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A domain containing all zeros of the partial theta function

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Abstract. We consider the partial theta function, i.e., the sum of the bivariate series $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2}z^j$ for $q \in (-1,1)$, $z \in \mathbb{C}$. We show that for any value of the parameter $q \in (0,1)$, all zeros of the function $\theta(q,.)$ belong to the domain $\{\mathrm{Re}\ z < 0, |\mathrm{Im}\ z| < 132\} \cup \{\mathrm{Re}\ z \geq 0, |z| < 18\}$. For $q \in (-1,0)$, all zeros belong to the strip $\{|\mathrm{Im}\ z| < 132\}$.

1. Introduction

We consider the bivariate series $\theta(q,z) := \sum_{j=0}^{\infty} q^{j(j+1)/2}z^j$ for $q \in (-1,1)$, $z \in \mathbb{C}$. We regard q as a parameter, and z as a variable. This series is convergent and defines an entire function called *partial theta function*. The terminology is explained by the resemblance of this formula with the one for the function

$$\Theta^*(q,z) := \sum_{j=-\infty}^{\infty} q^{j(j+1)/2}z^j,$$

because the latter is connected with the *Jacobi theta function*

$$\Theta(q,z) := \sum_{j=-\infty}^{\infty} q^{j^2}z^j,$$

by the formula $\Theta^*(q,z) = \Theta(q^{1/2}, q^{1/2}z)$. The word "partial" reminds that in the formula for θ the summation is performed from 0 to ∞, not from $-\infty$ to ∞.

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Studying the function θ is motivated by its applications in several domains, the most recent of which concerns section-hyperbolic polynomials, i.e., real univariate polynomials of degree ≥ 2 with all roots real negative and such that when their highest-degree monomial is deleted this gives again a polynomial having only real negative roots. The relationship between θ and such polynomials is explained in [13]. It is the spectrum of θ (see Section 2) which is involved in the explanation of this relationship, and having a good insight of the spectrum ultimately means having a deep understanding of the analytic properties of θ. Previous research on section-hyperbolic polynomials was performed in [8] and [14] which in turn was based on classical results of Hardy, Petrovitch and Hutchinson (see [6], [15] and [7]). Such polynomials being real means that the case when the parameter q is real is interesting in its own.

To be more precise, section-hyperbolic polynomials have all coefficients positive, which leads to the study of θ for $q \in (0,1)$. The cases $q \in (0,1)$ and $q \in (-1,0)$ are separated by the trivial case $q = 0$ (with $\theta(0,.) \equiv 1$). For $q \in (0,1)$, $\theta(q,.)$ has infinitely-many negative and no positive real zeros, for $q \in (-1,0)$, it has infinitely-many positive and infinitely-many negative zeros (for any q fixed, see, respectively, part (2) of Remarks 1 and part (1) of Theorem 3). This is a hint why the results in these two cases are formulated in different ways, see Theorem 1.

Other domains in which the partial theta function is used are statistical physics and combinatorics (see [16]), asymptotic analysis (see [2]), Ramanujan-type q-series (see [17]), and the theory of (mock) modular forms (see [4]); see also [1]. In the recent paper [3], several asymptotic results for Jacobi partial and false theta functions are proved. These results stem from Jacobi forms, and they are applied to study the asymptotic expansions of regularized characters and quantum dimensions of the $(1,p)$-singlet algebra modules. Another paper in which asymptotics, modularity and other aspects of partial and false theta functions are considered with regard to representation theory and conformal field theory is [5].

In the present paper, we prove the following theorem:

Theorem 1.

(1) *For* $\operatorname{Re} z \geq 0$ *and for any* $q \in (0,1)$, *the function* $\theta(q,.)$ *has no zeros outside the half-disk* $\{\operatorname{Re} z \geq 0, |z| < 18\}$.

(2) *For* $\operatorname{Re} z < 0$ *and for any* $q \in (0,1)$, *the function* $\theta(q,.)$ *has no zeros outside the half-strip* $\{\operatorname{Re} z < 0, |\operatorname{Im} z| < 132\}$.

(3) *For* $q \in (-1,0)$, *the function* $\theta(q,.)$ *has no zeros outside the strip* $\{|\operatorname{Im} z| < 132\}$.

In order to explain the importance of this theorem, we recall in Section 2 certain facts about the zeros of θ. We also give an example of a value of $q \in (0,1)$ for which $\theta(q,.)$ has a complex conjugate pair of zeros in the right half-plane. The proof of the theorem is given in Section 3.

2. Properties of the function θ

In the present section, we recall some results concerning the function θ. We denote by Γ the *spectrum* of θ, i.e., the set of values of q for which $\theta(q,.)$ has a multiple zero (the notion has been introduced by B. Z. Shapiro in [13]). Suppose first that $q \in (0,1)$. The following results are proved in [9]:

Theorem 2.

(1) *For $q \in (0,1)$, the spectrum Γ consists of countably-many values of q denoted by $0 < \tilde{q}_1 < \tilde{q}_2 < \cdots < \tilde{q}_N < \cdots < 1$ with $\lim_{j\to\infty} \tilde{q}_j = 1^-$.*

(2) *For $\tilde{q}_N \in \Gamma$, the function $\theta(\tilde{q}_N,.)$ has exactly one multiple real zero y_N which is negative, of multiplicity 2 and is the rightmost of its real zeros.*

(3) *For $q \in (\tilde{q}_N, \tilde{q}_{N+1}]$ (we set $\tilde{q}_0 := 0$), the function θ has exactly N complex conjugate pairs of zeros (counted with multiplicity). All its other zeros are real negative.*

Remarks 1. (1) It is proved in [13] that $\tilde{q}_1 = 0.3092\ldots$. Up to 6 decimals, the first 12 spectral numbers equal (see [13])

$$0.309249,\quad 0.516959,\quad 0.630628,\quad 0.701265,\quad 0.749269,\quad 0.783984,$$

$$0.810251,\quad 0.830816,\quad 0.847353,\quad 0.860942,\quad 0.872305,\quad 0.881949.$$

(2) It is shown in [9] that for $q \in (0, \tilde{q}_1)$, all zeros of θ are real, negative and distinct. For all $q \in (0,1)$, it is true that as q increases, the values of the local minima of θ between two negative zeros increase, and the values of its maxima between two negative zeros decrease. It is always the rightmost two negative zeros with a minimum of θ between them that coalesce to form a double zero of θ for $q = \tilde{q}_N$, and then a complex conjugate pair for $q = \tilde{q}_N^+$. For any $q \in (0,1)$, the function $\theta(q,.)$ has infinitely-many negative zeros and no positive ones; $\theta(q,.)$ is increasing for $x > 0$ and tends to ∞ as $x \to \infty$; there is no finite accumulation point for the zeros of $\theta(q,.)$.

(3) In [10], the following asymptotic expansions of $\tilde{q}_N$ and y_N are given:

$$\begin{aligned}\tilde{q}_N &= 1 - (\pi/2N) + (\log N)/8N^2 + O(1/N^2),\\ y_N &= -e^{\pi} e^{-(\log N)/4N + O(1/N)}.\end{aligned} \tag{2.1}$$

The importance of Theorem 1 lies in the fact that while the real zeros of θ remain all negative for any $q \in (0,1)$, no information was known about its complex conjugate pairs. It would be interesting to know whether all complex conjugate pairs remain (for all $q \in (\tilde{q}_1, 1)$) within some compact domain in $\mathbb{C}$ (independent of q).

Lemma 1. *The function $\theta(0.73,.)$ has exactly one complex conjugate pair of zeros inside the open half-disk $\tilde{D} := \{|z| < 3, \operatorname{Re} z > 0\}$.*

Proof. Consider the truncation of $\theta(0.73,.)$ of degree 20 w.r.t. z, i.e., the polynomial $\theta_{20} := \sum_{j=0}^{20} 0.73^{j(j+1)/2} z^j$. One checks numerically (say, using MAPLE) that θ_{20} has zeros $0.03356612894\ldots \pm 2.885381139\ldots i$. These are the only zeros of θ_{20} in the closure of $\tilde{D}$. Numerical check shows that the modulus of the restriction of θ_{20} to the border of $\tilde{D}$ is everywhere larger than 0.016. On the other hand, the sum $\sum_{j=21}^{\infty} |0.73^{j(j+1)/2} z^j|$ is $\leq \sum_{j=21}^{\infty} 0.73^{j(j+1)/2} 3^j < 3 \times 10^{-22}$. By the Rouché theorem, the functions $\theta(0.73,.)$ and θ_{20} have one and the same number of zeros inside the half-disk $\tilde{D}$. $\square$

Suppose now that $q \in (-1,0)$. The following results can be found in [12]:

Theorem 3.

(1) *For any $q \in (-1,0)$, the function $\theta(q,.)$ has infinitely-many negative and infinitely-many positive real zeros.*

(2) *There exists a sequence of values of q (denoted by q_j^*) tending to -1^+ such that $\theta(q_k^*,.)$ has a double real zero y_k^* (the rest of its real zeros being simple). For the remaining values of $q \in (-1,0)$, the function $\theta(q,.)$ has no multiple real zero.*

(3) *For k odd (resp. for k even), one has $y_k^* < 0$, $\theta(q_k^*,.)$ has a local minimum at y_k^* and y_k^* is the rightmost of the real negative zeros of $\theta(q_k^*,.)$ (resp. $y_k^* > 0$, $\theta(q_k^*,.)$ has a local maximum at y_k^* and for k sufficiently large y_k^* is the leftmost but one (second from the left) of the real negative zeros of $\theta(q_k^*,.)$).*

(4) *For k sufficiently large, one has $-1 < q_{k+1}^* < q_k^* < 0$.*

(5) *For k sufficiently large and for $q \in (q_{k+1}^*, q_k^*)$, the function $\theta(q,.)$ has exactly k complex conjugate pairs of zeros counted with multiplicity.*

(6) *One has $q_k^* = 1 - (\pi/8k) + o(1/k)$ and $|y_k^*| \to e^{\pi/2} = 4.810477382\ldots$.*

Remark 1. The approximative values of $-q_k^*$ and y_k^* for $k = 1, \ldots, 8$ are:

$-q_k^*$	0.72713332	0.78374209	0.84160192	0.86125727
y_k^*	−2.991	2.907	−3.621	3.523

$-q_k^*$	0.88795282	0.89790438	0.913191	0.9192012
y_k^*	−3.908	3.823	−4.08	4.002

3. Proof of Theorem 1

As for $q \in (0, \tilde{q}_1]$ all zeros of $\theta(q, .)$ are negative (see Remarks 1), we prove parts (1) and (2) of Theorem 1 only for $q \in (\tilde{q}_1, 1)$.

3.1. The Jacobi theta function. In the proof of Theorem 1, we use the Jacobi theta function $\Theta(q, z) := \sum_{j=-\infty}^{\infty} q^{j^2} z^j$. By the *Jacobi triple product* one has

$$\Theta(q, z^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + z^2 q^{2m-1})(1 + z^{-2} q^{2m-1}),$$

from which for the function $\Theta^*(q, z) := \Theta(q^{1/2}, q^{1/2} z) = \sum_{j=-\infty}^{\infty} q^{j(j+1)/2} z^j$, one deduces the formula

$$\Theta^*(q, z) = \prod_{m=1}^{\infty} (1 - q^m)(1 + z q^m)(1 + q^{m-1}/z). \tag{3.2}$$

Notation 1. We set

$$s_m := 1 + q^{m-1}/z, \qquad t_m := 1 + z q^m, \qquad Q := \prod_{m=1}^{\infty} (1 - q^m),$$
$$P := \prod_{m=1}^{\infty} t_m \quad \text{and} \quad R := \prod_{m=1}^{\infty} s_m.$$

Thus $\Theta^* = QPR$.

3.2. Proof of part (1). We begin with the observation that for any factor s_m (see (3.2) and Notation 1), one has

$$s_m = 1 + \bar{z} q^{m-1}/|z|^2, \quad \text{hence } |s_m| \geq \operatorname{Re} s_m \geq 1, \quad \text{for } \operatorname{Re} z \geq 0.$$

Clearly, for any factor t_m, it is true that $|t_m| \geq \operatorname{Re} t_m \geq 1$ and $|t_m| \geq |z q^m|$ for $\operatorname{Re} z \geq 0$.

Further, in the proof of Theorem 1, we subdivide the interval $(0, 1)$, to which q belongs, into intervals of the form

$$q \in (1 - 1/(n-1),\ 1 - 1/n], \quad n \in \mathbb{N},\, n \geq 3 \quad \text{and} \quad q \in (\tilde{q}_1,\ 1/2]. \tag{3.3}$$

Notation 2. We set $\theta := \Theta^* - G$, where $G := \sum_{j=-\infty}^{-1} q^{j(j+1)/2} z^j$, and $u := 2e^{(\pi^2/6)} = 10.36133664\ldots$.

Remark 2. Clearly, for $|z| > 1$, one has $|G| \leq \sum_{j=1}^{\infty} 1/|z|^j = 1/(|z|-1)$. In particular, for $|z| \geq 18$ (resp. for $|z| \geq u$) one has $|G| \leq 1/17$ (resp. $|G| \leq 1/(u-1)$).

Suppose first that $q \in (1/2, 1)$. We show that for $|z| \geq u$, $\mathrm{Re}\, z \geq 0$, one has $|\Theta^*| > |G|$, from which part (1) of the theorem follows.

Lemma 2. *For $q \leq 1 - 1/n$, $n \in \mathbb{N}$, $n \geq 2$, one has $Q \geq e^{(\pi^2/6)(1-n)}$.*

The lemma is a particular case of Lemma 4 in [11].

Consider the product $P_0 := \prod_{m=1}^{n} t_m$. It follows from $|t_m| \geq |zq^m|$ that $|P_0| \geq |z|^n q^{n(n+1)/2}$. The first line of conditions (3.3) implies

$$\begin{aligned} |P_0| &\geq |z|^n (1 - 1/(n-1))^{n(n+1)/2} = |z|^n (1 - 1/(n-1))^{(n-1)(n+2)/2+1} \\ &\geq |z|^n 4^{-(n+2)/2} (1 - 1/(n-1)) \geq |z|^n 4^{-(n+3)/2} |z|^n 2^{-(n+3)}; \end{aligned} \tag{3.4}$$

we use the inequalities

$$(1 - 1/(n-1))^{n-1} \geq 1/4 \tag{3.5}$$

and $1 - 1/(n-1) \geq 1/2$, which hold true for $n \geq 3$.

Set $P_1 := \prod_{m=n+1}^{\infty} t_m$. Hence we have $|P_1| \geq 1$, $|R| \geq 1$, and

$$|\Theta^*| = Q|P_0||P_1||R| \geq e^{(\pi^2/6)(1-n)} |z|^n 2^{-(n+3)} = \frac{e^{(\pi^2/6)}}{2^3} \times \left(\frac{|z|}{2e^{(\pi^2/6)}} \right)^n .$$

Obviously, for $|z| \geq u$, one has $|z|/(2e^{(\pi^2/6)}) \geq 1$. As $e^{(\pi^2/6)}/2^3 = 0.64\ldots > 1(u-1)$, one obtains the inequalities $|\Theta^*| > 1/(u-1) \geq |G|$, which proves part (1) of the theorem for $q \in (1/2, 1)$ (because $u < 18$).

Suppose that $q \in (\tilde{q}_1, 1/2]$. In this case, for $|z| \geq 18$ and $\mathrm{Re}\, z \geq 0$, one has $|t_1| \geq 18\tilde{q}_1$, $|t_m| \geq 1$, $|s_m| \geq 1$ for $m \in \mathbb{N}$ and (by Lemma 2 with $n = 2$) $Q \geq e^{-\pi^2/6}$, so $|\Theta^*| \geq e^{-\pi^2/6} 18\tilde{q}_1 > 1 > 1/(|z|-1) \geq |G|$.

3.3. Proof of part (2). The proof of part (2) is also based on formula (3.2). We aim to show that for $\mathrm{Re}\, z < 0$ and $|\mathrm{Im}\, z| \geq 132$ one has $|\Theta^*| > |G|$. The following technical result is necessary for the estimations and for the understanding of Figure 1:

Lemma 3. *For $x \in [0, 0.683]$ one has $\ln(1-x) \geq -x - x^2$ with equality only for $x = 0$.*

Proof. We set $\zeta(x) := \ln(1-x) + x + x^2$, so $\zeta(0) = 0$. As $\zeta' = -1/(1-x) + 1 + 2x = x(1-2x)/(1-x)$, which is nonnegative on $[0, 1/2]$ and positive on $(0, 1/2)$, one has $\zeta(x) > 0$ for $x \in (0, 1/2]$. On $(1/2, 1)$, one has $\zeta' < 0$, so ζ is decreasing. As $\lim_{x\to 1^-} \zeta = -\infty$, ζ has a single zero on $[1/2, 1)$. Numerical computation shows that this zero is > 0.683, which proves the lemma. □

To estimate the factor R, we use the following lemma:

Lemma 4. *For $q \in (1 - 1/(n-1), 1 - 1/n]$, $n \geq 2$, and $|\mathrm{Im}\ z| \geq b \geq 1.5$, one has $|R| \geq e^{-n(b+1)/b^2}$.*

Proof. Indeed, the condition $b \geq 1.5$ implies $1/|z| < 0.683$, so one can apply Lemma 3:

$$\ln |R| = \sum_{m=1}^{\infty} \ln |s_m| \geq \sum_{m=1}^{\infty} \ln(1 - (q^{m-1}/|z|)) \geq - \sum_{m=1}^{\infty} ((q^{m-1}/|z|) + (q^{2m-2}/|z|^2))$$
$$= -1/((1-q)|z|) - 1/((1-q^2)|z|^2) > -(|z|+1)/((1-q)|z|^2),$$

which, for $q \leq 1 - 1/n$, is $\geq -n(|z|+1)/|z|^2 \geq -n(b+1)/b^2$. □

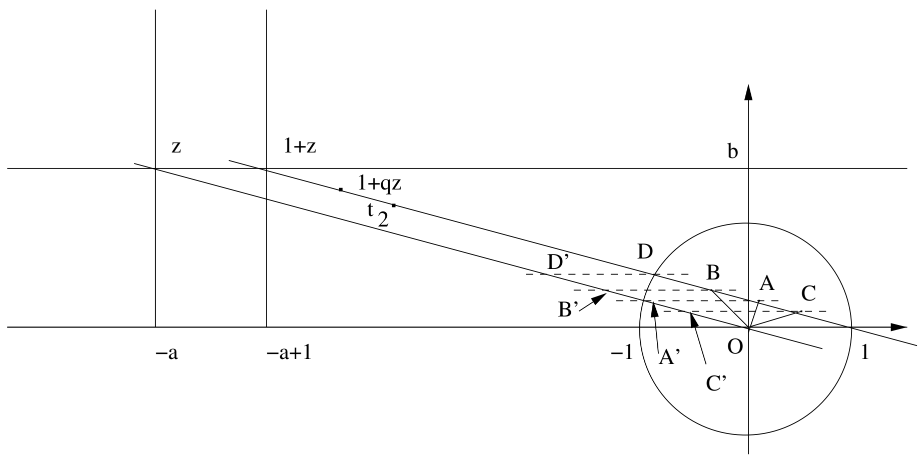

Figure 1. The points $1 + q^m z$.

We identify the complex numbers and the points in $\mathbb{R}^2$ representing them. In Figure 1, we represent the points

$$z = -a + bi \, (a > 0), \quad t_0 := 1 + z = 1 - a + bi,$$
$$t_1 := 1 + qz = 1 - qa + qbi \quad \text{and} \quad t_2 := 1 + q^2 z = 1 - q^2 a + q^2 bi.$$

The last three of them are situated on the straight line $\mathcal{L}$ passing through $1+z$ and $(1,0)$.

In what follows, we assume that $b \geq 0$. As the set of zeros of θ is symmetric w.r.t. the real axis, this leads to no loss of generality.

The point $A \in \mathcal{L}$ is such that the segment OA is orthogonal to $\mathcal{L}$. An easy computation shows that $A = (b^2/(a^2+b^2), ab/(a^2+b^2))$ (one has to use the fact that the vector $(-a,b)$ is collinear with $\mathcal{L}$). The points B and C belong to $\mathcal{L}$. The unit circumference intersects the line $\mathcal{L}$ at $(1,0)$ and D. Denote by $\tilde{\Delta}$ the length $\|[A,(1,0)]\|$ of the segment $[A,(1,0)]$. The points B and C are defined such that

$$\|[A,B]\| = \|[A,C]\| = 0.317\,\tilde{\Delta} \quad \text{and} \quad \|[C,(1,0)]\| = \|[D,B]\| = 0.683\,\tilde{\Delta}. \tag{3.6}$$

The line $\mathcal{L}'$ is parallel to $\mathcal{L}$. It passes through the points z and $O := (0,0)$. The points B', A', C' and D' belong to $\mathcal{L}'$. The lines BB', AA', CC' and DD' are parallel to the x-axis. Hence the segments $[D,B]$, $[C,(1,0)]$, $[D',B']$ and $[C',O]$ are of length $0.683\,\tilde{\Delta}$, while $[B,A]$, $[A,C]$, $[B',A']$ and $[A',C']$ are of length $0.317\,\tilde{\Delta}$.

Our aim is to estimate the product $|P| := \prod_{m=1}^{\infty} |t_m| = \prod_{m=1}^{\infty} |1+q^m z|$.

Notation 3. (1) We set

$$|P| := \tilde{P} P^{\ddagger} P^{\sharp} P^{\dagger}, \tag{3.7}$$

where $\tilde{P}$, $P^{\ddagger}$, $P^{\sharp}$ and $P^{\dagger}$ are the products of the moduli $|t_m|$, for which the point t_m belongs to the segment $[1+z,D]$, $[D,B]$, $[B,C]$ and $[C,(1,0)]$, respectively.

(2) We denote by $t_{m_0}, t_{m_0+1}, \ldots$ the points t_m belonging to the segment $[C,(1,0)]$, and we set $c_{m_0} := C$, $c_{m_0+k} := 1+(C-1)q^k$. Hence $|t_{m_0+k}| \geq |c_{m_0+k}|$ with equality only if $t_{m_0} = C$.

Lemma 5. *For $q \in (1-1/(n-1), 1-1/n]$, $n \geq 2$, one has $P^{\dagger} \geq e^{-1.149489n}$ and $P^{\ddagger} \geq e^{-1.149489n}$ (where $1.149489 = 0.683 + 0.683^2$).*

Proof. We notice first that the segment $[C,(1,0)]$ is of length < 0.683. Hence $|t_{m_0+k}| \geq |c_{m_0+k}| \geq (1-0.683q^k)$, $k = 0,1,\ldots$, so we can use Lemma 3 to get

$$\begin{aligned}\ln P^{\dagger} &\geq \ln(|c_{m_0}||c_{m_0+1}|\cdots) \geq \ln\left(\prod_{m=1}^{\infty}(1-0.683q^{m-1})\right)\\ &\geq -\sum_{m=1}^{\infty}(0.683q^{m-1} + 0.683^2 q^{2m-2})\\ &= -0.683/(1-q) - 0.683^2/(1-q^2) > -1.149489/(1-q).\end{aligned}$$

Thus $P^\dagger \geq e^{-1.149489n}$. Next, the distance between any two consecutive points $1+q^m z$ and $1+q^{m+1}z$ belonging to $[B,D]$ is greater than the distance between any two such points belonging to $[C,(1,0)]$. Denote by $U_1, U_2, \ldots, U_r$ the points t_m belonging to the segment $[B,D]$, where U_1 (resp. U_r) is closest to B (resp. to D). Then $|U_1| \geq |c_{m_0}|$, $|U_2| \geq |c_{m_0+1}|$, …, $|U_r| \geq |c_{m_0+r-1}|$. As $|c_m| \leq |t_m| < 1$ for $m \geq m_0$, one has $P^\ddagger \geq |c_{m_0}||c_{m_0+1}| \cdots |c_{m_0+r-1}| > |c_{m_0}||c_{m_0+1}| \cdots \geq e^{-1.149489n}$. □

Lemma 6. *There are* $\leq \mu_1$ *factors* $|t_m|$ *in* $P^\sharp$, *where*

$$\mu_1 := \ln \lambda_1 / \ln(1/q) + 1 \quad \text{and} \quad \lambda_1 := (0.634 + 0.683)/0.683 = 1.928\ldots,$$

with $\ln \lambda_1 = 0.6566\ldots$. *Hence*

$$P^\sharp \geq (b^2/(a^2+b^2))^{\mu_1/2} = (1/(\beta^2+1))^{\mu_1/2}, \quad \beta := a/b.$$

Proof. Consider the points C', A' and B', and the numbers $zq^{m_1} \in [B', A']$ and $zq^{m_2} \in [A', C']$ closest to B' and C', respectively. The lengths of the segments $[B', A']$, $[A', C']$ and $[C', O]$ (see (3.6)) imply $|zq^{m_1}|/|zq^{m_2}| \leq \lambda_1$, i.e., $m_2 - m_1 \leq \lambda_1/\ln(1/q)$. The number of factors $|t_m|$ in $P^\sharp$ equals $m_2 - m_1 + 1$, from which one deduces the first claim of the lemma. All factors $|t_m|$ in $P^\sharp$ are < 1 and $\geq \|[O,A]\| = b/(a^2+b^2)^{1/2}$, from which the second claim of the lemma follows. □

Remark 3. When $q \in (1/2, 1)$ and (3.3) holds true, then $n/(n-1) \leq 1/q < (n-1)/(n-2)$. As for $x \in (0,1)$, one has $x - x^2/2 < \ln(1+x) < x$ (by the Leibniz criterium for alternating series), one obtains the inequalities

$$\begin{aligned} (2n-3)/2(n-1)^2 \ &= \ 1/(n-1) - 1/2(n-1)^2 \ < \ \ln(1/q) < 1/(n-2), \text{ hence} \\ (\ln \lambda_1)(n-2) + 1 \ < \ \mu_1 \ &\leq \ \mu_1^0 \ := \ (\ln \lambda_1)(2(n-1)^2/(2n-3)) + 1. \end{aligned} \quad (3.8)$$

Lemma 7. *For* $q \in (\tilde{q}_1, 1)$ *and* $b \geq \max(a, 132)$, *one has* $|\Theta^*| > |G|$.

Proof. Prove first the lemma for $q \in (1/2, 1)$, see the first line of (3.3). Set again $P_0 := \prod_{m=1}^n t_m$. Hence

$$|t_m| \geq \operatorname{Im} t_m \quad \text{and} \quad |P_0| \geq \prod_{m=1}^n bq^m = b^n q^{n(n+1)/2} \geq b^n 2^{-(n+3)},$$

see (3.4) and (3.5). With Q, $P^\dagger$, $P^\ddagger$ and $P^\sharp$ defined in Notations 1 and 3, one has

$$|\Theta^*| \geq Q|P_0|P^\ddagger P^\sharp P^\dagger R. \quad (3.9)$$

Indeed, if $b \geq 132$, and if q satisfies the first line of conditions (3.3), then

$$q^m \geq q^n \geq (1-1/(n-1))^{n-1}(1-1/(n-1)) \geq (1/4)(1-1/(n-1)) \geq 1/8, \quad (3.10)$$

and $|bq^m| \geq |bq^n| \geq 132/8 > 1$. This means that all factors $|t_m|$ in $|P_0|$ are > 1. Moreover, some factors $|t_m|$ with $|t_m| > 1$ which are present in $|\Theta^*|$ (i.e., in $\tilde{P}$, see (3.7)) might be missing in the right-hand side of (3.9). Recall that each of the factors $P^\dagger$ and $P^\ddagger$ is minorized by $e^{-1.149489n}$, and that $|R| \geq e^{-(b+1)n/b^2}$, see Lemmas 5 and 4. Recall also that by Lemma 6,

$$P^\sharp \geq (1/(\beta^2+1))^{\mu_1/2} \geq 2^{-\mu_1/2} \quad \text{(because } \beta = a/b \leq 1 \text{)},$$

and that $\mu_1 \leq \mu_1^0$, see (3.8). Hence the right-hand side of (3.9) is

$$\geq H := e^{(\pi^2/6)(1-n)} b^n 2^{-(n+3)} e^{2(-1.149489n)}$$
$$\times\, e^{-(\ln 2)((\ln \lambda_1)2(n-1)^2/(2n-3)+1)/2} e^{-(b+1)n/b^2}.$$

Taking into account that

$$2(n-1)^2/(2n-3) = n - 1/2 + 1/2(2n-3), \quad (3.11)$$

we represent the expression H in the form $e^{K_1 n + K_0}$, where

$$K_1 := -\pi^2/6 + \ln b - \ln 2 - 2.298978 - (\ln 2)(\ln \lambda_1)/2 - (b+1)/b^2,$$
$$K_0 := \pi^2/6 - 3\ln 2 + (\ln 2)(\ln \lambda_1)/4 - (\ln 2)/2 - (\ln 2)(\ln \lambda_1)/4(2n-3).$$

Recall that $n \geq 3$, see the first line of (3.3). The sum K_0 is minimal for $n = 3$. For $b \geq 132$, one has $K_1 > 0$ and $K_0|_{n=3} > 0 > -\ln(b-1)$, which implies the inequalities

$$|\Theta^*| \geq Q|P_0|P^\ddagger P^\sharp P^\dagger R > 1/(b-1) \geq 1/(|z|-1) \geq |G|.$$

Prove the lemma for $q \in (\tilde{q}_1, 1/2]$. One has $Q \geq e^{-\pi^2/6}$ (Lemma 2 with $n=2$), $R \geq e^{-2(b+1)/b^2} \geq e^{-2\times 133/132^2}$ (Lemma 4 with $n = 2$), and $P^\dagger \geq e^{-2.298978}$ (Lemma 5 with $n = 2$).

Lemma 8. *For $q \in (\tilde{q}_1, 1/2]$, the product $P^\ddagger P^\sharp$ contains at most two factors.*

PROOF. Indeed, consider the line $\mathcal{L}'$, see Figure 1. One has $\|[O, C']\| = 0.683\,\tilde{\Delta}$ and $\|[O, D']\| < 4 \times 0.683\,\tilde{\Delta}$, see (3.6) and the lines that follow. Hence if the point zq^m belongs to the segment $[D', C']$, then this is not the case of the point zq^{m-2}, because for $q \in (\tilde{q}_1, 1/2]$ one has $q^{-2} \geq 4$ (but one could possibly have $zq^{m-1} \in [D', C']$). □

All factors $|t_m|$ of the product $P^{\ddagger}P^{\sharp}$ belong to $[b/(a^2+b^2)^{1/2},1)$, therefore by Lemma 8, $P^{\ddagger}P^{\sharp}\geq b^2/(a^2+b^2)$, which, for $a\leq b$, is $\geq 1/2$.

On the other hand, the moduli of the first three factors $|t_m|$ in $\tilde{P}$ are not less than, respectively,

$$132\,\tilde{q_1}-1>39.814,\quad 132\,\tilde{q_1}^2-1>11.619,\quad \text{and}\quad 132\,\tilde{q_1}^3-1>2.902,$$

and the moduli of all other factors $|t_m|$ in $\tilde{P}$ (if any) are ≥ 1, so for $b\geq 132$,

$$\begin{aligned}|\Theta^*|&\geq e^{-\pi^2/6}\times(39.814\times 11.619\times 2.902)\times(1/2)\times e^{-2.298978}\times e^{-2\times 133/132^2}\\&>12.8>|G|.\end{aligned}$$

□

Lemma 9. *For $a\geq b\geq 132$, one has $|\Theta^*|>|G|$.*

Proof. Suppose first that $q\in(1/2,1)$. We define $n\geq 3$ from conditions (3.3). Recall that the number μ_1 was defined in Lemma 6, and that inequalities (3.8) hold true. For $n\geq 3$, equality (3.11) implies

$$\begin{aligned}\mu_1&\leq(\ln\lambda_1)2(n-1)^2/(2n-3)+1\\&=(\ln\lambda_1)(n-1/2+1/2(2n-3))+1<(\ln\lambda_1)n+0.782,\end{aligned}\tag{3.12}$$

because $1/2(2n-3)\leq 1/6$ and $(\ln\lambda_1)(-1/2+1/6)+1=0.7811\ldots$.

Consider a factor t_m from $P_0:=\prod_{m=1}^{n}t_m$. One has

$$|t_m|^2=(aq^m-1)^2+b^2q^{2m}\geq 0.9(a^2+b^2)q^{2m};\tag{3.13}$$

this follows from

$$(a^2+b^2)q^{2m}/10-2aq^m+1=(aq^m-10)^2/10+(b^2q^{2m}-90)/10\geq 0;\tag{3.14}$$

the last inequality results from $b\geq 132$ and (3.10) (remember that if q satisfies conditions (3.3) with $n\geq 3$, then the inequality (3.5) holds true), so $b^2q^{2m}\geq(132/8)^2>90$. Set $A:=(a^2+b^2)^{(n-\mu_1)/2}$. Hence

$$\begin{aligned}&|P_0|\geq(a^2+b^2)^{n/2}q^{n(n+1)/2}(0.9)^{n/2}\geq(a^2+b^2)^{n/2}2^{-(n+3)}(0.9)^{n/2}\quad\text{and}\\&P^{\sharp}\geq b^{\mu_1}/(a^2+b^2)^{\mu_1/2},\quad\text{so}\quad|P_0|P^{\sharp}\geq Ab^{(\ln\lambda_1)(n-2)+1}2^{-(n+3)}(0.9)^{n/2},\end{aligned}$$

(we use inequalities (3.8)). As $(a^2+b^2)^{1/2}\geq 132\sqrt{2}$ and as $n-\mu_1>\omega_1 n-0.782$, $\omega_1:=1-(\ln\lambda_1)$, see (3.12), one obtains the minoration $|P_0|P^{\sharp}\geq e^M$, where

$$M:=(\omega_1 n-0.782)\ln(132\sqrt{2})+(\ln b)((\ln\lambda_1)(n-2)+1)-(n+3)\ln 2+(n/2)\ln 0.9.$$

To estimate $P^{\dagger}$ and $P^{\ddagger}$, we use Lemma 5. As in the proof of Lemma 7, one can minorize the right-hand side of (3.9) by

$$e^{(\pi^2/6)(1-n)}e^{M}e^{2(-1.149489n)}e^{-(b+1)n/b^2}.$$

This expression is of the form $e^{L_1 n+L_0}$, with

$$L_1 = -\pi^2/6+\omega_1\ln(132\sqrt{2})+(\ln b)(\ln\lambda_1)-\ln 2+(\ln 0.9)/2-2.298978-(b+1)/b^2,$$
$$L_0 = \pi^2/6-0.782\ln(132\sqrt{2})+(\ln b)(-2\ln\lambda_1+1)-3\ln 2.$$

For $a \geq b = 132$, one has $|z| \geq 132\sqrt{2}$, also, $L_1 > 0.3044 > 0$ and $L_0 = -6.0491\ldots$. For $a \geq b = 132$, $n \geq 3$, one has

$$L_1 n + L_0 \geq -5.136\ldots > -5.224\ldots = -\ln(132\sqrt{2}-1),$$

i.e., $|\Theta^*| > 1/(|z|-1) \geq |G|$. The functions $L_1 n+L_0$ and $e^{L_1 n+L_0}$ when considered as functions in b (for $n \geq 3$ fixed) are increasing, while the functions $-\ln(b-1)$ and $1/(b-1)$ are decreasing. Therefore, one has $|\Theta^*| > 1/(|z|-1) \geq |G|$ for $a \geq b \geq 132$, $n \geq 3$, from which for $q \in (1/2,1)$ the lemma follows.

Suppose that $q \in (\tilde{q}_1, 1/2]$. One deduces from Lemma 8 (as in the proof of Lemma 7) that $P^{\ddagger}P^{\sharp} \geq b^2/(a^2+b^2)$. On the other hand, consider the factors t_1 and t_2. One can apply to them inequality (3.13) with $m = 1$ and 2. Hence $|t_1| > 1$, $|t_2| > 1$,

$$\tilde{P} \geq |t_1||t_2| > 0.9(a^2+b^2)\tilde{q}_1^3 \quad \text{and} \quad \tilde{P}P^{\ddagger}P^{\sharp} > 0.9b^2\tilde{q}_1^3 > 463.$$

As in the proof of Lemma 7 we show that $R \geq e^{-2\times 133/132^2}$ and $P^{\dagger} \geq e^{-2.298978}$, so finally

$$|\Theta^*| = \tilde{P}P^{\ddagger}P^{\sharp}QP^{\dagger}R > 463e^{-\pi^2/6}e^{-2.298978}e^{-2\times 133/132^2} > 8.8 > |G|, \qquad (3.15)$$

which completes the proof of the lemma. □

Lemmas 7 and 9 together imply that for $q \in (0,1)$ and $z = -a+bi$, $b > 132$, $a > 0$, the function $\theta(q,.)$ has no zeros. Part (2) of Theorem 1 is proved.

3.4. Proof of part (3). We use the same notation and method of proof as for parts (1) and (2). Our aim is to show that outside the strip $\{|\mathrm{Im}\, z| < 132\}$ one has $|\Theta^*| > |G|$, from which part (3) follows.

For $q \in [q_1^*, 0)$ (hence for $q^2 \in (0, (q_1^*)^2]$), the function $\theta(q,.)$ has only real zeros, see Theorem 3 and Remark 1. Therefore, we prove part (3) only for $q \in (-1, q_1^*)$. As $(q_1^*)^2 > 1/2$, one has $q^2 \in (1/2, 1)$.

Notation 4. We set

$$Q^{\triangle} := \prod_{m=1}^{\infty}(1-q^{2m}), \qquad Q^{\bullet} := \prod_{m=1}^{\infty}(1-q^{2m-1}),$$

$$P^{\triangle} := \prod_{m=1}^{\infty} t_{2m}, \qquad P^{\bullet} := \prod_{m=1}^{\infty} t_{2m-1},$$

$$R^{\triangle} := \prod_{m=0}^{\infty} s_{2m}, \qquad R^{\bullet} := \prod_{m=1}^{\infty} s_{2m-1}.$$

As $q<0$, every factor in $Q^{\bullet}$ is >1, so $Q^{\bullet}>1$. To obtain an estimation for $Q^{\triangle}$, we make the same reasoning as the one used in the proofs of parts (1) and (2), but with q^2 instead of q. Thus for $1/2<q^2\leq 1-1/n$, one has $Q^{\triangle}\geq e^{(\pi^2/6)(1-n)}$, and hence $Q=Q^{\bullet}Q^{\triangle}>e^{(\pi^2/6)(1-n)}$ (by analogy with Lemma 2).

For the other factors of Θ^*, we apply the same kind of reasoning. Suppose first that $\operatorname{Re} z\geq 0$. Then:

1) One has $|t_{2m}|\geq 1$ (because $t_{2m}\geq 1$), so $|P^{\triangle}|\geq 1$.

2) One has $|s_{2m-1}|\geq 1$ (because $s_{2m-1}=1+\bar{z}(q^2)^{m-1}/|z|^2$ and $\operatorname{Re} s_{2m-1}\geq 1$), so $|R^{\bullet}|\geq 1$.

3) Recall that $s_m(q,z)=1+q^{m-1}/z$. Hence

$$s_{2m}=1+(q^2)^{m-1}/(z/q)=s_m(q^2,z/q).$$

As $|\operatorname{Im}(z/q)|\geq|\operatorname{Im} z|\geq 132$ and $\operatorname{Re}(z/q)\leq 0$, one can apply Lemma 4 and obtain that, for $q^2\in(1-1/(n-1),1-1/n]$, $n\geq 2$, one has $|R^{\triangle}|\geq e^{-n(b+1)/b^2}$.

4) Recall that

(i) if $|z|>18$, then $|G|<1/17$, see Remark 2, and

(ii) $t_m(q,z)=1+zq^m$.

Hence $t_{2m-1}(q,z)=1+(z/q)(q^2)^m=t_m(q^2,z/q)$. Therefore, by complete analogy with the proof of part (2) of the theorem, with $Q^{\triangle}$, $P^{\bullet}$ and $R^{\triangle}$ playing the roles of Q, P and R, one shows that $|Q^{\triangle}||P^{\bullet}||R^{\triangle}|>8.8>1/17>|G|$ (see (3.15)). Hence

$$|\Theta^*|=|Q^{\triangle}||P^{\bullet}||R^{\triangle}||Q^{\bullet}||P^{\triangle}||R^{\bullet}|>|Q^{\triangle}||P^{\bullet}||R^{\triangle}|>|G|.$$

Suppose now that $\operatorname{Re} z\leq 0$. Then

5) One has $|t_{2m-1}|\geq 1$ (because $\operatorname{Re} t_{2m-1}\geq 1$), so $P^{\bullet}\geq 1$.

6) One has $|s_{2m}|\geq 1$ (because $s_{2m}=1+\bar{z}(q^2)^m/(q|z|^2)$, so $\operatorname{Re} s_{2m}\geq 1$) hence $|R^{\triangle}|\geq 1$.

7) It is true that $s_{2m-1} = 1 + (q^2)^{m-1}/z = s_m(q^2, z)$. Thus one can apply Lemma 4 and obtain that for $q^2 \in (1 - 1/(n-1), 1 - 1/n]$, $n \geq 2$, one has $|R^\bullet| \geq e^{-n(b+1)/b^2}$.

8) As $t_{2m} = (q, z) = 1 + (q^2)^m z = t_m(q^2, z)$, one can perform the proof of part (2) of the theorem, with $Q^\triangle$, $P^\triangle$ and $R^\bullet$ playing the roles of Q, P and R, to show that $|Q^\triangle||P^\triangle||R^\bullet| > 8.8 > 1/17 > |G|$ (see (3.15)). Hence

$$|\Theta^*| = |Q^\triangle||P^\triangle||R^\bullet||Q^\bullet||P^\bullet||R^\triangle| > |Q^\triangle||P^\triangle||R^\bullet| > |G|.$$

References

[1] G. E. Andrews and B. C. Berndt, Ramanujan's Lost Notebook. Part II, *Springer*, *New York*, 2009.

[2] B. C. Berndt and B. Kim, Asymptotic expansions of certain partial theta functions, *Proc. Amer. Math. Soc.* **139** (2011), 3779–3788.

[3] K. Bringmann, A. Folsom and A. Milas, Asymptotic behavior of partial and false theta functions arising from Jacobi forms and regularized characters, *J. Math. Phys.* **58** (2017), 011702, 19 pp.

[4] K. Bringmann, A. Folsom and R. C. Rhoades, Partial theta functions and mock modular forms as q-hypergeometric series, *Ramanujan J.* **29** (2012), 295–310.

[5] T. Creutzig, A. Milas and S. Wood, On regularised quantum dimensions of the singlet vertex operator algebra and false theta functions, *Int. Math. Res. Not. IMRN* **2017** (2017), 1390–1432.

[6] G. H. Hardy, On the zeros of a class of integral functions, *Messenger of Mathematics* **34** (1904), 97–101.

[7] J. I. Hutchinson, On a remarkable class of entire functions, *Trans. Amer. Math. Soc.* **25** (1923), 325–332.

[8] O. M. Katkova, T. Lobova and A. M. Vishnyakova, On power series having sections with only real zeros, *Comput. Methods Funct. Theory* **3** (2003), 425–441.

[9] V. P. Kostov, On the zeros of a partial theta function, *Bull. Sci. Math.* **137** (2013), 1018–1030.

[10] V. P. Kostov, On the double zeros of a partial theta function, *Bull. Sci. Math.* **140** (2016), 98—111.

[11] V.P. Kostov, On the multiple zeros of a partial theta function, *Funct. Anal. Appl.* **50** (2016), 153–156.

[12] V. P. Kostov, On a partial theta function and its spectrum, *Proc. Roy. Soc. Edinburgh Sect. A* **146** (2016), 609–623.

[13] V. P. Kostov and B. Z. Shapiro, Hardy–Petrovitch–Hutchinson's problem and partial theta function, *Duke Math. J.* **162** (2013), 825–861.

[14] I. V. Ostrovskii, On zero distribution of sections and tails of power series, In: Israel Math. Conf. Proc., Vol. **15**, *Bar-Ilan University*, *Ramat Gan*, 2001, 297–310.

[15] M. Petrovitch, Une classe remarquable de séries entières, *Atti del IV Congresso Internazionale dei Matematici, Rome (Ser. 1)* **2** (1908), 36–43.

[16] A. Sokal, The leading root of the partial theta function, *Adv. Math.* **229** (2012), 2603–2621.

[17] S. O. Warnaar, Partial theta functions. I. Beyond the lost notebook, *Proc. London Math. Soc. (3)* **87** (2003), 363–395.

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