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Erratum and addendum to the paper: "On a class of projective Ricci flat Finsler metrics"

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Abstract. In this paper, we correct an error in the formula (24) in [1] and modify the formula of the projective Ricci curvature for Randers metrics. Based on these, we simplify and optimize the main results in [1].

1. Projective Ricci flat Randers metrics

By definition, the projective Ricci curvature of a Finsler metric is given by

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2. \tag{1.1}$$

For a Randers metric $F = \alpha + \beta$ on M, let G^i and ${}^{\alpha}G^i$ denote the geodesic coefficients of F and α, respectively. Then G^i and ${}^{\alpha}G^i$ are related by

$$G^i = {}^{\alpha}G^i + \alpha s^i{}_0 + \frac{1}{2F}\{-2\alpha s_0 + r_{00}\}y^i. \tag{1.2}$$

For the related definitions and notations, see [1]. The Ricci curvature of $F = \alpha+\beta$ is given by

$$\mathbf{Ric} = {}^{\alpha}\mathbf{Ric} + (2\alpha s^m{}_{0;m} - 2t_{00} - \alpha^2 t^m{}_m) + (n-1)\Xi, \tag{1.3}$$

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where

$$\Xi := \frac{3}{4F^2}(r_{00} - 2\alpha s_0)^2 + \frac{1}{2F}[4\alpha(q_{00} - \alpha t_0) - (r_{00;0} - 2\alpha s_{0;0})]. \tag{1.4}$$

Further, the S-curvature of $F = \alpha + \beta$ is given by

$$\mathbf{S} = (n+1)\left[\frac{e_{00}}{2F} - (s_0 + \rho_0)\right], \tag{1.5}$$

where $e_{00} = r_{00} + 2\beta s_0$. From (1.5), we have

$$\mathbf{S}_{y^m} = (n+1)\left[\frac{r_{m0} + b_m s_0 + \beta s_m}{F} - \frac{(\alpha^{-1}a_{mj}y^j + b_m)(r_{00} + 2\beta s_0)}{2F^2} - s_m - \rho_m\right], \tag{1.6}$$

where we have used $F_{y^m} = \alpha^{-1}a_{jm}y^j + b_m$.

By (1.2), we have

$$G^i_m = {}^{\alpha}G^i{}_m + \alpha_{y^m}s^i{}_0 + \alpha s^i{}_m - \frac{F_{y^m}}{2F^2}(-2\alpha s_0 + r_{00})y^i + \frac{1}{F}(-\alpha_{y^m}s_0 - \alpha s_m + r_{m0})y^i + \frac{1}{2F}(-2\alpha s_0 + r_{00})\delta^i{}_m.$$

Thus

$$\begin{aligned}\mathbf{S}_{|m}y^m &= y^m\frac{\partial \mathbf{S}}{\partial x^m} - G^l_m y^m \frac{\partial \mathbf{S}}{\partial y^l} = \mathbf{S}_{;m}y^m - \left[2\alpha s^m{}_0 + \frac{1}{F}(-2\alpha s_0 + r_{00})y^m\right]\frac{\partial \mathbf{S}}{\partial y^m} \\ &= \mathbf{S}_{;m}y^m - 2\alpha s^m{}_0\mathbf{S}_{y^m} - \frac{\mathbf{S}}{F}(-2\alpha s_0 + r_{00}).\end{aligned} \tag{1.7}$$

From (1.5) and (1.6), we obtain

$$\begin{aligned}\mathbf{S}_{;m}y^m &= (n+1)\left\{\frac{1}{2F}r_{00;0} + \frac{1}{F}r_{00}s_0 + \frac{1}{F}\beta s_{0;0} - \frac{1}{2F^2}e_{00}r_{00} - s_{0;0} - \rho_{0;0}\right\} \\ &= (n+1)\left\{\frac{1}{2F}r_{00;0} + \frac{1}{2F^2}(2\alpha s_0 - r_{00})r_{00} - \frac{1}{F}\alpha s_{0;0} - \rho_{0;0}\right\},\end{aligned} \tag{1.8}$$

$$\begin{aligned}2\alpha s^m{}_0\mathbf{S}_{y^m} = &\frac{2(n+1)}{F}\alpha q_{00} + \frac{2(n+1)}{F^2}\alpha^2 s_0^2 - \frac{2(n+1)}{F}\alpha^2 t_0 \\ &- \frac{(n+1)}{F^2}\alpha s_0 r_{00} - 2(n+1)\alpha(\rho_m s^m{}_0),\end{aligned} \tag{1.9}$$

$$\frac{\mathbf{S}}{F}(-2\alpha s_0 + r_{00}) = \frac{n+1}{F}\left\{-\frac{2}{F}\alpha r_{00}s_0 + \frac{2}{F}\alpha^2 s_0^2 + \frac{1}{2F}r_{00}^2 + (2\alpha s_0 - r_{00})\rho_0\right\}. \tag{1.10}$$

Here, we have corrected (24) in [1] as (1.9). By (1.8), (1.9) and (1.10) and (1.7), we obtain

$$\frac{n-1}{n+1}\mathbf{S}_{|m}y^m = (n-1)\left\{\frac{1}{2F}r_{00;0}+\frac{4}{F^2}\alpha r_{00}s_0-\frac{1}{F^2}r_{00}^2-\frac{1}{F}\alpha s_{0;0}-\rho_{0;0}-\frac{2}{F}\alpha q_{00}\right.$$
$$\left.-\frac{4}{F^2}\alpha^2 s_0^2+\frac{2}{F}\alpha^2 t_0+2\alpha(\rho_m s^m{}_0)-\frac{2}{F}\alpha s_0\rho_0+\frac{1}{F}\rho_0 r_{00}\right\}. \quad (1.11)$$

Here, we have corrected (26) in [1] and modified it into (1.11). Further, we have

$$\frac{n-1}{(n+1)^2}\mathbf{S}^2$$
$$= (n-1)\left\{\frac{1}{4F^2}r_{00}^2+\frac{1}{F^2}\alpha^2 s_0^2-\frac{1}{F^2}\alpha r_{00}s_0+\rho_0^2-\frac{1}{F}\rho_0 r_{00}+\frac{2}{F}\alpha\rho_0 s_0\right\}. \quad (1.12)$$

Substituting (1.3), (1.11) and (1.12) into (1.1), we obtain the following formula for the projective Ricci curvature of $F = \alpha + \beta$:

$$\mathbf{PRic} = {}^{\alpha}\mathbf{Ric}+2\alpha s^m{}_{0;m}-2t_{00}-\alpha^2 t^m{}_m+(n-1)\left\{2\alpha(\rho_m s^m{}_0)-\rho_{0;0}+\rho_0^2\right\}. \quad (1.13)$$

Here, (1.13) is the revised version of (28) in [1].

Now Theorem 1.1 in [1] can be simplified and optimized as follows.

Theorem 1.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$$ {}^{\alpha}\mathbf{Ric} = t^m{}_m\alpha^2 + 2t_{00} + (n-1)[\rho_{0;0} - \rho_0^2], \quad (1.14)$$
$$ s^m{}_{0;m} = -(n-1)(\rho_m s^m{}_0). \quad (1.15)$$

Proof. The proof of the sufficiency of the condition in Theorem 1.1 is immediate. To prove the necessity, let us assume that $\mathbf{PRic} = 0$. Equation (1.13) is equivalent to

$$\Xi_2\alpha^2 + \Xi_1\alpha + \Xi_0 = 0, \quad (1.16)$$

where

$$\Xi_2 = -t^m{}_m, \quad (1.17)$$
$$\Xi_1 = 2s^m{}_{0;m} + 2(n-1)(\rho_m s^m{}_0), \quad (1.18)$$
$$\Xi_0 = {}^{\alpha}\mathbf{Ric} - 2t_{00} + (n-1)[-\rho_{0;0} + \rho_0^2]. \quad (1.19)$$

Replacing y with $-y$ in (1.16), and then adding the obtained equation to (1.16) yields

$$0 = \Xi_2\alpha^2 + \Xi_0, \tag{1.20}$$

$$0 = \Xi_1\alpha. \tag{1.21}$$

By (1.20), we obtain

$$^{\alpha}\mathbf{Ric} = t^m_{\ m}\alpha^2 + 2t_{00} + (n-1)[\rho_{0;0} - \rho_0^2].$$

From (1.21), we obtain

$$s^m_{\ 0;m} = -(n-1)(\rho_m s^m_{\ 0}).$$

□

Further, we can restate Theorem 1.1 as follows.

Theorem 1.2. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$$^{\alpha}\mathbf{Ric} = t^m_{\ m}\alpha^2 + 2t_{00} - (n-1)\left[\frac{r_{0;0} + s_{0;0}}{1-b^2} + \frac{3(r_0+s_0)^2}{(1-b^2)^2}\right], \tag{1.22}$$

$$s^m_{\ 0;m} = \frac{n-1}{1-b^2}(q_0 + t_0). \tag{1.23}$$

We must point out that Theorem 1.1 and Theorem 1.2 here are just the revised versions of Theorem 1.1 and Theorem 1.2 in [1], respectively. We have cancelled condition (iii) in Theorem 1.1 and Theorem 1.2 in [1].

2. Application: projective Ricci flat Randers metrics with isotropic S-curvature

Let F be a Finsler metric on an n-dimensional manifold M. Assume that F is of isotropic S-curvature, i.e., $\mathbf{S} = (n+1)cF$. Then

$$\mathbf{S}_{|m} = (n+1)c_m F,$$

$$\mathbf{PRic} = \mathbf{Ric} + (n-1)c_0 F + (n-1)c^2F^2,$$

where $c_m := c_{x^m}$ and $c_0 := c_m y^m$.

Now, suppose that $F = \alpha + \beta$ is a Randers metric of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then, by Lemma 3.1 in [2], α and β satisfy

$$r_{00} + 2\beta s_0 = 2c(\alpha^2 - \beta^2), \tag{2.1}$$

that is,

$$r_{ij} = -b_i s_j - b_j s_i + 2c(a_{ij} - b_i b_j).$$

We have

$$r_i = -b^2 s_i + 2c(1-b^2)b_i, \tag{2.2}$$
$$q_i = -b^2 t_i + 2c(1-b^2)s_i \tag{2.3}$$

and

$$q_0 + t_0 = (1-b^2)(t_0 + 2cs_0). \tag{2.4}$$

From Theorem 1.1 and Theorem 1.2, we obtain the following result.

Theorem 2.1. *Let $F = \alpha + \beta$ be a Randers metric on a manifold M of dimension n. Assume that F is of isotropic S-curvature, $\mathbf{S} = (n+1)cF$. Then F is a projective Ricci flat metric if and only if α and β satisfy the following equations:*

$${}^{\alpha}\mathbf{Ric} = t^m{}_m \alpha^2 + 2t_{00} - (n-1)\left[s_{0;0} + s_0^2 + 4c^2\alpha^2 + 2c_0\beta\right]; \tag{2.5}$$
$$s^m{}_{0;m} = (n-1)(t_0 + 2cs_0). \tag{2.6}$$

Proof. By (2.2), we have

$$r_0 = -b^2 s_0 + 2c(1-b^2)\beta.$$

Further, we have

$$r_{0;0} = -b^2 s_{0;0} + 2(1-b^2)\left[-s_0^2 - 6c\beta s_0 + c_0\beta + 2c^2\alpha^2 - 6c^2\beta^2\right], \tag{2.7}$$
$$q_0 = -b^2 t_0 + 2c(1-b^2)s_0. \tag{2.8}$$

Then, from (1.22) and (1.23), we get (2.5) and (2.6). □

Actually, Theorem 2.1 is the revised version of Theorem 4.1 in [1]. We have simplified dramatically the conditions for a Randers metric with isotropic S-curvature to be a projective Ricci flat metric.

References

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