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# Isometric isomorphism of homogeneous space algebras

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**Abstract.** In this paper, we show that two homogeneous spaces with isometrically isomorphic algebras are topologically homeomorphic by themselves. This generalizes the well-known results of J. G. Wendel for group algebras and those of B. E. Johnson for measure algebras.

## 1. Introduction

Harmonic analysis on homogeneous spaces is a very powerful tool to study theoretical and applied physics, as well as engineering. Riemannian symmetric spaces, a very active area of research, are examples of homogeneous spaces. Various subjects related to the topic have been studied by many authors (see [2], [4], [7], [8]). Although homogeneous spaces essentially do not possess a group structure, they are locally compact Hausdorff spaces. Recall that the term "$X$ is a $G$-space" means that $X$ is a locally compact Hausdorff space on which the topological group $G$ acts by an action map transitively and continuously. Let $K$ and $H$ be two compact subgroups of a locally compact group $G$. In 2013, A. Ghaani Farashahi defined a convolution on $L^1(G/K)$ which makes it into a Banach algebra (see [9]). Now, consider two homogeneous spaces $G/K$ and $G/H$. We know that $G/K$ and $G/H$ are isomorphic as homogeneous spaces if and only if $K$ and $H$ are conjugate, strictly speaking, $H = g_0 K g_0^{-1}$ for some $g_0 \in G$ (see [5, Proposition 3.7]). In this paper, we aim to obtain the necessary

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and sufficient conditions on $G/K$ and $G/H$ for an isometrically isomorphism to exist between two Banach algebras $L^1(G/K)$ and $L^1(G/H)$.

In [28], J. G. Wendel showed that two locally compact topological groups with isometrically isomorphism group algebras are themselves isomorphic (algebraic and homeomorphic), and in [22], B. E. Johnson showed that the same is true for measure algebras. In this regard, we achieve the same and more general results for $G$-spaces, which have many applications in different areas, including computerized tomography, magnetic resonance imaging, radio astronomy, crystallographic analysis, etc. (see [1], [27]). The outline of the rest of this paper is as follows: in Section 2, we present some preliminaries and an overview on homogeneous spaces. Section 3 is allocated to prove some lemmas and propositions which are necessary to prove the main results. Finally, this section is ended by presenting the main theorem of this paper.

## 2. Preliminaries

In the sequel, $H$ is a closed subgroup of a locally compact group $G$, and $dx$, $dh$ are the left Haar measures on $G$ and $H$, respectively. We recall that the modular function $\triangle_G$ is a continuous homomorphism from $G$ into the multiplicative group $\mathbb{R}^+$. Furthermore,

$$\int_G f(y)dy = \triangle_G(x) \int_G f(yx)dy,$$

where $f \in C_c(G)$, the space of continuous functions on $G$ with compact support, and $x \in G$. A locally compact group $G$ is called unimodular if $\triangle_G(x) = 1$, for all $x \in G$. A compact group $G$ is always unimodular. Suppose that $\mu$ is a Radon measure on $G/H$. For $x \in G$, we denote by $\mu_x$ the translation of $\mu$ by $x$. Then $\mu$ is said to be $G$-invariant if $\mu_x = \mu$, for all $x \in G$, and is said to be strongly quasi-invariant, if there is a continuous function $\lambda : G \times G/H \to (0, +\infty)$ which satisfies

$$d\mu_x(yH) = \lambda(x, yH)d\mu(yH).$$

If the function $\lambda(x, .)$ reduces to a constant for each $x \in G$, then $\mu$ is called relatively invariant under $G$. We consider a rho-function for the pair $(G, H)$ as a continuous function $\rho : G \to (0, +\infty)$ for which $\rho(xh) = \triangle_H(h)\triangle_G(h)^{-1}\rho(x)$, for each $x \in G$ and $h \in H$. It is well-known that $(G, H)$ admits a rho-function, and for every rho-function $\rho$, there is a strongly quasi-invariant measure $\mu$ on

$G/H$ such that

$$\int_G f(x)\rho(x)dx = \int_{G/H}\int_H f(xh)dhd\mu(xH), \qquad (f \in C_c(G)). \tag{2.1}$$

This equation is called the quotient integral formula. The measure $\mu$ also satisfies

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)}, \qquad (x, y \in G).$$

Every strongly quasi-invariant measure on $G/H$ arises from a rho-function in this manner. All of these measures are strongly equivalent (see Proposition 2.54 and Theorem 2.56 of [6]). Therefore, if $\mu$ is a strongly quasi-invariant measure on $G/H$, then the measures $\mu_x$, $x \in G$, are all mutually absolutely continuous. It should be remarked that if $\mu$ is a strongly quasi invariant measure on $G/H$ which is associated with the rho$-$function $\rho$, then $\mu$ is relatively invariant if and only if $\rho(xy) = \frac{\rho(x)\rho(y)}{\rho(e)}$, $x, y \in G$. Also, $G/H$ has a $G$-invariant Radon measure if and only if the constant function $\rho(x) = 1$, $x \in G$, is a rho-function for the pair $(G, H)$.

Fix a strongly quasi-invariant measure $\lambda$ on $G/H$ which arises from the rho-function $\rho$. Put

$$C_c^\rho(G:H) = \{\varphi^\rho_{\pi_H} := \varphi \circ \pi_H \cdot \rho^{1/p} : \varphi \in C_c(G/H)\}. \tag{2.2}$$

Also take $L^p(G:H) = \overline{C_c^\rho(G:H)}^{\|\cdot\|_p}$, for all $1 \le p < \infty$. By a similar calculation in [3] and [9], one can see that $C_c^\rho(G:H)$ is a left ideal of the algebra $C_c(G)$. Consider the surjective bounded linear operator $T_H^p : C_c(G) \to C_c^\rho(G/H)$ is defined by $T_H^p(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)^{1/p}}dh$, and the norm is defined by $\|\varphi\|_p = \inf\{\|f\|_p : T_H^p(f) = \varphi\}$. Then we have the extension $T_H^p : L^p(G) \to (L^p(G/H), \lambda)$ that is a surjective and bounded linear operator with $\|T_H^p\| \le 1$. For all $1 \le p < \infty$, $T_H^p : L^p(G:H) \to (L^p(G/H), \lambda)$ is an isometric isomorphism between two Banach algebras. Since $T_H^p$ is surjective, for all $\varphi$ and $\psi$ in $L^p(G/H)$, there are $\varphi^\rho_{\pi_H}$ and $\psi^\rho_{\pi_H}$ in $L^p(G:H)$ such that $T_H^p(\varphi^\rho_{\pi_H}) = \varphi$ and $T_H^p(\psi^\rho_{\pi_H}) = \psi$. Further, $\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H}$ belongs to $L^p(G:H)$. So one can define

$$\begin{aligned}\varphi * \psi(xH) &= T_H^p(\varphi^\rho_{\pi_H} * \psi^\rho_{\pi_H})(xH)\\ &= \int_{G/H}\int_H \varphi(yH)\psi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p} dhd\lambda(yH),\end{aligned} \tag{2.3}$$

for all $\varphi \in (L^p(G/H), \lambda)$ and $\psi \in (L^p(G/H), \lambda)$. Let $\sigma \in M(G/H)$ and $\varphi \in L^p(G/H)$, then there exist $\sigma_{P_H} \in M(G:H)$ and $\varphi^\rho_{\pi_H} \in L^p(G:H)$ such that

$T_H^p(\varphi^\rho_{\pi_H})$ and $R_H(\sigma_{P_H}) = \sigma$. We now define the function $\sigma * \varphi$ in a natural way as follows:

$$\begin{aligned}\sigma * \varphi(xH) &= T_H^p(\sigma_{P_H} * \varphi^\rho_{\pi_H})(xH)\\ &= \int_{G/H}\int_H \varphi(hy^{-1}xH)\left(\frac{\rho(hy^{-1}x)}{\rho(x)}\right)^{1/p} dhd\sigma(yH). \qquad (2.4)\end{aligned}$$

Also we have

$$\varphi * \sigma(xH) = \int_{G/H}\Delta(y^{-1})\int_H \varphi(xhy^{-1}H)\left(\frac{\rho(xhy^{-1})}{\rho(x)}\right)^{1/p} dhd\sigma(yH). \qquad (2.5)$$

Suppose $M(G/H)$ denotes the space of all complex bounded Radon measures on $G/H$. Then $(M(G/H), *)$ is a Banach algebra, where $* : M(G/H)\times M(G/H) \longrightarrow M(G/H)$ is defined by $\sigma_1 * \sigma_2(\varphi) = R_H\left(\sigma_{1_{P_H}} * \sigma_{2_{P_H}}\right)(\varphi)$, $\sigma_{i_{P_H}} = \sigma \circ P_H$, and $R_H : M(G) \longrightarrow M(G/H)$ is defined by $R_H\mu(\varphi) = \mu\left(\varphi^\rho_\pi\right)$. Moreover, one can show that for all $1 \leq p < \infty$, $(L^p(G/H), \lambda)$ is a left Banach $M(G/H)$-module via the module action $(\sigma, \varphi) \longrightarrow \sigma * \varphi$ (see [3], [9], [26]).

Recall that for a Banach algebra $\mathcal{A}$, the bounded linear operator $\varpi$ of $\mathcal{A}$ into itself is called a centralizer if $a(\varpi b) = (\varpi a)b$, for all $a, b \in \mathcal{A}$. One can show that the space of all centralizers of $\mathcal{A}$ is a commutative sub-algebra of the algebra of all bounded linear operators on $\mathcal{A}$ (see [25]). Also the bounded linear operator $\varpi$ of $\mathcal{A}$ into itself is called a left centralizer if $\varpi(ab) = (\varpi a)b$, for all $a, b \in \mathcal{A}$. As an example, any left translation on a group algebra is a right centralizer, since $L_x(f * g) = (L_x f) * g$, for all $f, g \in L^1(G)$.

## 3. Main results

Throughout this section, $H$ and $K$ will denote two compact subgroups of a locally compact group $G$. Also let $dh$, $dk$ and $dx$ be the left Haar measures on them, respectively. From now on, all $G$-spaces provided with a relatively invariant measure $\lambda$ correspond to the rho-function $\rho$. The starting point is the following definition.

*Definition 3.1.* A bounded linear operator $\varpi : \big(L^1(G/H), \lambda\big) \longrightarrow \big(L^1(G/H), \lambda\big)$ is called the right centralizer of $L^1(G/H)$ if $\varpi(\varphi * \psi) = \varpi(\varphi) * \psi$, for all $\varphi$ and $\psi$ in $L^1(G/H)$.

If $x \in G$, define the left and right translations of $\varphi \in L^1(G/H)$ by $\ell_x\varphi := T_H\left(L_x\left(\varphi_\pi^\rho\right)\right)$ and $\Re_x\varphi := T_H\left(R_x\left(\varphi_\pi^\rho\right)\right)$, where $L_x$ and $R_x$ are the known left and right translations of functions in the group algebra $L^1(G)$. To avoid confusion, left translations of functions in $L^1(G/H)$ and $L^1(G/K)$ through $x$ will be denoted by $\ell_{xH}$ and $\ell_{xK}$, respectively.

The following lemma gives an example of a right centralizer in $L^1(G/H)$.

**Lemma 3.2.** *Let $x \in G$ and $\varphi \in L^1(G/H)$. Then $\ell_{xH}(\varphi) \in L^1(G/H)$ and the following equalities are satisfied:*

(i) $\ell_{xH}(\varphi * \psi) = \ell_{xH}(\varphi) * \psi$ *and* $xH \in G/H$, $\varphi, \psi \in L^1(G/H)$;

(ii) $\|\ell_{xH}(\varphi)\|_1 = \|\varphi\|_1$ *and* $xH \in G/H$, $\varphi \in L^1(G/H)$.

Proof.

(i) Let $\varphi$ and $\psi$ be in $L^1(G/H)$, then by the definitions of left translation and convolution on $L^1(G/H)$, we have

$$\begin{aligned}\ell_{xH}(\varphi * \psi) &= T_H\left(L_x\left(\varphi * \psi\right)_{\pi_H}^\rho\right) = T_H\left(L_x\left(T_H\left(\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho\right)\right)_{\pi_H}^\rho\right)\\ &= T_H\left(L_x\left(\varphi_{\pi_H}^\rho * \psi_{\pi_H}^\rho\right)\right) = T_H\left(\left(L_x\varphi_{\pi_H}^\rho\right) * \psi_{\pi_H}^\rho\right)\\ &= T_H\left(L_x\varphi_{\pi_H}^\rho\right) * T_H\left(\psi_{\pi_H}^\rho\right) = \ell_{xH}\left(\varphi\right) * \psi.\end{aligned}$$

Note that $L_x$ is a right centralizer of $L^1(G)$ (for more details, see [6, p. 51]).

(ii) Let $\varphi$ be in $L^1(G/H)$ and $xH \in G/H$. Then by the definition of left translation $\ell_{xH}$ and by the relatively invariant property of the measure $\lambda$, we have

$$\begin{aligned}\|\ell_{xH}(\varphi)\|_1 &= \int_{G/H} \left|\ell_{xH}\varphi(yH)\right| d\lambda(yH)\\ &= \int_{G/H} \left|T_H\left(L_x\varphi_{\pi_H}^\rho\right)(yH)\right| d\lambda(yH)\\ &= \int_{G/H} \left|\int_H \frac{L_x\varphi_{\pi_H}^\rho(yh)}{\rho(yh)} dh\right| d\lambda(yH)\\ &= \int_{G/H} \left|\int_H \frac{\varphi_{\pi_H}(x^{-1}yh)\rho(x^{-1}yh)}{\rho(yh)} dh\right| d\lambda(yH)\\ &= \int_{G/H} \left|\varphi(x^{-1}yH)\right| \int_H \frac{\rho(x^{-1}yh)}{\rho(yh)} dh d\lambda(yH)\\ &= \int_{G/H} \left|\varphi(yH)\right| \int_H \frac{\rho(yh)}{\rho(xyh)} dh d\lambda(xyH)\\ &= \int_{G/H} \left|\varphi(yH)\right| d\lambda(yH) = \|\varphi\|_1.\end{aligned}$$

□

**Lemma 3.3.** *Consider the isometric isomorphism $T_H^p$ between two Banach algebras $L^p(G:H)$ and $(L^p(G/H),\lambda)$. Then $\varpi$ is a right centralizer of $L^p(G:H)$ if and only if $T_H^p\varpi\,(T_H^p)^{-1}$ is a right centralizer of $(L^p(G/H),\lambda)$.*

Proof. Let $\varpi$ be a right centralizer of $L^p(G:H)$, and put $\varpi_{G/H} := T_H^p\varpi\,(T_H^p)^{-1}$. Then the linearity and boundedness of $\varpi_{G/H}$ will be deduced from that of $\varpi$, $T_H^p$ and $(T_H^p)^{-1}$. Furthermore, for all $\varphi,\psi\in L^p(G/H)$, we have

$$\begin{aligned}\varpi_{G/H}\,(\varphi*\psi) &= T_H^p\varpi\,(T_H^p)^{-1}\,(\varphi*\psi) = T_H^p\varpi\,(T_H^p)^{-1}\,\big(T_H^p\,(\varphi^\rho_{\pi_H}*\psi^\rho_{\pi_H})\big)\\ &= T_H^p\varpi\,(\varphi^\rho_{\pi_H}*\psi^\rho_{\pi_H}) = T_H^p\,\big[(\varpi\varphi^\rho_{\pi_H})*\psi^\rho_{\pi_H}\big]\\ &= T_H^p\,\big[\varpi\varphi^\rho_{\pi_H}\big]*T_H^p\,\big[\psi^\rho_{\pi_H}\big] = T_H^p\,\Big[\varpi\,(T_H^p)^{-1}\,\varphi\Big]*\psi = \big(\varpi_{G/H}\varphi\big)*\psi,\end{aligned}$$

and note that $T_H^p\,\big[\psi^\rho_{\pi_H}\big]=\psi$. The reverse implication can be obtained in a similar way. $\square$

Now, we prepare some preliminary lemmas which we need in the sequel.

**Lemma 3.4.** *Suppose $\{e_\alpha\}$ is a left approximate identity of $L^1(G)$, and put $\zeta_\alpha := T_H(e_\alpha)$. Then $\{\zeta_\alpha\}$ is a left approximate identity of $\big(L^1(G/H),\lambda\big)$.*

Proof. See [9, Proposition 3.3]. $\square$

In the next lemma, we need the fact that $\big(L^1(G/H),\lambda\big)$ can be embedded into $M(G/H)$ via the embedding $\psi\mapsto\lambda_\psi$, where $d\lambda_\psi(xH)=\psi(xH)d\lambda(xH)$, $xH\in G/H$. Further, $\|\psi\|_1=\|\lambda_\psi\|$ (see [9]).

**Lemma 3.5.** *Let $\{\psi_\alpha\}$ be a net in $\big(L^1(G/H),\lambda\big)$. Then there exists a unique complex regular measure of bounded variation $\sigma$ such that*

(i) $\sigma\,(\gamma)=\lim_\alpha\lambda_{\psi_\alpha}\,(\gamma)$, $\gamma\in C_0(G/H)$;

(ii) $\|\sigma\|=\lim_\alpha\|\psi_\alpha\|$.

Proof. Since $\|\psi_\alpha\|_1\le 1$ and $\|\psi_\alpha\|_1=\|\lambda_{\psi_\alpha}\|$, we conclude that $\{\lambda_{\psi_\alpha}\}$ is a net of bounded positive linear functionals on $C_0(G/H)$. So $\{\lambda_{\psi_\alpha}\}$ is a subset of the closed unit ball in $C_0(G/H)^*$, which is weak$-*$ compact by the Banach–Alaoglu Theorem. Then the set $\{\lambda_{\psi_\alpha}\}$ has a weak$-*$ limit point $F$. According to the Riesz–Markov Theorem, there exists a unique compex regular measure of bounded variation $\sigma$ corresponding to the positive linear functional $F$ such that $F(\varphi)=\int_{G/H}\varphi(xH)d\sigma(xH),\varphi\in C_0(G/H)$. Furthermore,

$$\|\sigma\|=\|F\|=\|\lim_\alpha\lambda_{\psi_\alpha}\|=\lim_\alpha\|\psi_\alpha\|.$$

Since $F = W^* - \lim_{\alpha} \lambda_{\psi_\alpha}$, we obtain

$$\lim_{\alpha} \lambda_{\psi_\alpha}(\gamma) = F(\gamma) = \int_{G/H} \gamma(xH) d\sigma(xH) = \sigma(\gamma),$$

for all $\gamma \in C_0(G/H)$. $\square$

The next theorem asserts that any right centralizer of $L^1(G/H)$ can be represented as a convolution with a complex regular measure of bounded variation on $C_c(G/H)$.

**Theorem 3.6.** *Let $\varpi$ be a right centralizer of $L^1(G/H)$, then there exists a unique complex regular measure of bounded variation $\sigma$ on $C_c(G/H)$ such that $\varpi(\varphi) = \sigma * \varphi$, $\varphi \in L^1(G/H)$ and $\|\varpi\| = \|\sigma\|$.*

Proof. If $\{e_\alpha\}$ is a left approximate identity of $L^1(G)$, then by Lemma 3.4, $\{\zeta_\alpha := T_H(e_\alpha)\}$ is a left approximate identity of $L^1(G/H)$. Without loss of generality, we may assume that $\|\zeta_\alpha\| = 1$. Let $\varphi \in L^1(G/H)$, then

$$\varpi\varphi = \varpi(\lim_{\alpha} \zeta_\alpha * \varphi) = \lim_{\alpha} \varpi(\zeta_\alpha * \varphi) = \lim_{\alpha} (\varpi(\zeta_\alpha) * \varphi) = \lim_{\alpha}(\psi_\alpha * \varphi), \tag{3.1}$$

where $\psi_\alpha = \varpi(\zeta_\alpha)$. According to Lemma 3.5, there exists a unique complex regular measure of bounded variation $\sigma$ on $C_0(G/H)$ such that $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_\alpha}(\gamma)$, $\gamma \in C_0(G/H)$ and $\|\sigma\| = \lim_{\alpha} \|\psi_\alpha\|$. Since $\|\psi_\alpha\| = \|\varpi(\zeta_\alpha)\| \leq \|\varpi\|\|\zeta_\alpha\| = \|\varpi\|$,

$$\|\sigma\| \leq \|\varpi\|. \tag{3.2}$$

Now, if $\Upsilon, \Psi \in C_c(G/H)$, then so is $\gamma$ defined by

$$\gamma(yH) = \int_{G/H} \int_H \Upsilon(xH)\Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH).$$

Let $\varepsilon > 0$. Since $\sigma(\gamma) = \lim_{\alpha} \lambda_{\psi_\alpha}(\gamma)$, for all $\gamma$ in $C_0(G/H)$, it follows that for any $\alpha_0$, there exists an $\alpha_1 > \alpha_0$ such that $\left|\lambda_{\psi_{\alpha_1}}(\gamma) - F(\gamma)\right| < \varepsilon$, for all $\alpha_1 > \alpha_0$. Hence

$$\left| \int_{G/H} \gamma(yH)\psi_{\alpha_1}(yH) d\sigma(xH) - \int_{G/H} \gamma(yH) d\sigma(xH) \right| < \varepsilon.$$

Then by substituting,

$$\left| \int_{G/H} \int_{G/H} \int_H \Upsilon(xH)\Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \psi_{\alpha_1}(yH) d\lambda(yH) \right.$$
$$\left. - \int_{G/H} \int_{G/H} \int_H \Upsilon(xH)\Psi(hy^{-1}xH) \frac{\rho(hy^{-1}x)}{\rho(x)} dh d\lambda(xH) \gamma(yH) d\sigma(xH) \right| < \varepsilon.$$

By Fubini's Theorem,

$$\left|\int_{G/H}\Upsilon(xH)\int_{G/H}\int_H \Psi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\psi_{\alpha_1}(yH)d\lambda(yH)d\lambda(xH)\right.$$
$$\left.-\int_{G/H}\Upsilon(xH)\int_{G/H}\int_H \Psi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\gamma(yH)d\sigma(xH)d\lambda(xH)\right|<\varepsilon.$$

Using (2.3) and (2.4), we obtain

$$\left|\int_{G/H}\Upsilon(xH)(\psi_{\alpha_1}*\Psi)(xH)d\lambda(xH)-\int_{G/H}\Upsilon(xH)(\sigma*\Psi)(xH)\lambda(xH)d\lambda(xH)\right|<\varepsilon.$$

By 3.1, $\psi_\alpha * \varphi$ tends to $\varpi\varphi$ in $L^1(G/H)$, and therefore these converge as linear functionals. This means that by passing to the limit through a suitable cofinal subset $\{\alpha_1\}$ of $\{\alpha\}$, for any $\varphi \in L^1(G/H)$, we obtain

$$\left|\int_{G/H}\Upsilon(xH)\varpi\varphi(xH)d\lambda(xH)-\int_{G/H}\Upsilon(xH)\sigma*\varphi(xH)\lambda(xH)d\lambda(xH)\right|<\varepsilon.$$

Thus $\varpi\varphi$ and $\sigma * \varphi$ are equal as linear functionals on $C_c(G/H)$, so they are equal easily on $C_0(G/H)$ by $\overline{C_c(G/H)} = C_0(G/H)$. Then by an approximation treatment, these are equal on $L^1(G/H)$, so $\|\varpi\varphi\|_1 = \|\sigma * \varphi\|_1$. Hence by [9, Proposition 2.18], $\|\varpi\| \le \|\sigma\|$. Considering 3.2, we finally conclude that $\|\sigma\| = \|\varpi\|$. To prove the uniqueness, let $\sigma_1$ and $\sigma_1$ satisfy the desired condition of this theorem. Then $\sigma_1 * \varphi = \varpi\varphi = \sigma_2 * \varphi$, for all $\varphi \in C_c(G/H)$, and this implies that $\sigma_1 = \sigma_2$. □

The next two lemmas are both needed to prove Theorem 3.9.

**Lemma 3.7.** *Let $\sigma$ be a complex regular measure of bounded variation on $C_c(G/H)$, and*

$$\left|\int_{G/H}\varphi(yH)d\sigma(yH)\right| = \int_{G/H}|\varphi(yH)|\,d\,|\sigma|\,(yH), \qquad \varphi\in C_c\,(G/H)\,.$$

*Then*

$$\left|\int_{G/H}\varphi(yH)d\sigma(yH)\right| = \int_{G/H}|\varphi(yH)|\,d\sigma(yH), \qquad \varphi\in C_c\,(G/H)\,.$$

Proof. Let

$$\Theta(\varphi) := \left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)|\, d\,|\sigma|\,(yH).$$

Consider a non-negative function $\varphi$ in $C_c(G/H)$ such that $\Theta(\varphi) \neq 0$. Using the assumption, there exists a unique $c_\varphi \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c_\varphi \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH).$$

Similarly, for a non-negative function $\psi$ in $C_c(G/H)$ with $\Theta(\psi) \neq 0$, there exists a unique $c_\psi \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \psi(yH) d\sigma(yH) = c_\psi \int_{G/H} \psi(yH) d\,|\sigma|\,(yH).$$

Also for $\varphi + \psi$, there exists a unique $c_{\varphi+\psi} \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} (\varphi + \psi)(yH) d\sigma(yH) = c_{\varphi+\psi} \int_{G/H} (\varphi + \psi)(yH) d\,|\sigma|\,(yH).$$

Hence we have

$$\begin{aligned} & c_\varphi \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH) + c_\psi \int_{G/H} \psi(yH) d\,|\sigma|\,(yH) \\ & = c_{\varphi+\psi} \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH) + c_{\varphi+\psi} \int_{G/H} \psi(yH) d\,|\sigma|\,(yH). \end{aligned}$$

Given the uniqueness of constants, we deduce that $c_\varphi = c_\psi = c_{\varphi+\psi}$ so that the coefficient is independent of the choice of $\varphi \in C_c(G/H)$. Then for any $\varphi$ in $C_c(G/H)$ with $\Theta(\varphi) \neq 0$, there exists a unique $c \in \mathbb{C}$ with absolute value 1 such that

$$\int_{G/H} \varphi(yH) d\sigma(yH) = c \int_{G/H} \varphi(yH) d\,|\sigma|\,(yH).$$

This means that $d\sigma(yH) = d\,|\sigma|\,(yH)$, and finally

$$\left| \int_{G/H} \varphi(yH) d\sigma(yH) \right| = \int_{G/H} |\varphi(yH)|\, d\sigma(yH); \varphi \in C_c(G/H).$$

In fact, this result holds for all real continuous functions which have a limit at infinity. □

**Lemma 3.8.** *Fix $\sigma$ as a normalized measure in $M(G/H)$ with $|\sigma(\varphi)| = \sigma\,|\varphi|$, $\varphi \in C_0(G/H)$. If $F : C_0(G/H) \longrightarrow \mathbb{C}$ is the mapping defined by $F(\varphi) := \sigma(\varphi)$, then $F$ is a point functional.*

Proof. It is clear that $F$ is a positive linear functional. Using the Riesz Representation Theorem, $\|F\| = \|\sigma\| = 1$. Further, if $\min(\varphi, \psi) = 0$ for all non-negative $\varphi, \psi \in C_0(G/H)$. Then $|\varphi + \psi| = |\varphi - \psi|$, so

$$\begin{aligned}\int_{G/H} (\varphi(yH) + \psi(yH))\, d\sigma(yH) &= \int_{G/H} |\varphi(yH) - \psi(yH)|\, d\sigma(yH)\\ &= \left|\int_{G/H} (\varphi(yH) - \psi(yH))\, d\sigma(yH)\right|.\end{aligned}$$

Using the hypothesis $|\sigma(\varphi)| = \sigma\,|\varphi|$, we get $\left|\int_{G/H} \varphi(yH)d\sigma(yH) + \int_{G/H} \psi(yH)\times d\sigma(yH)\right| = \left|\int_{G/H} \varphi(yH)d\sigma(yH) - \int_{G/H} \psi(yH)d\sigma(yH)\right|$. Thus $\min(F(\varphi), F(\psi)) = 0$. Finally, by the Kakutani Theorem, $F$ is a point functional (see [23]), that is, there exists $x_\sigma \in G$ such that $F(\varphi) = \sigma(\varphi) = \varphi(x_\sigma)$, $\varphi \in C_0(G/H)$. □

In the next theorem, we present necessary and sufficient conditions on a bounded linear operator $A$ to be a right centralizer in $L^1(G/H)$.

**Theorem 3.9.** *Let $\varpi : L^1(G/H) \longrightarrow L^1(G/H)$ be a bounded linear operator, then there exists $x_0 \in G$ such that $\varpi = c\ell_{x_0H}$ for some $c \in \mathbb{C}(|c| = 1)$ if and only if the following statements are satisfied:*

(i) *$\varpi$ is a right centralizer on $L^1(G/H)$;*

(ii) *$\varpi$ preserves the norm.*

Proof. Let $\varpi = c\ell_{x_0H}$ for some $x_0 \in G$ and $c \in \mathbb{C}(|c| = 1)$. Then it is obvious to see that $\varpi$ is a right centralizer on $L^1(G/H)$ and it preserves the norm by Lemma 3.2. Conversely, suppose that $\varpi$ is a right centralizer and it preserves the norm. Therefore, by Theorem 3.8, there exists a unique complex regular measure of bounded variation $\sigma$ on $C_c(G/H)$ such that $\varpi\varphi = \sigma * \varphi$ and $\|\varpi\| = \|\sigma\|$, $\varphi \in C_c(G/H)$. So for all $\varphi \in C_c(G/H)$, we have $\|\varphi\|_1 = \|\varpi\varphi\|_1 = \|\sigma * \varphi\|_1 \le \|\sigma\|\|\varphi\|_1$. The last inequality follows by [9, Proposition 2.18]. So

$$\begin{aligned}&\int_{G/H} |\sigma * \varphi(xH)|\, \lambda(xH)\\ &\quad = \int_{G/H} \left|\int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)} dh\sigma(yH)\right| d\lambda(xH) = \|\varphi\|_1. \qquad (3.3)\end{aligned}$$

Since $|\sigma|$ is a regular measure,

$$\left|\int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dhd\sigma(yH)\right| \leq \int_{G/H}\left|\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right| d\,|\sigma|\,(yH). \tag{3.4}$$

If strict inequality holds in 3.4 on a set of positive measure, then by 3.3 we get

$$\begin{aligned}\|\varphi\|_1 &< \int_{G/H}\int_{G/H}\left|\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right| d\,|\sigma|\,(yH)d\lambda(xH)\\ &= \int_{G/H}\int_{G/H}\left|\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right| d\,|\sigma|\,(yH)d\lambda(xH).\end{aligned}$$

By replacing $x$ by $yh^{-1}x$, we get

$$\begin{aligned}\|\varphi\|_1 &< \int_{G/H}\int_{G/H}\left|\int_H \varphi(xH)dh\right| d\lambda(xH)d\,|\sigma|\,(yH)\\ &= \int_{G/H}\int_{G/H}|\varphi(xH)|\,d\lambda(xH)d\,|\sigma|\,(yH) = \|\varphi\|_1 d\|\sigma\|.\end{aligned}$$

Using the equality $\|\varpi\| = \|\sigma\|$ and knowing that $\|\varpi\| = 1$, we conclude that $\|\varphi\|_1 < \|\varphi\|_1\|\varpi\| = \|\varphi\|_1$. But this is a contradiction. Thus inequality 3.3 is an equality for a.e. $xH \in G/H$. But since both sides of the this equality are continuous functions of $xH$, the equality holds everywhere:

$$\left|\int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dhd\sigma(yH)\right| = \int_{G/H}\left|\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dh\right| d\,|\sigma|\,(yH).$$

If $x \in H$, and if we replace $\varphi(yH)$ by $\varphi(y^{-1}H)$,

$$\left|\int_{G/H}\int_H \varphi(hyH)\frac{\rho(hyx)}{\rho(x)}dhd\sigma(yH)\right| = \int_{G/H}\left|\int_H \varphi(hyH)\frac{\rho(hyx)}{\rho(x)}dh\right| d\,|\sigma|\,(yH).$$

Thus

$$\left|\int_{G/H}\varphi(yH)d\sigma(yH)\right| = \int_{G/H}|\varphi(yH)|\,d\,|\sigma|\,(yH).$$

By Lemma 3.7, we obtain

$$\left|\int_{G/H} \varphi(yH)d\sigma(yH)\right| = \int_{G/H} |\varphi(yH)|\, d\sigma(yH).$$

Now, let us define $F : C_0(G/H) \longrightarrow \mathbb{C}$ by $F(\varphi) := \sigma(\varphi)$, $\varphi \in C_0(G/H)$. Then, since $\sigma(|\varphi|) = |\sigma(\varphi)|$, Lemma 3.8 assures us that $F$ is a point functional. Then there exists $x_0 = x_\sigma \in G$ such that $F(\varphi) = \varphi(x_0H)$, $\varphi \in C_0(G/H)$, so $\sigma(\varphi) = F(\varphi) = \varphi(x_0H)$. But $\varpi(\varphi)(xH) = \sigma \star \varphi(xH) = \int_{G/H}\int_H \varphi(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)} \times dhd\sigma(yH) = \int_H \varphi(hx_0^{-1}xH)\frac{\rho(hx_0^{-1}x)}{\rho(x)}dh = \ell_{x_0H}\varphi(xH)$. □

The following lemma has a significant role in the proof of the main Theorem 3.13.

**Lemma 3.10.** *Let $T : L^1(G/K) \longrightarrow L^1(G/H)$ be an algebraic isomorphism, both algebras are real or complex, which does not increase norms, and also let $\ell_{xK}$ be a left translation on $L^1(G/K)$. Put $\gamma_{xK} := T\ell_{xK}T^{-1}$. Then there exist $c_{xK}$ in $\mathbb{C}$ ($|c_{xK}| = 1$) and a unique $yH = \tau(xK)$ in $G/H$ such that $\gamma_{xK} = c_{xK}\ell_{yH}$.*

Proof. Considering Theorem 3.9 and the boundedness of the linear operator $\gamma_{xK}$, it suffices to show that $\gamma_{xK}$ is a norm-preserving right centralizer in $L^1(G/H)$. To this end, let $\varphi_H$ and $\psi_H$ be in $L^1(G/H)$, then

$$\begin{aligned}\gamma_{xK}(\varphi_H \star \psi_H) &= T\ell_{xK}T^{-1}(\varphi_H \star \psi_H) = T\ell_{xK}\left(T^{-1}\varphi_H \star T^{-1}\psi_H\right)\\ &= T\left(\ell_{xK}\left(T^{-1}\varphi_H\right) \star T^{-1}\psi_H\right) = T\left(\ell_{xK}\left(T^{-1}\varphi_H\right) \star T^{-1}\psi_H\right)\\ &= T\ell_{xK}T^{-1}(\varphi_H) \star \psi_H = (\gamma_{xK}\varphi_H) \star \psi_H.\end{aligned}$$

Then $\gamma_{xK}$ is a right centralizer. Now, we want to show that $\gamma_{xK}$ preserves the norm in $L^1(G/H)$. First, assume that $\{e_\alpha\}$ is a left approximate identity of $L^1(G/K)$, and put $\zeta_\alpha := T_H(e_\alpha)$. By surjectivity of $T$, it is concluded that to any $\varphi_H$ in $L^1(G/H)$ there exists $\varphi_K \in L^1(G/K)$ such that $\varphi_H = T\varphi_K$. Thus

$$\begin{aligned}\lim_\alpha \|\varphi_K \star \zeta_\alpha - \varphi_K\| &= \lim_\alpha \|T\varphi_K \star Te_\alpha - T\varphi_K\|\\ &= \lim_\alpha \|T(\varphi_K \star e_\alpha - \varphi_K\| \le \lim_\alpha \|\varphi_K \star e_\alpha - \varphi_K\|.\end{aligned}$$

The last term of the previous inequality converges to 0, so $\{\zeta_\alpha\}$ is a left approximate identity of $L^1(G/H)$. Now, we have

$$\begin{aligned}\gamma_{xK}\varphi_H &= \gamma_{xK}\left(\lim_\alpha \zeta_\alpha \star \varphi_H\right) = \lim_\alpha (\gamma_{xK}\zeta_\alpha) \star \varphi_H\\ &= \lim_\alpha \left(T\ell_{xK}T^{-1}\zeta_\alpha\right) \star \varphi_H = \lim_\alpha (T\ell_{xK}e_\alpha) \star \varphi_H.\end{aligned}$$

Then we have

$$\begin{aligned}\|\lim_{\alpha}\gamma_{xK}\varphi_H\| &= \|\lim_{\alpha}(T\ell_{xK}e_\alpha)\star\varphi_H\| \leq \lim_{\alpha}\|(T\ell_{xK}e_\alpha)\|\|\varphi_H\|\\ &\leq \lim_{\alpha}\|(T\ell_{xK}e_\alpha)\|\|\varphi_H\| \leq \lim_{\alpha}\|(\ell_{xK}e_\alpha)\|\|\varphi_H\|\\ &\leq \lim_{\alpha}\|(e_\alpha)\|\|\varphi_H\| = \|\varphi_H\|.\end{aligned}$$

Then $\gamma_{xK}$ is a contraction in $L^1(G/H)$, and since $(\gamma_{xK})^{-1} = \gamma_{x^{-1}K}$, so it is a contraction in $L^1(G/H)$ in a similar way. So it preserves the norm. Thus by Theorem 3.9, there exists $c_{xK}$ in $\mathbb{C}$ ($|c_{xK}| = 1$), and there exists $yH \in G/H$ such that $\gamma_{xK} = c_{xK}\ell_{yH}$. It is worth noting that $yH$ corresponds to $xK$. □

**Lemma 3.11.** *The mappings $xK \mapsto c_{xK}$ and $xK \mapsto \tau(xK)$ defined in Lemma 3.10 are of continuous homomorphism-type of $G/K$ to, respectively, the circle group $\mathbb{T}$ and the G-space $G/H$; $\tau$ is $1-1$.*

Proof. Let $x_1, x_2 \in G$ and $\ell_{x1H}$, $\ell_{x2H}$ denote the left translations in $L^1(G/H)$. Then the two mappings are of homomorphism-type by Lemma 3.10, and we have the following equations:

$$\begin{aligned}c_{x_1x_2K}\ell_{y_1y_2H} &= T\ell_{x_1x_2K}T^{-1} = T\ell_{x_1K}\ell_{x_2K}T^{-1} = T\ell_{x_1K}T^{-1}T\ell_{x_2K}T^{-1}\\ &= c_{x_1K}\ell_{y_1H}c_{x_2K}\ell_{y_2H} = c_{x_1K}c_{x_2K}\ell_{y_1H}\ell_{y_2H},\end{aligned}$$

Now, $\tau$ is the product of $\tau_1 : xK \mapsto \ell_{xK}$, $\tau_2 : \ell_{xK} \mapsto T\ell_{xK}T^{-1} = c_{xK}\ell_{yH}$ and $\tau_3 : c_{xK}\ell_{yH} \mapsto yH$. Due to [6, Proposition 2.41], the mapping $x \mapsto L_x$ is continuous in strong operator topology. So $\|L_xf - L_{x_0}f\|_1 \to 0$, as $x \to x_0$, for all $f \in L^1(G)$. Then $\|\ell_x\varphi - \ell_{x_0}\varphi\|_1 = \|T_H\left(L_x\left(\varphi^\rho_{\pi_H}\right)\right) - T_H\left(L_{x_0}\left(\varphi^\rho_{\pi_H}\right)\right)\|_1 = \|L_x\left(\varphi^\rho_{\pi_H}\right) - L_{x_0}\left(\varphi^\rho_{\pi_H}\right)\|_1 \to 0$, as $x \to x_0$, for all $\varphi \in L^1(G/H)$. Hence the first mapping is continuous in strong operator topology. The second one is continuous by boundedness of $T$ and $T^{-1}$. So it suffices to show that the third mapping is continuous. In the first step, we prove $L_y \mapsto y$ is a continuous homomorphism of the group of all left translations to $G$; let $V$ be a neighborhood of $1 \in G$, then there exists a symmetric neighborhood $W$ of 1 with finite measure $\omega$ such that $WW^{-1} \subseteq V$. Put $N_I := \{L_y : \|L_y\chi_W - \chi_W\|_1 < \omega\}$. If $L_y \in N_I$, then $y \in V$. To verify this, suppose $y \notin V$, then $y \notin WW^{-1}$, so $yW$ and $W$ are disjoint sets. Hence

$$\begin{aligned}\|L_y\chi_W - \chi_W\|_1 &= \int_G \left|\chi_W(y^{-1}x) - \chi_W(x)\right| dx\\ &= \int_G \left|\chi_W(y^{-1}x)\right| dx + \int_G |\chi_W(x)|\, dx = \|L_y\chi_W\|_1 + \|\chi_W\|_1 = 2\omega.\end{aligned}$$

This contradiction completes the proof of continuity of $L_y \mapsto y$. For the rest of the proof of the lemma, let $V'$ be an arbitrary neighborhood of $\pi_H(y_0) = y_0H \in G/H$, and put $U := \pi_H^{-1}(V')$ that includes $y_0 \in G$. Then by the continuity of $L_y \mapsto y$, there exists $N_{L_{y_0}}$ such that $y \in U$ for all $L_y \in N_{L_{y_0}}$. Now, it suffices to take

$$\mathcal{N}_{\ell_{y_0}} := \left\{\ell_y : L_y \in N_{L_{y_0}}\right\}.$$

Since $N_{L_{y_0}}$ is open, for any $L_y \in N_{L_{y_0}}$, there exists $\varepsilon > 0$ such that $B\left(L_y, \varepsilon\right) \subseteq N_{L_{y_0}}$. Suppose $\ell_{y_1} \in \mathcal{N}_{\ell_{y_0}}$, then $\mathcal{B} := \left\{\ell_y : L_y \in B\left(L_{y_1}, \varepsilon\right)\right\} \subseteq \mathcal{N}_{\ell_{y_0}}$. So $\mathcal{N}_{\ell_{y_0}}$ is open. Finally, if $yH \notin V'$, then $y \notin U$. So $L_y \notin N_{L_{y_0}}$, hence $\ell_y \notin \mathcal{N}_{\ell_{y_0}}$, and this implies that $\ell_y \mapsto y$ is continuous. Finally, $xK \mapsto c_{xK}$ is continuous, since $c_{xK}I$ is the product of the uniformly bounded and continuous functions $T\ell_{xK}T^{-1}$ and $\ell_{x^{-1}H}$. To show that $\tau$ is $1-1$, let $x_1, x_2 \in G$ and $\tau(x_1K) = \tau(x_2K)$. Then by Lemma 3.10, we have $c_{x_1K}T\ell_{x_1K}T^{-1} = c_{x_2K}T\ell_{x_2K}T^{-1}$ so that $c_{x_2^{-1}K}\ell_{x_1K} = c_{x_1^{-1}K}\ell_{x_2K}$, then $c_{x_1K}c_{x_2^{-1}K}\ell_{x_2^{-1}x_1K} = I_K$, and this implies $x_1K = x_2K$.

Then we are done, the proof is completed. □

The following lemma states that the right translations span the space of all right centralizers.

**Lemma 3.12.** *Let $\varpi$ be a right centralizer of $L^1(G/H)$, then $\varpi$ is a strong limit point of the set of finite linear combinations of left translations.*

Proof. Considering the Hahn–Banach Theorem, it is enough to show that if $\Lambda$ is an arbitrary strongly continuous linear functional on the operators on $L^1(G/H)$, which vanishes on the left translations, then $\Lambda(\varpi) = 0$. It is well known that any strongly continuous linear functional $\Lambda$ on the space of all operators $\{T\}$ on a Banach space $L^1(G/H)$ can be written as $\Lambda(T) = \sum_{j=1}^n \varphi_j^*\left(T(\varphi_j)\right)$, where $\varphi_j \in L^1(G/H)$ and $\varphi_j^* \in \left(L^1(G/H)\right)^*$. Since $\varphi_j^* \in \left(L^1(G/H)\right)^*$, and $\left(L^1(G/H)\right)^*$ is the Banach space of all locally measurable functions which a.e. are bounded. We have

$$\Lambda(T) = \sum_{j=1}^n \int_{G/H} \psi_j(xH)T(\varphi_j)(xH)d\lambda(xH),$$

where $\psi_j$ are locally measurable functions which a.e. are bounded, and $\lambda$ is a relatively invariant measure on $G/H$ which arises from the rho-function $\rho$. If $\Lambda \equiv 0$ on the space of all left translations, then

$$\sum_{j=1}^n \int_{G/H} \psi_j(xH)\ell_{yH}\varphi_j(xH)d\lambda(xH) = 0, \qquad (yH \in G/H),$$

or, equivalently,

$$\sum_{j=1}^{n}\int_{G/H}\psi_j(xH)\int_H\varphi_j(y^{-1}xH)\frac{\rho(y^{-1}xh)}{\rho(xh)}dhd\lambda(xH)=0,\quad (yH\in G/H)\,. \tag{3.5}$$

Now, by Theorem 3.6, there exists a unique complex regular measure of bounded variation $\sigma$ on $C_c(G/H)$ such that $\varpi(\varphi)=\sigma*\varphi$, $\varphi\in L^1(G/H)$ and $\|\varpi\|=\|\sigma\|$. Then

$$\begin{aligned}\Lambda(\varpi)&=\sum_{j=1}^{n}\int_{G/H}\psi_j(xH)\varpi(\varphi_j)(xH)d\lambda(xH)=\sum_{j=1}^{n}\int_{G/H}\psi_j(xH)\sigma*\varphi_j(xH)d\lambda(xH)\\&=\sum_{j=1}^{n}\int_{G/H}\psi_j(xH)\int_{G/H}\int_H\varphi_j(hy^{-1}xH)\frac{\rho(hy^{-1}x)}{\rho(x)}dhd\sigma(yH)d\lambda(xH)\\&=\sum_{j=1}^{n}\int_{G/H}\psi_j(xH)\int_{G/H}\int_H\varphi_j(y^{-1}xH)\frac{\rho(y^{-1}xh)}{\rho(xh)}dhd\sigma(yH)d\lambda(xH)\\&=\sum_{j=1}^{n}\int_{G/H}\int_{G/H}\psi_j(xH)\int_H\varphi_j(hy^{-1}xH)\frac{\rho(y^{-1}xh)}{\rho(xh)}dhd\lambda(xH)d\sigma(yH)=0.\end{aligned}$$

Hence the proof is complete. □

**Theorem 3.13.** *Let $T:L^1(G/K)\longrightarrow L^1(G/H)$ be an algebraic isomorphism, both algebras real or complex, which does not increase norms. The mapping $\tau:G/K\longrightarrow G/H$ defined in Lemma 3.10 is a homeomorphism.*

Proof. By Lemma 3.10, $\tau$ is a continuous injection, then it is enough to show that $\tau$ is surjective and $\tau^{-1}$ is continuous. The mapping $\tau^{-1}$ is the product of $\tau_1^{-1}:\ell_{xK}\mapsto xK$, $\tau_2^{-1}:T\ell_{xK}T^{-1}=c_{xK}\ell_{yH}\mapsto\ell_{xK}$ and $\tau_3^{-1}:yH\mapsto c_{xK}\ell_{yH}$. Considering Lemma 3.10, the proofs of the continuity of $\tau_1^{-1}$, $\tau_2^{-1}$ and $\tau_3^{-1}$ are the same as those of $\tau_3$, $\tau_2$ and $\tau_1$, respectively.

To see that $\tau$ is surjective, as our first step, we shall show that $\tau(G/K)$ is closed in $G/H$. Let $\{\tau(x_\alpha K)\}\subseteq\tau(G/K)$ be a directed sequence in $G/H$ that converges to $yH$ in $G/H$. Since $yH\mapsto\ell_{yH}$ is continuous, $\ell_{\tau(x_\alpha K)}$ tends to $\ell_{yH}$. So $T^{-1}\ell_{\tau(x_\alpha K)}T=c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$, $c_{x_\alpha K}\in\mathbb{C},|c_{x_\alpha K}|=1$ (see Lemma 3.10). Considering Theorem 3.9, $A=\text{strong}-\lim_\alpha c_{x_\alpha K}^{-1}\ell_{x_\alpha K}$ is an isometric right centralizer, so it has the form $c_{xK}\ell_{xK}$, for some $xK\in G/K$ and $c_{xK}\in\mathbb{C},|c_{xK}|=1$. Therefore, $T^{-1}\ell_{yH}T=c_{xK}\ell_{xK}$, so $\ell_{yH}=c_{xK}T\ell_{xK}T^{-1}$, which, using Lemma 3.10, implies that $yH=\tau(xK)$.

As the final step, we show that $\tau(G/K) = G/H$. Suppose that there is no preimage in $G/K$ for some $yH$ in $G/H$. Then $\varpi := T^{-1}\ell_{yH}T$ is a right centralizer of $L^1(G/K)$. Thus by Theorem 3.12, it holds that $\varpi = \text{strong} - \lim_{\alpha} \ell_\alpha$, where $\ell_\alpha = \sum_{i=1}^{n_\alpha} c_{\alpha_i}\ell_{x_{\alpha_i}K}$, $c_{\alpha_i} \in \mathbb{C}, |c_{\alpha_i}| = 1$. Then $T\varpi T^{-1} = \ell_{yH} = \text{strong} - \lim_{\alpha} \sum_{i=1}^{n_\alpha} c_{\alpha_i} T\ell_{x_{\alpha_i}K}T^{-1} = \text{strong} - \lim_{\alpha} \sum_{i=1}^{n_\alpha} c'_{\alpha_i}\ell_{y_{\alpha_i}H}$, for some $y_{\alpha_i}H \in G/H$ and $c'_{\alpha_i} \in \mathbb{C}, \left|c'_{\alpha_i}\right| = 1$. Now, since $\tau(G/K)$ is closed in $G/H$, there exists a neighborhood $\omega$ of $H$ in $G/H$ such that $\lambda(\omega) < \infty$ and $y\omega\omega^{-1} \cap \tau(G/K)$ is empty, where $\lambda$ is a relatively invariant measure on $G/H$. Put $\varphi := \chi_\omega$ and $\varphi_\alpha := \ell_\alpha(\varphi)$. Let $zH \in \omega$. If $y_{\alpha_i}H \in \tau(\omega)$, or equivalently, $y^{-1}y_{\alpha_i}H \notin \omega\omega^{-1}$, then $y_{\alpha_i}^{-1}zH \in \omega(z^{-1}y_{\alpha_i}H \in \omega^{-1})$ implies that $y^{-1}y_{\alpha_i}H = y^{-1}zz^{-1}y_{\alpha_i}H \in \omega\omega^{-1}$, which is a contradiction. Therefore, $\varphi(y^{-1}zH) = 1$ concludes that $\varphi_{\alpha_i}(zH) = 0$. Hence we have

$$\begin{aligned}\|\ell_\alpha(\varphi) - \ell_{yH}(\varphi)\|_1 &\geq \|\varphi_\alpha - \ell_{yH}(\varphi)\|_1 \\ &= \int_{y\omega} \left|\varphi_\alpha(zH) - \varphi(y^{-1}zH)\right| d\lambda(zH) = \int_{y\omega} 1 d\lambda(zH) = \lambda(y\omega),\end{aligned}$$

which contradicts that $\{\ell_\alpha\}$ tends to $\ell_{yH}$ in strong operator topology. With this, the proof is completed. □

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