ, and then using (2.13), $r=2n(\kappa-2n)$ (follows from Lemma 4.2), we achieve

$$\begin{aligned} g(R(X,Y)D\lambda,\xi) &= \{2n(\kappa+1)-\kappa\}\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}\\ &\quad+\lambda\{g(Y,(\nabla_X Q)\xi)-g(X,(\nabla_Y Q)\xi)\}, \end{aligned} \tag{4.17}$$

for all vector fields X, Y on M. Since the metric satisfying (1.1) has constant scalar curvature (see [13]) and $r=2n(\kappa-2n)$, it follows that κ is constant. Hence equation (4.7) is also valid here. Now, making use of (4.7) and (2.8) in (4.17), we immediately infer that

$$\begin{aligned} g(R(X,Y)D\lambda,\xi) &= \{2n(\kappa+1)-\kappa\}\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}\\ &\quad+\lambda\{g(Q\varphi hX,Y)-g(X,Q\varphi hY)\}, \end{aligned} \tag{4.18}$$

for all vector fields X, Y on M. Next, replacing X by φX, and Y by φY in (4.18) and noting that $g(R(\varphi X,\varphi Y)D\lambda,\xi)=0$ (follows from (2.10)) and $h\varphi=-\varphi h$, we have

$$\lambda\{g(Qh\varphi^2X,\varphi Y)-g(\varphi X,Qh\varphi^2Y)\}=0,$$

for all vector fields X, Y on M. Therefore, using (4.2) and $h\varphi=-\varphi h$, we have $\lambda\mu h^2\varphi^2X=0$ for any vector field X on M. Thus, by virtue of (2.1) and (2.12), the last equation yields $(\kappa+1)\lambda\mu\varphi^2X=0$ for any vector field X on M. Since $\kappa<-1$, the foregoing equation provides $\lambda\mu=0$.

We suppose that $\lambda \neq 0$ in some open set $\mathcal{O}$ in M. Then on $\mathcal{O}$, $\mu = 0$. Now, replacing X by ξ in (2.10) and then taking the scalar product of the resulting equation with $D\lambda$ gives

$$g(R(\xi, Y)D\lambda, \xi) = \kappa g(D\lambda - (\xi\lambda)\xi, Y). \tag{4.19}$$

Moreover, substituting X by ξ in (4.18) and using (2.13), $h\xi = 0$ and $\varphi\xi = 0$, we have

$$g(R(\xi, Y)D\lambda, \xi) = \{2n(\kappa + 1) - \kappa\}\{(\xi\lambda)\eta(Y) - (Y\lambda)\}.$$

Combining this with (4.19), we obtain

$$(\kappa + 1)\{D\lambda - (\xi\lambda)\xi\} = 0.$$

Since $h \neq 0$, i.e., $\kappa < -1$, the last equation gives $D\lambda - (\xi\lambda)\xi = 0$. Taking covariant derivative of $D\lambda = (\xi\lambda)\xi$ along an arbitrary vector field X and using (2.1), (2.9), we deduce

$$\nabla_X D\lambda = X(\xi\lambda)\xi + (\xi\lambda)(X - \eta(X)\xi - \varphi hX). \tag{4.20}$$

Since the scalar curvature is $2n(\kappa - 2n)$, equation (3.3) transforms into

$$\nabla_X D\lambda = \lambda\{QX + (2n - \kappa)X\} - \frac{1}{2n}X. \tag{4.21}$$

Making use of (4.21) in (4.20), it follows that

$$\lambda QX = \left\{(\kappa - 2n)\lambda + (\xi\lambda) + \frac{1}{2n}\right\}X + X(\xi\lambda)\xi - (\xi\lambda)(\eta(X)\xi + \varphi hX).$$

By virtue of (4.2), the preceding equation transforms into

$$\begin{aligned}\{\kappa\lambda + (\xi\lambda) + \frac{1}{2n}\}X + X(\xi\lambda)\xi + \{(\xi\lambda) + 2(n-1)\lambda\}h\varphi X \\ - \{2n(\kappa + 1)\lambda + (\xi\lambda)\}\eta(X)\xi = 0.\end{aligned} \tag{4.22}$$

Now, tracing (4.22) over X and noting that $\operatorname{Tr} h\varphi = 0$, we have

$$(2n + 1)\left\{\kappa\lambda + (\xi\lambda) + \frac{1}{2n}\right\} + \xi(\xi\lambda) - \{2n(\kappa + 1)\lambda + (\xi\lambda)\} = 0. \tag{4.23}$$

Next, substituting X by ξ in (4.21) and using $Q\xi = 2n\kappa\xi$, and then taking the scalar product of the resulting equation with ξ, we get $\xi(\xi\lambda) = \lambda\{2n(\kappa + 1) - \kappa\} - \frac{1}{2n}$. Making use of this, (4.23) reduces to

$$\kappa\lambda + (\xi\lambda) + \frac{1}{2n} = 0. \tag{4.24}$$

Further, operating (4.22) by φ and using (4.24) and $h\varphi = -\varphi h$, we get $\{2(n-1)\lambda + (\xi\lambda)\}h\varphi^2 X = 0$. Moreover, using (2.1) and recalling $\varphi\xi = 0$, the last equation provides $\{2(n-1)\lambda+(\xi\lambda)\}hX = 0$. Making use of (4.24), the preceding equation transforms into $\left\{2(n-1)\lambda - \kappa\lambda - \frac{1}{2n}\right\} hX = 0$. Since $h \neq 0$, the last equation shows that $2n\{2(n-1) - \kappa\}\lambda - 1 = 0$. As $\kappa < -1$, this shows that λ is constant. Therefore, it follows from equation (1.1) that $S = -\frac{1}{\lambda}g$ on $\mathcal{O}$, since $\lambda \neq 0$ on $\mathcal{O}$. The g-trace gives $-\frac{1}{\lambda} = \frac{r}{2n+1} = \frac{2n(\kappa-2n)}{2n+1}$. Therefore, $S = \frac{2n(\kappa-2n)}{2n+1}g$. Since $Q\xi = 2n\kappa\xi$, from the foregoing equation we deduce that $\kappa = -1$ on $\mathcal{O}$, which is a contradiction. Hence λ is trivial on M. This completes the proof. □

5. Examples

In this section, we shall exhibit some examples of almost Kenmotsu manifolds that satisfy the Miao–Tam equation.

Example 5.1. Consider the warped product $(M, g) = (\mathbb{R} \times_f N, g_0 + f^2\tilde{g})$, where $f^2 = ce^{2t}$, $(N, J, \tilde{g})$ is strictly almost Kähler Einstein manifold, g_0 is the Euclidean metric on $\mathbb{R}$, and c is a positive constant. We set $\eta = dt$, $\xi = \frac{\partial}{\partial t}$, and the tensor field φ is defined on $\mathbb{R}\times_f N$ by $\varphi X = JX$ for vector field X on N, and $\varphi X = 0$ if X is tangent to $\mathbb{R}$. Then it is easy to verify (see [7]) that the warped product $\mathbb{R} \times_f N$, $f^2 = ce^{2t}$, with the structure (φ, ξ, η, g) is an almost Kenmotsu manifold. Thus, if we take $\lambda = ce^t + \frac{1}{2n}$ on M, then it is easy to see that λ is a solution of (1.1).

Remark 5.1. Oguro and Sekigawa (see [14]) constructed a strictly almost Kähler structure on the Riemannian product $\mathbb{H}^3 \times \mathbb{R}$. By virtue of this, it is possible to obtain a 5-dimensional strictly almost Kenmotsu manifold on the warped product $\mathbb{R} \times_{f^2} (\mathbb{H}^3 \times \mathbb{R})$, where $f^2 = ce^{2t}$.

Example 5.2. Let $(N^{2n}, J, \tilde{g})$ be a Kähler Einstein manifold with negative scalar curvature, i.e., $\tilde{S} = -2n\tilde{g}$. We consider the warped product $(M, g) = (\mathbb{R}\times_{f^2} N, dt^2 + f^2\tilde{g})$ with coordinate t on $\mathbb{R}$, where $f = \cosh t$. Let $\lambda = K \sinh t + \frac{1}{2n}$, where K is a positive constant. Then from [12] it follows that λ is a solution of (1.1). It remains to prove that the warped product $\mathbb{R}\times_{f^2} N$, with $f = \cosh t$, is an almost Kenmotsu manifold. Defining ξ, η and φ as in Example 5.1, we see that (M, g) admits an almost contact metric structure. Moreover, from Lemma 2.1, it is obvious that the warped product under consideration is a β-Kenmotsu manifold with $\beta = \tanh t$, which is also an almost Kenmotsu manifold.

Example 5.3. Suppose that $(N^{2n}, J, \tilde{g})$ is a Kähler Einstein manifold with $\tilde{S} = -2n\tilde{g}$. Let $f : \mathbb{R} \to \mathbb{R}^+$ be a smooth function defined by

$$f(t) = A \sinh t + B \cosh t,$$

where A, B are constants, not simultaneously zero. Consider the warped product $M = \mathbb{R} \times_f N$ of dimension $2n + 1$ endowed with the metric

$$g = dt^2 + f^2 \tilde{g}.$$

Then it follows from [17, Lemma 1.1] that (M^{2n+1}, g) is an Einstein manifold. Defining ξ, η and φ as in Example 5.1, it is easy to see that (M, g) admits an almost contact metric structure (φ, ξ, η, g). Further, from Lemma 2.1 it is obvious that the manifold M with the almost contact structure is almost Kenmotsu (in particular, β-Kenmotsu). Now if we take $\lambda = A \cosh t + B \sinh t + \frac{1}{2n}$, then it is straightforward to verify that (g, λ) is a solution of the Miao–Tam equation.

Acknowledgements. We are very much thankful to the reviewer for his/her valuable comments towards the improvement of the paper.

References

[1] P. Alegre, D. E. Blair and A. Carriazo, Generalized Sasakian-space-forms, *Israel J. Math.* **141** (2004), 157–183.

[2] A. Barros, R. Diogenes and E. Ribeiro Jr., Bach-flat critical metrics of the volume functional on 4-dimensional compact manifolds with boundary, *J. Geom. Anal.* **25** (2015), 2698–2715.

[3] A. Besse, Einstein Manifolds, *Springer-Verlag*, *New York*, 2008.

[4] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, *Birkhäuser Boston*, *Boston, MA*, 2002.

[5] D. E. Blair, T. Koufogiorgos and B. J. Papantoniou, Contact metric manifolds satisfying a nullity condition, *Israel J. Math.* **91** (1995), 189–214.

[6] C. P. Boyer and K. Galicki, Einstein manifolds and contact geometry, *Proc. Amer. Math. Soc.* **129** (2001), 2419–2430.

[7] G. Dileo and A. M. Pastore, Almost Kenmotsu manifolds and local symmetry, *Bull. Belg. Math. Soc. Simon Stevin* **14** (2007), 343–354.

[8] G. Dileo and A. M. Pastore, Almost Kenmotsu manifolds and nullity distributions, *J. Geom.* **93** (2009), 46–61.

[9] A. Ghosh, Ricci soliton and Ricci almost soliton within the framework of Kenmotsu manifold, preprint.

[10] A. Ghosh and D. S. Patra, The critical point equation and contact geometry, *J. Geom.* **108** (2017), 185–194.

[11] K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tôhoku Math. J. (2)* **24** (1972), 93–103.

[12] P. Miao and L.-F. Tam, Einstein and conformally flat critical metrics of the volume functional, *Trans. Amer. Math. Soc.* **363** (2011), 2907–2937.

[13] P. Miao and L.-F. Tam, On the volume functional of compact manifolds with boundary with constant scalar curvature, *Calc. Var. Partial Differential Equations* **36** (2009), 141–171.

[14] T. Oguro and K. Sekigawa, Almost Kahler structures on the Riemannian product of a 3-dimensional hyperbolic space and a real line, *Tsukuba J. Math.* **20** (1996), 151–161.

[15] A. M. Pastore and V. Saltarelli, Generalized nullity distributions on almost Kenmotsu manifolds, *Int. Electron. J. Geom.* **4** (2011), 168–183.

[16] D. S. Patra and A. Ghosh, Certain contact metrics satisfying Miao–Tam critical condition, *Ann. Polon. Math.* **116** (2016), 263–271.

[17] S. Pigola, M. Rigoli, M. Rimold and A. Setti, Ricci almost solitons, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **10** (2011), 757–799.

[18] Y. Wang and X. Liu, On almost Kenmotsu manifolds satisfying some nullity distributions, *Proc. Nat. Acad. Sci. India, Sect. A* **86** (2016), 347–353.

[19] K. Yano, Integral Formulas in Riemannian Geometry, *Marcel Dekker*, *New York*, 1970.

AMALENDU GHOSH
DEPARTMENT OF MATHEMATICS
CHANDERNAGORE COLLEGE
HOOGHLY: 712 136 (W. B.)
INDIA

E-mail: aghosh_70@yahoo.com

DHRITI SUNDAR PATRA
DEPARTMENT OF MATHEMATICS
BIRLA INSTITUTE OF TECHNOLOGY MESRA
RANCHI: 835215, JHARKHAND
INDIA

E-mail: dhritimath@gmail.com

(Received June 26, 2017; revised February 8, 2018)