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A condition for a Landsberg space to be Berwaldian

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Abstract. I show that a Landsberg space whose mean Berwald curvature has vanishing trace is a Berwald space.

1. Introduction

More than ten years ago Shen pointed out, in [7], that it was not known whether a Landsberg space whose mean Berwald curvature vanishes is a Berwald space. That question has at last been resolved, in the affirmative, by Ming Li in [4]. But in fact, as I shall establish in this paper, rather less is required of the Berwald curvature of a Landsberg space in order for the space to be Berwaldian.

I denote by B^i_{jkl} the Berwald curvature of a Finsler space. Its vanishing is the necessary and sufficient condition for the space to be a Berwald space. The mean Berwald curvature is $E_{ij} = B^k_{kij}$. I call the scalar $g^{ij}E_{ij}$ the trace of the mean Berwald curvature, the g^{ij} being the components of the contravariant form of the fundamental tensor of the Finsler structure. I shall show that:

Theorem. *A Landsberg space for which the trace of the mean Berwald curvature vanishes is a Berwald space.*

I shall also derive the following results, of interest in their own right, about Finsler spaces which are not necessarily Landsbergian.

Theorem. *A Finsler space for which $g^{kl}B^i_{jkl} = 0$ is a Berwald space.*

Theorem. *In any Finsler space, if the trace of the mean Berwald curvature vanishes, then the mean Berwald curvature itself vanishes.*

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It is important to note that the tensor $g^{kl}B^i_{jkl}$ is in general distinct from the mean Berwald curvature. However, in a Landsberg space the two effectively coincide. Thus, in fact, the first of these theorems follows from the third via the second.

The well-known texts by Bao, Chern and Shen [1] and Shen alone [6] are comprehensive background references. The summation convention is in force throughout.

2. The key theorem

My results about Finsler spaces follow from a result about Minkowski spaces which I regard as key. I shall show that a smooth globally-defined function ϕ on $\mathbb{R}^n_0 = \mathbb{R}^n - \{0\}$ which is positively homogeneous of degree 1 and which satisfies

$$g^{ij}\frac{\partial^2\phi}{\partial y^i \partial y^j} = 0,$$

where the g^{ij} are the components of the contravariant form of the fundamental tensor of any Minkowski norm on $\mathbb{R}^n_0$, must be linear (see Theorem 1 below).

I begin with some remarks about divergences, which will be needed in the proof of Theorem 1.

Let Ω be a volume form on a manifold M. Then for any vector field X on M, $\mathcal{L}_X\Omega = d(i_X\Omega)$ is a scalar multiple of Ω, and the scalar factor is $\mathrm{div}_\Omega X$, the divergence of X with respect to Ω. If M is compact without boundary, then for any X,

$$\int_M (\mathrm{div}_\Omega X)\Omega = \int_M d(i_X\Omega) = 0,$$

the divergence lemma.

Let $\iota : \bar{M} \to M$ be an embedding of $\bar{M}$ as a submanifold of M of codimension 1. Suppose that N is a vector field on M which is everywhere transversal to $\iota(\bar{M})$ – a normal field. Set $\Theta = i_N\Omega$. Then $\bar{\Theta} = \iota^*\Theta$ is a volume form on $\bar{M}$. Let $\bar{X}$ be a vector field on $\bar{M}$, and suppose that there is a vector field X on M which is tangent to $\iota(\bar{M})$ and which coincides with $\iota_*\bar{X}$ there. I wish to find the relationship between $\mathrm{div}_{\bar{\Theta}}\bar{X}$ and $\mathrm{div}_\Omega X$. I shall make the simplifying assumption that $[N, X]$ is tangent to $\iota(\bar{M})$. Then

$$\begin{aligned} d(i_X\Theta) &= \mathcal{L}_X\Theta - i_X d\Theta = \mathcal{L}_X(i_N\Omega) - i_X d(i_N\Omega) \\ &= i_N(\mathcal{L}_X\Omega) - i_{[N,X]}\Omega - (\mathrm{div}_\Omega N) i_X\Omega \\ &= (\mathrm{div}_\Omega X)\Theta - i_{[N,X]}\Omega - (\mathrm{div}_\Omega N) i_X\Omega. \end{aligned}$$

Since X and $[N, X]$ are both tangent to $\iota(\bar{M})$, $\iota^*\big(i_X\Omega\big) = \iota^*\big(i_{[N,X]}\Omega\big) = 0$. Thus

$$(\mathrm{div}_{\bar{\Theta}}\bar{X})\bar{\Theta} = d(i_{\bar{X}}\bar{\Theta}) = \iota^*(d(i_X\Theta)) = \iota^*((\mathrm{div}_\Omega X)\Theta),$$

so that

$$\mathrm{div}_{\bar{\Theta}}\bar{X} = \iota^*(\mathrm{div}_\Omega X).$$

Consider now a Minkowski space, that is, the punctured vector space $\mathbb{R}^n_0$ equipped with a Minkowski norm. I shall work with the standard linear coordinate system on $\mathbb{R}^n$, with coordinates y^i, as one clearly may. I denote by F the Minkowski norm, $\mathcal{I} = \{y : F(y) = 1\}$ the indicatrix, $E = \frac{1}{2}F^2$ the energy, $g_{ij} = \partial^2 E/\partial y^i \partial y^j$ the metric or fundamental tensor, $C_{ijk} = \partial g_{ij}/\partial y^k$ the Cartan tensor or torsion. Indices are raised with g^{ij} and lowered with g_{ij}.

The key result is this:

Theorem 1. *A smooth globally-defined function ϕ on $\mathbb{R}^n_0$ which is positively homogeneous of degree 1 and which satisfies*

$$g^{ij}\frac{\partial^2\phi}{\partial y^i \partial y^j} = 0$$

must be linear.

Proof. The proof involves an application of the divergence lemma. I take for Ω the coordinate volume form with respect to the fixed linear coordinate system:

$$\Omega = dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n,$$

so that for a vector field $X = X^i\partial/\partial y^i$,

$$\mathrm{div}_\Omega X = \frac{\partial X^i}{\partial y^i}.$$

I take for $\bar{M}$ (or strictly $\iota(\bar{M})$) the indicatrix $\mathcal{I}$, and for the normal field the Liouville field $\Delta = y^i\partial/\partial y^i$. And for X, I take the vector field whose components with respect to the chosen coordinates are

$$X^i = (\det g)g^{ij}g^{kl}\frac{\partial^2\phi}{\partial y^j \partial y^k}\frac{\partial\phi}{\partial y^l}.$$

I first confirm that X is tangent to $\mathcal{I}$. In fact, $X(F) = 0$, because

$$g^{ij}\frac{\partial^2\phi}{\partial y^j \partial y^k}\frac{\partial F}{\partial y^i} = \frac{1}{F}y^j\frac{\partial^2\phi}{\partial y^j \partial y^k} = 0,$$

since ϕ is homogeneous of degree 1. Secondly, taking note of the homogeneity degrees of the terms occuring in X^i, one sees that $[\Delta, X] = -2X$, which is of course tangent to $\mathcal{I}$. So I can compute $\mathrm{div}_\Theta X$ (where $\Theta = i_\Delta \Omega$) simply by computing $\partial X^i/\partial y^i$. For this purpose, I make several preliminary calculations. I note first that

$$\frac{\partial g^{kl}}{\partial y^i} = -C_i^{kl},$$

and in particular

$$\frac{\partial g^{ij}}{\partial y^i} = -C_i^{ij} = -C^j, \quad \text{where } C_k = C_{kl}^l = g^{ij} C_{ijk}.$$

On the other hand,

$$\frac{\partial (\det g)}{\partial y^i} = (\det g) g^{jk} \frac{\partial g_{jk}}{\partial y^i} = (\det g) C_i,$$

and therefore

$$\frac{\partial}{\partial y^i}\big((\det g) g^{ij}\big) = 0.$$

Furthermore,

$$g^{ij}\frac{\partial^3 \phi}{\partial y^i \partial y^j \partial y^k} = \frac{\partial}{\partial y^k}\left(g^{ij}\frac{\partial^2 \phi}{\partial y^i \partial y^j}\right) - \left(-C_k^{ij}\frac{\partial^2 \phi}{\partial y^i \partial y^j}\right) = C_k^{ij}\frac{\partial^2 \phi}{\partial y^i \partial y^j}.$$

Thus

$$\begin{aligned}
\frac{\partial X^i}{\partial y^i} &= (\det g)\left(-g^{ij} C_i^{kl}\frac{\partial^2 \phi}{\partial y^j \partial y^k}\frac{\partial \phi}{\partial y^l} + g^{kl} C_k^{ij}\frac{\partial^2 \phi}{\partial y^i \partial y^j}\frac{\partial \phi}{\partial y^l} + g^{ij} g^{kl}\frac{\partial^2 \phi}{\partial y^j \partial y^k}\frac{\partial^2 \phi}{\partial y^i \partial y^l}\right)\\
&= (\det g)\left(-C^{jkl}\frac{\partial^2 \phi}{\partial y^j \partial y^k}\frac{\partial \phi}{\partial y^l} + C^{ijl}\frac{\partial^2 \phi}{\partial y^i \partial y^j}\frac{\partial \phi}{\partial y^l} + g^{ij} g^{kl}\frac{\partial^2 \phi}{\partial y^i \partial y^k}\frac{\partial^2 \phi}{\partial y^j \partial y^l}\right)\\
&= (\det g) g^{ij} g^{kl}\frac{\partial^2 \phi}{\partial y^j \partial y^k}\frac{\partial^2 \phi}{\partial y^i \partial y^l}
\end{aligned}$$

(since C^{ijk}, like C_{ijk}, is symmetric). Now the value of the function

$$\Phi = g^{ij} g^{kl}\frac{\partial^2 \phi}{\partial y^j \partial y^k}\frac{\partial^2 \phi}{\partial y^i \partial y^l}$$

is never negative; and if $\Phi(y) = 0$ for any $y \in \mathbb{R}_0^n$, then $\partial^2\phi/\partial y^i \partial y^j(y) = 0$. (The matrix $T(y)$ whose components are

$$T_k^i(y) = g^{ij}(y)\frac{\partial^2 \phi}{\partial y^j \partial y^k}(y)$$

has real eigenvalues, since it is self-adjoint with respect to $g(y)$; and $\Phi(y)$, which is the trace of $T(y)^2$, is the sum of their squares.) In addition, $\det g > 0$. Thus $\mathrm{div}_\Omega X$ is never negative, and vanishes only if $\partial^2\phi/\partial y^i\partial y^j = 0$. On the other hand, by the divergence lemma,

$$\int_{\mathcal{I}} (\mathrm{div}_\Theta X)\Theta = 0.$$

But the integrand, which is the restriction of $\mathrm{div}_\Omega X$ to $\mathcal{I}$, is never negative, and the integral can be zero only if $\mathrm{div}_\Omega X = 0$ everywhere on $\mathcal{I}$. We conclude that

$$\frac{\partial^2\phi}{\partial y^i\partial y^j} = 0$$

everywhere on $\mathcal{I}$, and then on $\mathbb{R}^n_0$. So $\phi(y) = a + b_i y^i$ for some constants a and b_i. But ϕ is homogeneous of degree 1: so $a = 0$, and ϕ is linear. □

The theorem is most often used in the form

$$g^{ij}\frac{\partial^2\phi}{\partial y^i\partial y^j} = 0 \text{ implies } \frac{\partial^2\phi}{\partial y^i\partial y^j} = 0$$

when the given conditions hold.

The assumptions of the theorem, and its conclusion, are unchanged if one makes a linear change of coordinates. However, the object X defined in the proof, and called there a vector field, is strictly a vector density with respect to linear coordinate transformations, because of the presence of the factor $\det g$. This, of course, makes no difference, provided one works with a fixed linear coordinate system. Nevertheless, with a view to later applications, it is worth considering what effect a linear transformation of coordinates might have. The answer is effectively none, since it will at worst introduce certain factors which are powers of the determinant of the coordinate transformation matrix, which is of course constant.

In the proof one has to assume only that

$$\frac{\partial}{\partial y^k}\left(g^{ij}\frac{\partial^2\phi}{\partial y^i\partial y^j}\right) = 0,$$

which may appear to be weaker than the assumption in the statement of the theorem, namely, that the function in the brackets must vanish (rather than be constant). But this is illusory, since by the assumed degree 1 homogeneity of ϕ, this function must be homogeneous of degree -1, and if constant, can only be zero.

I shall illustrate the potential applicability of this theorem by using it to give a short proof of a well-known result in Minkowski geometry: Deicke's theorem.

Theorem 2. *If in a Minkowski space the determinant of the fundamental tensor is constant, then its coefficients are constant.*

Proof. If $\det g$ is constant, then (with respect to standard coordinates as before)

$$0 = \frac{1}{(\det g)}\frac{\partial(\det g)}{\partial y^k} = g^{ij}\frac{\partial g_{ij}}{\partial y^k} = g^{ij}\frac{\partial}{\partial y^k}\left(\frac{\partial^2 E}{\partial y^i \partial y^j}\right) = g^{ij}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial E}{\partial y^k}\right).$$

Of course, $\partial E/\partial y^k$ is homogeneous of degree 1. It follows from Theorem 1, with $\phi = \partial E/\partial y^k$, that

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial E}{\partial y^k}\right) = \frac{\partial^2}{\partial y^i \partial y^j}(g_{kl}y^l) \\ &= \frac{\partial}{\partial y^i}\left(\frac{\partial g_{kl}}{\partial y^j}y^l + g_{jk}\right) = \frac{\partial}{\partial y^i}\left(\frac{\partial g_{jk}}{\partial y^l}y^l + g_{jk}\right) = \frac{\partial g_{jk}}{\partial y^i}, \end{aligned}$$

since g_{jk} is homogeneous of degree 0. □

In this application ϕ is, of course, not a scalar, but a component of a linear-tensorial object. This certainly makes no difference if one sticks to a single linear coordinate system; and clearly the assumption of the theorem and its conclusion, and indeed the argument, remain valid if one is working in any linear coordinate system.

3. Applications in Finsler geometry

I shall now show how to deduce the results stated in the Introduction from Theorem 1.

Consider a Finsler space over a manifold M, with Finsler function F. For each $x \in M$ the space $T^\circ_x M$, the tangent space to M at x with its origin deleted, is a Minkowski space with Minkowski norm $F(x,\cdot)$. The associated quantities g_{ij}, etc., defined on $T^\circ_x M$ in terms of linear coordinates corresponding to some coordinates about x on M may be regarded equally as linear tensors over $T^\circ_x M$ or as values of tensor fields along the projection $T^\circ M \to M$; I shall use the term 'tensor' indiscriminately for either, hoping that the context will make it clear which is intended.

For my purposes, the important extra ingredient is a connection. Of the various options I choose to work with the Berwald connection. There is a canonical horizontal distribution on $T^\circ M$, where

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial y^j}$$

is the horizontal lift to $T^\circ M$ of the coordinate vector field $\partial/\partial x^i$ on M, the coefficients Γ_j^i being homogeneous of degree 1 in the y^k. One sets

$$\Gamma_{ij}^k = \frac{\partial \Gamma_j^k}{\partial y^i} = \frac{\partial \Gamma_i^k}{\partial y^j}.$$

These coefficients define a covariant derivative of tensor fields in horizontal directions: for example, for a covector field C, the components $C_{i;j}$ of the covariant derivative in the direction H_j are given by

$$C_{i;j} = H_j(C_i) - \Gamma_{ij}^k C_k.$$

The Berwald curvature of a Finsler space is the tensor given by

$$B_{jkl}^i = \frac{\partial^2 \Gamma_j^i}{\partial y^k \partial y^l}.$$

The space is a Berwald space if and only if the coefficients Γ_{jk}^i are independent of y^l, that is, if and only if the Berwald curvature vanishes.

Theorem 3. *A Finsler space for which $g^{kl} B_{jkl}^i = 0$ is a Berwald space.*

Proof. For any $x \in M$, on the Minkowski space $T_x^\circ M$ we have

$$g^{kl} \frac{\partial^2 \Gamma_j^i}{\partial y^k \partial y^l} = 0.$$

Now for fixed x, each component $\Gamma_j^i(x, \cdot)$ is a smooth globally-defined function on $T_x^\circ M$ which is homogeneous of degree 1. It follows therefore from Theorem 1 that

$$\frac{\partial^2 \Gamma_j^i}{\partial y^k \partial y^l} = 0 = B_{jkl}^i$$

at every $x \in M$, and the space is a Berwald space. □

Here I have of course used the linear coordinates on $T_x^\circ M$ corresponding to some coordinates about x on M; and one might have a slightly queasy feeling about the possible effects on the Γ^i_j of a coordinate transformation in M. But on changing from coordinates x^i to coordinates $\bar{x}^i$, these coefficients transform as

$$J^k_j\bar{\Gamma}^i_k = J^i_k\Gamma^k_j - \frac{\partial J^i_j}{\partial x^k}y^k, \qquad J^i_j = \frac{\partial \bar{x}^i}{\partial x^j};$$

the important point is that the transformation involves addition of a term linear in y^i. Thus the assumption of the theorem is unaffected: the Berwald curvature is a tensor after all. And so is the conclusion, from Theorem 1, that $B^i_{jkl} = 0$.

The mean Berwald curvature is $E_{ij} = B^k_{kij}$.

Theorem 4. *In any Finsler space, if the mean Berwald curvature satisfies $g^{ij}E_{ij} = 0$, then it vanishes: $E_{ij} = 0$.*

Proof. The mean Berwald curvature is given by

$$E_{ij} = B^k_{kij} = \frac{\partial^2 \Gamma^k_k}{\partial y^i \partial y^j}.$$

Thus for any $x \in M$, on the Minkowski space $T^\circ_x M$ we have

$$g^{ij}\frac{\partial^2 \Gamma^k_k}{\partial y^i \partial y^j} = 0.$$

Again, for fixed x, $\Gamma^k_k(x,\cdot)$ is a smooth globally-defined function on $T^\circ_x M$ which is homogeneous of degree 1. It follows therefore from Theorem 1 that

$$E_{ij} = \frac{\partial^2 \Gamma^k_k}{\partial y^i \partial y^j} = 0$$

at every $x \in M$. □

I now specialize to Landsberg spaces. One succinct way of defining a Landsberg space is that it is a Finsler space whose fundamental tensor satisfies $g_{ij;k} = 0$.

As I pointed out at the beginning, the tensor $g^{kl}B^i_{jkl}$ is in general not directly related to the mean Berwald curvature. However, in a Landsberg space the two do effectively coincide, for the following reason. It is known (see, for example, [3] and [9]) that a Finsler space is a Landsberg space if and only if $B_{ijkl} = g_{im}B^m_{jkl}$ is symmetric in all its four indices. Thus in a Landsberg space

$$g^{kl}B^i_{jkl} = g^{im}g^{kl}B_{mjkl} = g^{im}g^{kl}B_{klmj} = g^{im}B^l_{lmj} = g^{im}E_{mj}.$$

That is to say:

Lemma. *In a Landsberg space* $g^{kl}B^i_{jkl} = 0$ *if and only if the mean Berwald curvature vanishes.*

The main theorem follows.

Theorem 5. *A Landsberg space for which the mean Berwald curvature satisfies* $g^{ij}E_{ij} = 0$ *is a Berwald space.*

PROOF. If $g^{ij}E_{ij} = 0$, then $E_{ij} = 0$ by Theorem 4; $g^{kl}B^i_{jkl} = 0$ by the lemma above; and the space is a Berwald space by Theorem 3. □

This result has an alternative formulation in terms of the mean Cartan torsion. The value of the mean Cartan torsion of a Finsler space over a manifold M at any $x \in M$ is the covector field $C_i = g^{jk}C_{ijk}$ on the punctured vector space $T^\circ_x M$ coming from the Minkowski structure there, as given, for example, in the proof of Theorem 1. But as was pointed out above, it can equally be regarded as a covector field along the projection $T^oM \to M$, and as such has a covariant derivative in horizontal directions, whose components are $C_{i;j}$. Likewise, the full Cartan torsion has a covariant derivative in horizontal directions. Now in a Landsberg space

$$C_{ijk;l} = B_{ijkl} + B_{jikl} = 2B_{ijkl}$$

(see [3]). Moreover, in a Landsberg space $g^{ij}{}_{;k} = 0$. Thus in a Landsberg space the mean Cartan torsion satisfies

$$C_{k;l} = 2g^{ij}B_{ijkl} = 2E_{kl}.$$

So we have the following corollary of Theorem 5.

Corollary. *A Landsberg space whose mean Cartan torsion satisfies*

$$C^i{}_{;i} = g^{ij}C_{i:j} = 0$$

is a Berwald space.

4. Relation with some previous results

I hit upon the results above through reading the very interesting recent paper by MING LI [4] mentioned briefly in the Introduction. This author observes that Minkowski spaces may be studied by using the methods of affine differential geometry, and by applying a theorem of Schneider concerning hyperovaloids to the case of the indicatrices of Minkowski norms obtains an equivalence theorem for Minkowski spaces. It is of course a matter of considerable interest to see how geometrical results about Minkowski spaces may be derived by considering these spaces as particular cases of more general geometrical structures (the geometry of Hessian manifolds [8] might be another source of such insights). Indeed, Theorem 1 appears to be related to the Blaschke–Schneider Theorem of affine differential geometry, which will be found in the book on that subject by Nomizu and Tanaka [5]; moreover, the method of the proof of Theorem 1 was suggested by the proof of the Blaschke–Schneider Theorem given there, though the details differ considerably.

It is, however, in my view equally interesting to derive such results directly, using the special properties of Minkowski spaces, and this is what I have attempted to do here.

I shall now obtain a version of Ming Li's equivalence theorem of Minkowski spaces using Theorem 1. This is concerned with the following situation. Suppose that we have two Minkowski structures on $\mathbb{R}^n_0$, with norms F, $\bar{F}$, which are homogeneously isometric: that is to say, there is a diffeomorphism $f : \mathbb{R}^n_0 \to \mathbb{R}^n_0$ such that $f(ty) = tf(y)$ for all $t > 0$, which is an isometry of the metrics associated with the Minkowski norms. Ming Li shows that if a certain additional condition is satisfied, then a homogeneous isometry must be linear, in which case the Minkowski spaces are said, reasonably enough, to be equivalent.

By assumption,

$$\bar{g}_{kl}(f(y))\frac{\partial f^k}{\partial y^i}(y)\frac{\partial f^l}{\partial y^j}(y) = g_{ij}(y),$$

or more succinctly

$$(\bar{g}_{kl} \circ f)\frac{\partial f^k}{\partial y^i}\frac{\partial f^l}{\partial y^j} = g_{ij},$$

where, of course, $\bar{g}_{kl}$ are the components of the metric associated with $\bar{F}$. Now from the homogeneity assumption on f, applied componentwise, we see that

$$g_{ij}y^i y^j = (\bar{g}_{kl} \circ f)\frac{\partial f^k}{\partial y^i}y^i\frac{\partial f^l}{\partial y^j}y^j = (\bar{g}_{kl} \circ f)f^k f^l = 2\bar{E} \circ f.$$

That is to say, the energy functions are related by $\bar{E} \circ f = E$, and consequently, the Minkowski norms by $\bar{F} \circ f = F$. So f is necessarily norm preserving.

The fact that the metrics are derived from Finsler norms imposes a condition of f which is not evident at the outset, which I shall now derive. Note that

$$\frac{\partial g_{ij}}{\partial y^k} = \left(\frac{\partial \bar{g}_{pq}}{\partial y^r} \circ f\right)\frac{\partial f^p}{\partial y^i}\frac{\partial f^q}{\partial y^j}\frac{\partial f^r}{\partial y^k} + (\bar{g}_{pq} \circ f)\left(\frac{\partial^2 f^p}{\partial y^i \partial y^k}\frac{\partial f^q}{\partial y^j} + \frac{\partial f}{\partial y^i}\frac{\partial^2 f^q}{\partial y^j \partial y^k}\right).$$

Both sides must be symmetric in j and k: thus f must satisfy

$$(\bar{g}_{pq}\circ f)\frac{\partial^2 f^p}{\partial y^i\partial y^k}\frac{\partial f^q}{\partial y^j}=(\bar{g}_{pq}\circ f)\frac{\partial^2 f^p}{\partial y^i\partial y^j}\frac{\partial f^q}{\partial y^k}.$$

By homogeneity it follows that f must also satisfy

$$(\bar{g}_{kl}\circ f)\frac{\partial^2 f^k}{\partial y^i\partial y^j}f^l=0;$$

in fact, the two conditions are easily seen to be equivalent. This is the condition, in either of its forms, referred to above. (It is a particular case of this condition in its latter version that forms the basis of BRICKELL's original proof, in [2], that a Minkowski norm which is absolutely homogeneous and whose metric is flat is an inner product.)

Then writing $\bar{C}_{pqr}$ for $\partial\bar{g}_{pq}/\partial y^r$, etc., we find that

$$C_{ijk}=(\bar{C}_{pqr}\circ f)\frac{\partial f^p}{\partial y^i}\frac{\partial f^q}{\partial y^j}\frac{\partial f^r}{\partial y^k}+2(\bar{g}_{pq}\circ f)\frac{\partial^2 f^p}{\partial y^i\partial y^j}\frac{\partial f^q}{\partial y^k}.$$

This may be written equivalently as

$$\tfrac{1}{2}C^k_{ij}\frac{\partial f^r}{\partial y^k}=\tfrac{1}{2}(\bar{C}^r_{pq}\circ f)\frac{\partial f^p}{\partial y^i}\frac{\partial f^q}{\partial y^j}+\frac{\partial^2 f^r}{\partial y^i\partial y^j},$$

and recalling that $\frac{1}{2}C^k_{ij}$ is the connection coefficient of the Levi–Civita connection of the metric whose components are g_{ij}, we see that this just says that f transforms the connection appropriately. Clearly,

$$\bar{C}^k_{ij}\frac{\partial f^r}{\partial y^k}=(C^r_{pq}\circ f)\frac{\partial f^p}{\partial y^i}\frac{\partial f^q}{\partial y^j}$$

if and only if f is linear.

Note, however, that (with $\bar{C}^r=\bar{g}^{pq}\bar{C}^r_{pq}$, etc.)

$$C^k\frac{\partial f^r}{\partial y^k}=\bar{C}^r\circ f+g^{ij}\frac{\partial^2 f^r}{\partial y^i\partial y^j}.$$

We thus have the following version of Ming Li's equivalence theorem of Minkowski spaces.

Theorem 6. *Suppose given two Minkowski structures on* $\mathbb{R}^n_0$ *which are homogeneously isometric, with homogeneous isometry* f*: then*

$$\bar{C}^r\circ f=C^k\frac{\partial f^r}{\partial y^k}$$

if and only f *is linear.*

Proof. When f is a homogeneous isometry,

$$\bar{C}^r \circ f = C^k \frac{\partial f^r}{\partial y^k}$$

if and only if

$$g^{ij} \frac{\partial^2 f^r}{\partial y^i \partial y^j} = 0,$$

and so by Theorem 1 if and only if f is linear. □

Ming Li goes on to use his equivalence theorem to show that a Landsberg space whose mean Berwald curvature vanishes is a Berwald space, thus answering the question raised by Shen in [7]. His result is not as strong as mine, but his argument is geometrically interesting, and worth repeating. It goes as follows.

Take a pair of points x, x' in M, and let $\sigma : [0,1] \to M$ be a curve with $\sigma(0) = x$, $\sigma(1) = x'$. Define a map $\Sigma : T^\circ_x M \to T^\circ_{x'} M$ in the following way. For each $y \in T^\circ_x M$, there is a unique curve $\sigma^H_y : [0,1] \to T^\circ M$ which is horizontal (its tangent vector at each point on it is a horizontal vector), projects onto σ, and is such that $\sigma^H_y(0) = y$. Then $\Sigma(y) = \sigma^H_y(1)$. This map is often called nonlinear parallel transport. Nonlinear parallel transport is positively-homogeneous of degree 1, in the sense that $\Sigma(ty) = t\Sigma(y)$, $t > 0$, and preserves the Finsler function F, so that $F(\Sigma(y)) = F(y)$. The space is a Landsberg space when nonlinear parallel transport is an isometry $T^\circ_x M \to T^\circ_{x'} M$ (with respect to the metrics on those spaces induced by the Finsler structure), for every pair of points x, x' and every curve σ joining them. Suppose that in addition the mean Cartan torsion is invariant under nonlinear parallel transport: that is to say, for every pair of points x, x' and every curve σ joining them, the pullback by Σ of the mean Cartan torsion at x' is the mean Cartan torsion at x. Then by Theorem 6, with Σ in place of f, nonlinear parallel transport is in fact everywhere linear, which is to say that the Finsler space is a Berwald space. The condition of invariance of the mean Cartan torsion has an infinitesimal equivalent, which is just that $C_{i;j} = 0$; as indeed has the condition that nonlinear parallel transport is an isometry, namely that $g_{ij;k} = 0$. As was pointed out earlier, in a Landsberg space $C_{i;j} = 2E_{ij}$.

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