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$*$-Ricci solitons on Sasakian 3-manifolds

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Abstract. In the present paper, we study $*$-Ricci solitons and prove that if a Sasakian 3-manifold M admits $*$-Ricci soliton, then it has constant scalar curvature, and the flow vector field V is Killing. Furthermore, the potential vector field V is an infinitesimal automorphism of the contact metric structure on M. Besides, we study $*$-gradient Ricci solitons on Sasakian 3-manifolds. As a consequence of the main theorem, we obtain several results.

1. Introduction

The roots of contact geometry lie in differential equations, as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has multiple connections with other fields of pure mathematics, and with substantial applications in applied areas such as mechanics, optics, phase space of dynamical systems, thermodynamics and control theory.

The Sasakian structure, which is defined on an odd dimensional manifold, is, in a sense, the closest possible analogue of the Kaehler geometry of even dimension. It was introduced by Sasaki [24] in 1965, who considered it as a special kind of contact geometry. A Sasakian structure consists, in particular, of the contact 1-form η and the Riemannian metric g. The differential of η defines a 2-form, which constitutes an analogue of the fundamental form of Kaehler geometry.

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Sasakian and Sasaki–Einstein structures appear in physics in the context of string theory. Moreover, Sasaki–Einstein manifolds in dimension $2k+1$ and Sasakian manifolds with three Sasakian structures in dimension $2k+3$ are related to the MALDACENA conjecture ([21]). It turns out that they are one of very few structures which can serve as a compact factor M in the (anti-de Sitter)$\times M$ background for classical field theories, which, via the Maldacena conjecture, corresponds to the large N limit of certain quantum field theories.

The Ricci tensor S of type (0,2) in a Riemannian manifold is given by

$$S(X,Y) = g(QX,Y) = \text{Trace}\{Z \longrightarrow R(Z,X)Y\}, \tag{1.1}$$

where Q is the Ricci operator.

Definition 1. A contact metric manifold of dimension $n > 2$ is called Einstein if the Ricci tensor S of type (0,2) satisfies the relation

$$S = \lambda g, \tag{1.2}$$

where λ is a constant.

In 1959, TACHIBANA [25] introduced the notion of $*$-Ricci tensor on almost Hermitian manifolds. Later in [14], HAMADA studied $*$-Ricci flat real hypersurfaces in non-flat complex space forms and BLAIR [2] defined $*$-Ricci tensor in contact metric manifolds given by

$$S^*(X,Y) = g(Q^*X,Y) = \text{Trace}\{\phi \circ R(X,\phi Y)\}, \tag{1.3}$$

where Q^* is the $*$-Ricci operator.

We recall several sentences from the Introduction of our paper [20]: In 1982, R. S. HAMILTON [15] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial_t} g_{ij} = -2R_{ij}. \tag{1.4}$$

Ricci solitons are special solutions of the Ricci flow equation (1.4) of the form $g_{ij} = \sigma(t)\psi_t^* g_{ij}$ with the initial condition $g_{ij}(0) = g_{ij}$, where ψ_t are diffeomorphisms of M, and $\sigma(t)$ is the scaling function.

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [4]. On the manifold M, a Ricci soliton is a triple

(g, V, λ) with g, a Riemannian metric, V a vector field, called potential vector field, and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.5}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.5) are interesting and useful in physics and are often referred to as quasi-Einstein ([5], [6]). A Ricci soliton is said to be shrinking, steady and expanding whereas λ is negative, zero and positive, respectively. Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t} g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan [13], who discusses some aspects of it.

In a recent paper, Wang *et al.* [27] studied Ricci solitons on three dimensional η-Einstein almost Kenmotsu manifolds. Also Ricci solitons and gradient Ricci solitons on some kinds of almost contact metric manifolds of dimension three were studied by several authors such as ([4], [7]–[11], [26], [28]) and many others. In this connection, we can also mention the works of ([16], [17], [22]), and the references therein.

Definition 2 ([19]). A Riemannian metric g on M is called a $*$-Ricci soliton, if

$$\pounds_V g + 2S^* + 2\lambda g = 0, \tag{1.6}$$

where λ is a constant.

Definition 3 ([19]). A Riemannian metric g on M is called a $*$-gradient Ricci soliton if

$$\nabla\nabla f = S^* + \lambda g. \tag{1.7}$$

Definition 4. A contact metric manifold of dimension $n > 2$ is called $*$-Einstein if the $*$-Ricci tensor S^* of type (0,2) satisfies the relation

$$S^* = \mu g, \tag{1.8}$$

where μ is a constant.

If a Sasakian 3-manifold M satisfies relation (1.6), then we say that M admits a $*$-Ricci soliton.

In a recent paper, [23] the authors study $*$-Ricci solitons in para-Sasakian manifolds of dimension n. It may be mentioned that a Sasakian manifold is a contact metric manifold with Riemannian metric, whereas a para-Sasakian manifold is a para-contact manifold with pseudo-Riemannian metric. Therefore, these two notions are complectly different.

The present paper focuses on the study of Sasakian 3-manifolds M admitting a $*$-Ricci soliton. More precisely, the following theorems are proved.

Theorem 1.1. *If a Sasakian 3-manifold M admits $*$-Ricci soliton, then it has constant scalar curvature, and the flow vector field V is Killing. Furthermore, V is an infinitesimal automorphism of the contact metric structure on M.*

As a consequence of the Theorem, we obtain several remarks.

Theorem 1.2. *A $*$-gradient Ricci soliton on a Sasakian 3-manifold is $*$-Einstein.*

2. Preliminaries

An odd dimensional smooth manifold M^{2n+1} $(n \geq 1)$ is said to admit an almost contact structure, sometimes called a (ϕ, ξ, η)-structure, if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying ([1], [2]):

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

The first relation and one of the remaining three relations in (2.1) imply the other two relations in (2.1). An almost contact structure is said to be normal if the induced almost complex structure J on $M^n \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right) \tag{2.2}$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$, and f is a smooth function on $M^n \times \mathbb{R}$. Let g be a compatible Riemannian metric with (ϕ, ξ, η) structure, that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y)$$

and

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for all vector fields X, Y tangent to M. Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g).

An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X, Y), \tag{2.5}$$

for all X, Y tangent to M. The 1-form η is then a contact form, and ξ is its characteristic vector field.

We define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ denotes the Lie derivative. Then h is symmetric and satisfies the conditions $h\phi = -\phi h$, $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$\nabla_X \xi = -\phi X - \phi h X \tag{2.6}$$

holds in a contact metric manifold. A normal contact manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

where $X, Y \in \chi(M)$, and ∇ is the Levi–Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is Killing is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However, a 3-dimensional K-contact manifold is Sasakian [18].

Example 2.1 ([3]). The 3-dimensional Heisenberg group nil^3 may be represented as the group of upper triangular matrices $\begin{bmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R}$ and has the standard Sasakian structure

$$\eta = \frac{1}{2}(dz - ydx), \quad \xi = 2\frac{\partial}{\partial z}, \tag{2.8}$$

and the left invariant metric

$$g = \eta \otimes \eta + \frac{1}{4}((dx)^2 + (dy)^2). \tag{2.9}$$

For more detail, we refer to Blair [2].

On the other hand, on a Sasakian manifold the following relations hold:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$\nabla_X \xi = -\phi X, \tag{2.11}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$\begin{aligned}\tilde{R}(X,Y,\phi Z,\phi W) = \tilde{R}(X,Y,Z,W) + \Phi(X,Z)\Phi(Y,W) + \Phi(Y,Z)\Phi(X,W)\\ - g(X,W)g(Y,Z) + g(Y,W)g(X,Z),\end{aligned} \tag{2.14}$$

$$Q\xi = 2n\xi, \tag{2.15}$$

where ∇, R and Q denote, respectively, the Riemannian connection, the curvature tensor of type $(1,3)$, and the $(1,1)$-tensor metrically equivalent to the Ricci tensor S. Also $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$, and Φ is the fundamental 2-form of M defined by $\Phi(X,Y) = g(X,\phi Y)$.

The Ricci tensor S of a 3-dimensional Sasakian manifold is given in [3] by

$$S(X,Y) = \frac{1}{2}(r-2)g(X,Y) + \frac{1}{2}(6-r)\eta(X)\eta(Y), \tag{2.16}$$

where r is the scalar curvature of the manifold.

Now we prove the following Lemma which will be used later.

Lemma 1. *In a Sasakian 3-manifold the $*$-Ricci tensor is given by*

$$S^*(X,Y) = S(X,Y) - g(X,Y) - \eta(X)\eta(Y), \tag{2.17}$$

where S and S^* are Ricci tensor and $*$-Ricci tensor of type $(0,2)$, respectively.

Proof. Let us consider a Sasakian 3-manifold, and $\{e_i\}$, $i = 1,2,3$, be an orthonormal basis of the tangent space at each point of the manifold. Therefore, from (1.3) and using (2.14), we infer

$$\begin{aligned}S^*(Y,Z) &= \sum_{i=1}^{3} \tilde{R}(e_i,Y,\phi Z,\phi e_i)\\ &= \sum_{i=1}^{3}[\tilde{R}(e_i,Y,Z,e_i) + \Phi(e_i,Z)\Phi(e_i,Y) + \Phi(Y,Z)\Phi(e_i,e_i)\\ &\quad - g(e_i,e_i)g(Y,Z) + g(Y,e_i)g(e_i,Z)].\\ &= S(Y,Z) - g(Y,Z) - \eta(Y)\eta(Z).\end{aligned} \tag{2.18}$$

Hence, for a Sasakian 3-manifold the $*$-Ricci tensor is

$$S^*(Y,Z) = S(Y,Z) - g(Y,Z) - \eta(Y)\eta(Z), \tag{2.19}$$

for all vector fields Y and Z on M. This completes the proof. □

From the above Lemma, the $(1,1)$ ∗-Ricci operator Q^* and the ∗-scalar curvature r^* are given by

$$Q^*X = QX - X - \eta(X)\xi, \tag{2.20}$$
$$r^* = r - 4. \tag{2.21}$$

Remark 2.1. The authors in [23] obtain the expression of a ∗-Ricci tensor S^* in a para-Sasakian manifold of dimension n which is different from (2.17).

3. ∗-Ricci solitons on Sasakian 3-manifolds

In view of equation (2.16), the ∗-Ricci tensor is given by

$$S^*(X,Y) = \frac{1}{2}(r-4)\{g(X,Y) - \eta(X)\eta(Y)]. \tag{3.1}$$

Again from the equation of a ∗-Ricci soliton, we have

$$\begin{aligned}(\pounds_V g)(X,Y) &= -2S^*(X,Y) - 2\lambda g(X,Y)\\ &= -(r-4+2\lambda)g(X,Y) + (r-4)\eta(X)\eta(Y).\end{aligned} \tag{3.2}$$

Taking covariant differentiation with respect to Z, we get

$$\begin{aligned}(\nabla_Z \pounds_V g)(X,Y) = &-(Zr)[g(X,Y) - \eta(X)\eta(Y)]\\ &+ (r-4)[-g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)].\end{aligned} \tag{3.3}$$

Following Yano ([29, p. 23]), the following formula

$$(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) - g((\pounds_V \nabla)(X,Z),Y)$$

is well known for any vector fields X, Y, Z on M. As g is parallel with respect to the Levi–Civita connection ∇, the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y),Z) + g((\pounds_V \nabla)(X,Z),Y) \tag{3.4}$$

for any vector fields X, Y, Z. Since $\pounds_V \nabla$ is a symmetric tensor of type $(1,2)$, that is, $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)$, it follows from (3.4) that

$$\begin{aligned}g((\pounds_V \nabla)(X,Y),Z) = &\frac{1}{2}(\nabla_X \pounds_V g)(Y,Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(X,Z)\\ &- \frac{1}{2}(\nabla_Z \pounds_V g)(X,Y).\end{aligned} \tag{3.5}$$

Using (3.3) in (3.5) yields

$$\begin{aligned}
&2g((\pounds_V\nabla)(X,Y),Z)\\
&=(Zr)[g(X,Y)-\eta(X)\eta(Y)]+(r-4)[g(X,\phi Z)\eta(Y)+g(Y,\phi Z)\eta(X)]\\
&\quad-(Yr)[g(X,Z)-\eta(X)\eta(Z)]-(r-4)[g(Z,\phi Y)\eta(X)+g(X,\phi Y)\eta(Z)]\\
&\quad-(Xr)[g(Y,Z)-\eta(Y)\eta(Z)]-(r-4)[g(Y,\phi X)\eta(Z)+g(Z,\phi X)\eta(Y)]. \quad (3.6)
\end{aligned}$$

Removing Z from the above equation (3.6), it follows that

$$\begin{aligned}
&2(\pounds_V\nabla)(X,Y)\\
&=(Dr)[g(X,Y)-\eta(X)\eta(Y)]+(r-4)[-\phi X\eta(Y)-\phi Y\eta(X)]\\
&\quad-(Yr)[X-\eta(X)\xi]-(r-4)[-\phi Y\eta(X)+g(X,\phi Y)\xi]\\
&\quad-(Xr)[Y-\eta(Y)\xi]-(r-4)[g(Y,\phi X)\xi+\phi X\eta(Y)], \quad (3.7)
\end{aligned}$$

where $X\alpha=g(D\alpha,X)$, D denotes the gradient operator with respect to g. Substituting $Y=\xi$ in the foregoing equation and making use of the fact that ξ is Killing (i.e., (ξr)=0), we have

$$(\pounds_V\nabla)(X,\xi)=-(r-4)\phi X. \quad (3.8)$$

Taking the covariant derivative of (3.8) with respect to Y, we infer

$$(\nabla_Y\pounds_V\nabla)(X,\xi)=(\pounds_V\nabla)(X,\phi Y)-(Yr)\phi X-(r-4)[g(X,Y)\xi-\eta(X)Y]. \quad (3.9)$$

Again,

$$(\pounds_V R)(X,Y)Z=(\nabla_X\pounds_V\nabla)(Y,Z)-(\nabla_Y\pounds_V\nabla)(X,Z). \quad (3.10)$$

Therefore, (3.9) and (3.10) yield

$$(\pounds_V R)(X,\xi)\xi=(r-4)[X-\eta(X)\xi]. \quad (3.11)$$

Setting $Y=\xi$ in (3.2), it follows that $(\pounds_V g)(X,\xi)-2\lambda\eta(X)=0$. Lie-differentiating equation (2.4) along V and by virtue of the last equation, we have

$$(\pounds_V\eta)(X)-g(\pounds_V\xi,X)+2\lambda\eta(X)=0. \quad (3.12)$$

From (3.12), we also we obtain $\eta(\pounds_V\xi)=0$ and $(\pounds_V\eta)(\xi)=-2\lambda$, where we have used the Lie-differentiation of $\eta(\xi)=1$ along V. Now, Lie-differentiating the equation $R(X,\xi)\xi = X - \eta(X)\xi$ and taking into account $\eta(\pounds_V \xi) = 0$ and $(\pounds_V \eta)(\xi) = -2\lambda$, we obtain

$$(\pounds_V R)(X,\xi)\xi = 2\lambda\eta(X)\xi. \tag{3.13}$$

Thus from (3.11) and (3.13), we get

$$(r-4)[X - \eta(X)\xi] = 2\lambda\eta(X)\xi. \tag{3.14}$$

Putting $X = \xi$ in the foregoing equation gives

$$\lambda = 0. \tag{3.15}$$

Also, using (3.15) in (3.14), we have $r = 4$, therefore the scalar curvature is constant. Moreover, using $\lambda = 0$ and $r = 4$ in (3.2) yields $(\pounds_V g)(X,Y) = 0$, and hence the potential vector field V is Killing.

As V is Killing, we also conclude that $\pounds_V \xi = 0$. Finally, Lie-differentiating the equation $\eta(X) = g(X,\xi)$ along V, and noting that Lie-derivation commutes with exterior derivation, we conclude $\pounds_V \phi = 0$. Thus, V is an infinitesimal automorphism of the contact metric structure on M. This completes the proof of Theorem 1.1. □

Remark 3.1. For the 3-dimensional Sasakian manifold, an easy computation shows that the ϕ-sectional curvature (sectional curvature with respect to a plane section orthogonal to ξ) is equal to $\frac{r-4}{2}$. Under the hypothesis of Theorem 1.1, we concluded that $r(=4)$ is constant. Hence the ϕ-sectional curvature is zero, which is constant, and thus M is a 3-dimensional Sasakian space-form (see Blair [2, p. 149]).

Remark 3.2. Since the ϕ-sectional curvature is 0, for every point $p \in M$, the mapping $\exp_p : B_\epsilon(0) \subset T_pM \to B_\epsilon(p)$ is an isometry ([12, p. 119]), where $B_\epsilon(p)$ is a normal ball at p.

Remark 3.3. It is evident from the conclusion $\pounds_V \xi = 0$ of Theorem 1.1 that, if g is not of constant curvature, and V pointwise non-collinear with ξ, then the pair (V,ξ) spans a foliation, and ϕV is normal to those leaves. From equation (2.11), we have $\nabla_V \xi = -\phi V$. Using this, and denoting the Riemannian connection induced on a leaf $\sum$ by D, we find $D_V \xi = 0$. Also, as $\nabla_\xi \xi = 0$, Gauss equation implies that $D_\xi \xi = 0$ and ξ is an asymptotic direction. A straightforward computation shows that the sectional curvature of $\sum$ with respect to the plane section spanned by V and $\sum$ vanishes. Hence $\sum$ is intrinsically flat. Furthermore, the conclusion

$\pounds_V \xi = 0$ implies the existence of a function f on M such that $V = f\xi - \frac{1}{2}\phi Df$ (see [2, p. 72]). Since V is Killing, we find that $\xi f = 0$. If V is pointwise collinear with ξ, then it follows that V is a constant multiple of ξ. On the other hand, V cannot be orthogonal to ξ unless $V = 0$. We also observe that $Vf = 0$. Consequently, Df is orthogonal to ξ and V, and hence normal to $\sum$.

4. ∗-gradient Ricci solitons on Sasakian 3-manifolds

Let M be a Sasakian 3-manifold with g as a ∗-gradient Ricci soliton. Then equation (1.7) can be written as

$$\nabla_Y Df = Q^* Y + \lambda Y \tag{4.1}$$

for all vector fields Y in M, where D denotes the gradient operator of g. From (4.1) it follows that

$$R(X,Y)Df = (\nabla_X Q^*)Y - (\nabla_Y Q^*)X, \qquad X, Y \in TM. \tag{4.2}$$

Using (2.10), we have

$$g(R(\xi, X)Df, \xi) = (Xf) - X(\xi f). \tag{4.3}$$

Moreover, in view of (2.20), we infer

$$g((\nabla_\xi Q^*)Y - (\nabla_Y Q^*)\xi, \xi) = 0. \tag{4.4}$$

From (4.3) and (4.4) we get

$$(Xf) = X(\xi f), \tag{4.5}$$

for all vector fields X. Therefore, either $f = 0$ or f is constant. Thus from (4.1) it follows that

$$S^*(X,Y) = -\lambda g(X,Y), \tag{4.6}$$

for all vector fields X and Y. This completes the proof of Theorem 1.2. □

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