

Characterization of the Euler gamma function with the aid of an arbitrary mean

By JANUSZ MATKOWSKI (Zielona Góra)

Abstract. We prove that a continuous function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying the functional equation

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1,$$

is the Euler gamma function iff for some $a > 0$ and a strict and continuous mean $M : (a, \infty)^2 \rightarrow (a, \infty)$, the following inequality holds:

$$f(M(x, y)) f\left(\frac{xy}{M(x, y)}\right) \leq f(x) f(y), \quad x, y \in (a, \infty).$$

Taking for M the geometric mean $G(x, y) = \sqrt{xy}$, we obtain the result of [2] generalizing the classical BOHR–MOLLERUP theorem [1]. For $M = A$, where $A(x, y) = \frac{x+y}{2}$ is the arithmetic mean, the assumed inequality reduces to $f(A(x, y)) f(H(x, y)) \leq f(x) f(y)$ for all $x, y > a$, where H is the harmonic mean, and the result gives a new characterization of the gamma function, involving the arithmetic and harmonic means.

1. Introduction

According to the celebrated result of BOHR and MOLLERUP [1], the Euler gamma function Γ is the only function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying the functional equation

$$f(x+1) = xf(x) \quad \text{for all } x > 0, \quad f(1) = 1, \quad (1)$$

and such that $\log \circ f$ is convex.

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This theorem has been improved in [2], where it is shown that Γ is the only function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying (1) and such that $\log \circ f \circ \exp$ is convex in a vicinity of ∞ . Interpreting this result, note that [2], for a positive real function f defined in an interval $I \subset (0, \infty)$ and continuous at least at one point, the function $\log \circ f \circ \exp$ is convex in the interval $\log(I)$, iff f is *Jensen geometrically convex* in I , i.e., iff

$$f(G(x, y)) \leq G(f(x) f(y)), \quad x, y \in I,$$

where $G(x, y) := \sqrt{xy}$ is the geometric mean.

In the present paper, we show that any two-variable strict and continuous mean defined in a vicinity of ∞ can be used in a characterization of the Euler gamma function. The main results imply that: *a continuous function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying (1) is the Euler gamma function iff there is an $a > 0$ and a strict continuous mean $M : (a, \infty)^2 \rightarrow (a, \infty)$ such that, for all $x, y > a$,*

$$f(M(x, y)) f\left(\frac{xy}{M(x, y)}\right) \leq f(x) f(y).$$

Taking $M = G$, one gets the result of [2]; taking $M = A$, where A is the arithmetic mean, we obtain the following characterization of the gamma function: *a continuous function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying (1) is the Euler gamma function, if and only if there is an $a > 0$ such that*

$$f(A(x, y)) f(H(x, y)) \leq f(x) f(y), \quad x, y \in (a, \infty),$$

where H is the harmonic mean.

2. Auxiliary results

Let $I \subset \mathbb{R}$ be an interval. A function $M : I \times I \rightarrow I$ is called a *bivariable mean in I* if

$$\min(x, y) \leq M(x, y) \leq \max(x, y), \quad x, y \in I, \quad (2)$$

and the mean M is called *strict*, if these inequalities are sharp for all distinct $x, y \in I$.

We shall need the following:

Lemma 1 ([5, Theorem 1]). *Let $M : I^2 \rightarrow I$ and $N : I^2 \rightarrow I$ be continuous means in I . If, for all $x, y \in I$, $x \neq y$,*

$$\max(M(x, y), N(x, y)) - \min(M(x, y), N(x, y)) < \max(x, y) - \min(x, y), \quad (3)$$

then there exists a unique mean $K : I^2 \rightarrow I$ that is invariant with respect to the mean-type mapping $(M, N) : I^2 \rightarrow I^2$, i.e., such that

$$K(M(x, y), N(x, y)) = K(x, y), \quad x, y \in I;$$

moreover, the sequence $((M, N)^n)_{n \in \mathbb{N}}$ of iterates of the mapping (M, N) converges pointwise in I^2 and

$$\lim_{n \rightarrow \infty} (M, N)^n(x, y) = (K(x, y), K(x, y)), \quad (x, y) \in I^2.$$

Remark 1. It is obvious that condition (3) is satisfied if one of the means M or N is strict.

From Lemma 1, we obtain

Lemma 2. *Let $I \subseteq (0, \infty)$ be an interval, $M : I^2 \rightarrow I$ be a (strict) mean, and let $N : I^2 \rightarrow I$ be given by*

$$N(x, y) := \frac{xy}{M(x, y)}, \quad x, y \in I.$$

Then

- (i) *the function N is a (strict) mean in I ;*
- (ii) *the geometric mean G is invariant with respect the mean-type mapping (M, N) , i.e., $G \circ (M, N) = G$;*
- (iii) *for every $n \in \mathbb{N}$, the mapping $(M, N)^n$, the n -th iterate of (M, N) , is a mean-type mapping;*
- (iv) *if M is a continuous and strict mean, then the sequence $((M, N)^n)_{n \in \mathbb{N}}$ of iterates of (M, N) converges pointwise in I^2 and*

$$\lim_{n \rightarrow \infty} (M, N)^n(x, y) = (\sqrt{xy}, \sqrt{xy}), \quad (x, y) \in I^2.$$

PROOF. To prove (i), it is enough to observe that condition (2) is equivalent to

$$\min(x, y) \leq \frac{xy}{M(x, y)} \leq \max(x, y), \quad x, y \in I,$$

and these inequalities are sharp iff so are inequalities (2).

To verify (ii), note that for all $x, y \in I$, we have

$$G \circ (M(x, y), N(x, y)) = \sqrt{M(x, y) \frac{xy}{M(x, y)}} = \sqrt{xy} = G(x, y).$$

Part (iii) is an obvious consequence of the definition of mean. Part (iv) follows from (ii) and Lemma 1. \square

Remark 2. In [4], the mean N such that $G \circ (M, N) = G$ is referred to as the complementary to M with respect to M .

Let us also quote

Lemma 3 ([2, Corollary 1]). *If a function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying (1) is continuous at a point (or bounded above at a point) and there is $a > 0$ such that f is Jensen geometrically convex in the interval (a, ∞) , i.e.,*

$$f(G(x, y)) \leq G(f(x), f(y)), \quad x, y > a,$$

then $f = \Gamma$.

In view of the Bernstein–Doetsch theorem, if f is bounded from above in a neighborhood of a point ([3, KUCZMA, p. 145]), the function $\log \circ f \circ \exp$ is Jensen convex in the interval $\log(I)$ iff

$$f(x^t y^{1-t}) \leq [f(x)]^t [f(y)]^{1-t} \quad \text{for all } x, y \in I, \quad t \in (0, 1),$$

that is, iff f is convex with respect to the family of weighted geometric means I , and referred to as the *geometric convexity* of f ([2]).

3. Main results

Theorem 1. *Let a function $f : (0, \infty) \rightarrow (0, \infty)$ be continuous and such that*

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1.$$

If there is an $a > 0$ and a strict continuous mean $M : (a, \infty)^2 \rightarrow (a, \infty)$ such that

$$f(M(x, y)) f\left(\frac{xy}{M(x, y)}\right) \leq f(x) f(y), \quad x, y \in (a, \infty), \quad (4)$$

then f is the Euler gamma function.

PROOF. In view of Lemma 2, the function $N : (a, \infty)^2 \rightarrow (a, \infty)$ defined by

$$N(x, y) := \frac{xy}{M(x, y)}, \quad x, y \in (a, \infty), \quad (5)$$

is a continuous and strict mean. Since

$$M(x, y)N(x, y) = xy, \quad x, y \in (a, \infty),$$

taking the square root of both sides, we get

$$G \circ (M, N) = G,$$

where $G : (0, \infty)^2 \rightarrow (0, \infty)$, $G(x, y) = \sqrt{xy}$, $x, y > 0$, is the geometric mean. Thus the geometric mean G is invariant with respect to the mean-type mapping $(M, N) : (0, \infty)^2 \rightarrow (0, \infty)^2$.

For every $n \in \mathbb{N}$, denote by (M_n, N_n) the n -th iterate $(M, N)^n$ of the mean-type mapping (M, N) .

From (4), we have

$$f(M(x, y))f(N(x, y)) \leq f(x)f(y), \quad x, y \in (a, \infty).$$

Replacing here x by $M(x, y)$ and y by $N(x, y)$, we have

$$f(M(M(x, y), N(x, y)))f(N(M(x, y), N(x, y))) \leq f(M(x, y))f(N(x, y))$$

for all $x, y \in (a, \infty)$, that is,

$$f(M_2((x, y)))f(N_2((x, y))) \leq f(M(x, y))f(N(x, y)) \leq f(x)f(y)$$

for $x, y \in (a, \infty)$, whence

$$f(M_2((x, y)))f(N_2((x, y))) \leq f(x)f(y), \quad x, y \in (a, \infty).$$

Similarly, applying the induction, we obtain

$$f(M_n(x, y))f(N_n(x, y)) \leq f(x)f(y), \quad n \in \mathbb{N}, x, y \in (a, \infty). \quad (6)$$

The invariance of G with respect to the mean-type mapping (M, N) and Lemma 2 imply that

$$\lim_{n \rightarrow \infty} M_n(x, y) = G(x, y) = \lim_{n \rightarrow \infty} N_n(x, y), \quad x, y \in (a, \infty).$$

Therefore, letting $n \rightarrow \infty$ in (6), and making use of the continuity of f , we obtain

$$[f(G(x, y))]^2 \leq f(x)f(y), \quad x, y \in (a, \infty),$$

or, equivalently,

$$f(G(x, y)) \leq G(f(x)f(y)), \quad x, y \in (a, \infty),$$

which proves that f is Jensen geometrically convex in (a, ∞) . Applying Lemma 3, we conclude that $f = \Gamma$. \square

Theorem 2. For every $a > a_0 := 1.462$ and every strict mean $M : (a, \infty)^2 \rightarrow (a, \infty)$, the Euler gamma function satisfies the inequality

$$\Gamma(M(x, y)) \Gamma\left(\frac{xy}{M(x, y)}\right) \leq \Gamma(x) \Gamma(y), \quad x, y \in (a, \infty).$$

PROOF. Since Γ is logarithmically convex and increasing in (a_0, ∞) , where $a_0 = 1.462$, it is geometrically convex in every interval (a, ∞) with $a > a_0$. Thus, for an arbitrarily fixed $a > a_0$, and for all $x, y \in (a, \infty)$ and $t \in (0, 1)$, we have

$$\Gamma(x^t y^{1-t}) \leq [\Gamma(x)]^t [\Gamma(y)]^{1+t}. \quad (7)$$

Let $M : (a, \infty)^2 \rightarrow (a, \infty)$ be a strict mean. Taking arbitrary $x, y \in (a, \infty)$, $x \neq y$, and putting

$$t = t(x, y) := \frac{\log M(x, y) - \log y}{\log x - \log y},$$

we have

$$0 < t(x, y) < 1, \quad M(x, y) = x^{t(x, y)} y^{1-t(x, y)}$$

and, by (5),

$$N(x, y) = x^{1-t(x, y)} y^{t(x, y)}.$$

Hence, applying (7), we get

$$\Gamma(M(x, y)) = \Gamma\left(x^{t(x, y)} y^{1-t(x, y)}\right) \leq [\Gamma(x)]^{t(x, y)} [\Gamma(y)]^{1-t(x, y)}$$

and

$$\Gamma(N(x, y)) = \Gamma\left(x^{1-t(x, y)} y^{t(x, y)}\right) \leq [\Gamma(x)]^{1-t(x, y)} [\Gamma(y)]^{t(x, y)},$$

whence, multiplying the respective sides of these inequalities, we obtain

$$\begin{aligned} \Gamma(M(x, y)) \Gamma(N(x, y)) &\leq \left([\Gamma(x)]^{t(x, y)} [\Gamma(y)]^{1-t(x, y)}\right) \left([\Gamma(x)]^{1-t(x, y)} [\Gamma(y)]^{t(x, y)}\right) \\ &= \Gamma(x) \Gamma(y), \end{aligned}$$

that is, for all $x, y \in (a, \infty)$, $x \neq y$,

$$\Gamma(M(x, y)) \Gamma\left(\frac{xy}{M(x, y)}\right) \leq \Gamma(x) \Gamma(y).$$

Since this inequality is obvious for all $x, y \in (a, \infty)$ such that $x = y$, the proof is completed. \square

Remark 3. Theorem 1 with $M = G$ reduces to the main result of [2].

Indeed, taking $M = G$ in inequality (4), we get $[f(G(x, y))]^2 \leq f(x)f(y)$ or, equivalently, $f(G(x, y)) \leq G(f(x)f(y))$ for all $x, y \in (a, \infty)$, which means that f is Jensen geometrically convex.

Finally note that the arithmetic and harmonic means can be used for a characterization of the gamma function. Namely, we have the following:

Theorem 3. *A continuous function $f : (0, \infty) \rightarrow (0, \infty)$ satisfying the functional equation*

$$f(x+1) = xf(x), \quad x > 0, \quad f(1) = 1,$$

is the Euler gamma function Γ , if and only if, there is an $a > 0$ such that

$$f(A(x, y))f(H(x, y)) \leq f(x)f(y), \quad x, y \in (a, \infty),$$

where $A(x, y) = \frac{x+y}{2}$ is the arithmetic mean and $H(x, y) = \frac{2xy}{x+y}$ is the harmonic mean.

PROOF. Define $M : (0, \infty)^2 \rightarrow (0, \infty)$ by

$$M(x, y) := A(x, y) = \frac{x+y}{2}, \quad x, y > 0.$$

Then

$$\frac{xy}{M(x, y)} = \frac{2xy}{x+y} = H(x, y), \quad x, y > 0,$$

therefore the result is a consequence of Theorem 1 and Theorem 2. \square

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JANUSZ MATKOWSKI
FACULTY OF MATHEMATICS
COMPUTER SCIENCE AND ECONOMETRICS
UNIVERSITY OF ZIELONA GÓRA
SZAFRANA 4A
PL 65-516, ZIELONA GÓRA
POLAND

E-mail: J.Matkowski@wmie.uz.zgora.pl

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