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Existence and uniqueness theorem for slant immersion in cosymplectic space forms

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Abstract. In this paper, we have established a general existence and uniqueness theorem for slant immersions in a non-flat cosymplectic space form $\overline{M}(c)$.

1. Introduction

B. Y. Chen [4] has defined slant immersions as a natural generalization of both holomorphic and totally real immersions and since then this topic has attracted the attention of Mathematicians. In 1996, A. Lotta [2] introduced the notion of slant immersion of a Riemannian manifold into an almost contact metric manifold and obtained some useful results. B. Y. Chen and Y. Tazawa [7] have shown that there exist several examples of n-dimensional proper slant submanifolds in the complex Euclidean n-space C^n. On the other hand, Chen and Vrancken [5] have established the existence of n-dimensional proper slant submanifolds in a non-flat complex-space form $\overline{M}^n(4c)$.

Let $\overline{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors (φ, ξ, η, g), where φ is a (1,1) tensor field, ξ a vector field,

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η a 1-form and g the Riemannian metric on $\overline{M}$. These tensors satisfy [8]

$$\begin{cases} \varphi^2 X = -X + \eta(X)\xi, \ \varphi\xi = 0, \ \eta(\xi) = 1, & \eta(\varphi) = 0; \\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), & \eta(X) = g(X,\xi) \end{cases} \tag{1.1}$$

for any $X, Y \in T\overline{M}$. A normal almost contact metric manifold is called a cosymplectic manifold [1] if

$$(\overline{\nabla}_X \varphi)(Y) = 0, \quad \overline{\nabla}_X \xi = 0 \tag{1.2}$$

where $\overline{\nabla}$ denotes the Levi–Civita connection of $\overline{M}$. The curvature tensor $\overline{R}$ of a cosymplectic space form $\overline{M}(c)$ is given by [1]

$$\begin{aligned} \overline{R}(X,Y)Z = \frac{c}{4}\big\{&g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi - g(\varphi X, Z)\varphi Y \\ &+ g(\varphi Y, Z)\varphi X + 2g(X, \varphi Y)\varphi Z\big\}. \end{aligned} \tag{1.3}$$

Let M be an m-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. Let TM be the tangent bundle of M and $T^{\perp}M$ be the set of all vector fields normal to M.

For any $X \in TM$ and $N \in T^{\perp}M$, we write

$$\varphi X = PX + FX \quad \text{and} \quad \varphi N = tN + fN \tag{1.4}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of φX, and tN (resp. fN) denotes the tangential (resp. normal) component of φN.

In what follows, we suppose that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we can consider the orthogonal direct decomposition $TM = D \oplus \{\xi\}$.

For each non zero X tangent to M at x such that X is not proportional to ξ_x, we denote by $\theta(X)$ the Wirtinger angle of X, that is, the angle between φX and $T_x M$.

The submanifold M is called slant if the Wirtinger angle $\theta(X)$ is a constant, which is independent of the choice of $x \in M$ and $X \in T_x M - \{\xi_x\}$ [2]. The Wirtinger angle θ of a slant immersion is called the slant angle of the immersion. Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

Let ∇ be the Riemannian connection on M. Then the Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1.5}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{1.6}$$

where h and A_N are the second fundamental forms related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{1.7}$$

and $\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M, for $X, Y \in TM$ and $N \in T^{\perp}M$. Let the curvature tensor corresponding to $\overline{\nabla}$, ∇ and $\nabla^{\perp}$ be denoted by $\overline{R}$, R, and $R^{\perp}$ respectively. The Gauss, Codazzi and Ricci equations are, respectively

$$\begin{aligned}\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) &- g(h(X, W), h(Y, Z)) \\ &+ g(h(X, Z), h(Y, W))\end{aligned} \tag{1.8}$$

$$\big[\overline{R}(X, Y)Z\big]^{\perp} = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z) \tag{1.9}$$

and

$$\overline{R}(X, Y, N_1, N_2) = R^{\perp}(X, Y, N_1, N_2) - g([A_{N_1}, A_{N_2}]X, Y) \tag{1.10}$$

where $\big[\overline{R}(X, Y)Z\big]^{\perp}$ denotes the normal component of $\overline{R}(X, Y)Z$ and $\big(\overline{\nabla}_X h\big)(Y, Z)$ is given by

$$\big(\overline{\nabla}_X h\big)(Y, Z) = \nabla^{\perp}_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

If P is the endomorphism defined by (1.4), then

$$g(PX, Y) + g(X, PY) = 0 \tag{1.11}$$

Thus P^2 which is simply denoted by Q, is self adjoint.

We define

$$(\nabla_X P)Y = \nabla_X(PY) - P(\nabla_X Y) \tag{1.12}$$

and

$$(\nabla_X F)Y = \nabla^{\perp}_X FY - F(\nabla_X Y) \tag{1.13}$$

for any $X, Y \in TM$.

Using Gauss and Weingarten formulae and equations (1.2) and (1.10), we have

$$(\nabla_X P)Y = A_{FY}X + th(X,Y) \tag{1.14}$$

$$\nabla^{\perp}_X(FY) - F(\nabla_X Y) = fh(X,Y) - h(X,PY) \tag{1.15}$$

for any $X, Y \in TM$.

For each $X \in TM$, we put

$$X^* = \frac{FX}{\sin\theta}. \tag{1.16}$$

We define the symmetric bilinear TM-valued form ρ on M by

$$\rho(X,Y) = th(X,Y). \tag{1.17}$$

Moreover, from (1.2), we have

$$\rho(X,\xi) = 0. \tag{1.18}$$

Also, from (1.4), (1.16) and (1.17), we get

$$\varphi\rho(X,Y) = P\rho(X,Y) + \sin\theta\rho^*(X,Y) \tag{1.19}$$

Using (1.4) and (1.17), we can write

$$\varphi h(X,Y) = \rho(X,Y) + \sigma^*(X,Y) \tag{1.20}$$

where σ is a symmetric bilinear D-valued form on M. Operating φ on (1.20) and using (1.19) together with (1.4), we find

$$-h(X,Y) = P\rho(X,Y) + \sin\theta\rho^*(X,Y) + t\sigma^*(X,Y) + f\sigma^*(X,Y). \tag{1.21}$$

On comparing the tangential and normal parts, we get

(i) $$P\rho(X,Y) + t\sigma^*(X,Y) = 0$$

and

$$\text{(ii)} \qquad -h(X,Y) = \sin\theta\rho^*(X,Y) + \frac{fF\sigma(X,Y)}{\sin\theta}.$$

Also,

$$\varphi^2\sigma(X,Y) = -\sigma(X,Y) = P^2\sigma(X,Y) + FP\sigma(X,Y) + tF\sigma(X,Y) + fF\sigma(X,Y).$$

Comparing the tangential and normal parts, we get

$$\text{(iii)} \qquad -\sigma(X,Y) = P^2\sigma(X,Y) + tF\sigma(X,Y)$$

and

$$\text{(iv)} \qquad FP\sigma(X,Y) + fF\sigma(X,Y) = 0.$$

Now, from (i), we have

$$P\rho(X,Y) = -t\sigma^*(X,Y) = -\frac{tF\sigma(X,Y)}{\sin\theta}.$$

Using (iii) in the above equation, we get

$$\begin{aligned} -\sigma(X,Y) &= P^2\sigma(X,Y) - P\rho(X,Y)\sin\theta \\ &= -\sigma(X,Y)\cos^2\theta - P\rho(X,Y)\sin\theta \end{aligned}$$

which gives that

$$\sigma(X,Y) = \csc\theta P\rho(X,Y). \tag{1.22}$$

Now, from (ii) and (iv), we have

$$-h(X,Y) = \sin\theta\rho^*(X,Y) - \frac{FP\sigma(X,Y)}{\sin\theta}$$

and using (1.22), we get

$$-h(X,Y) = \sin\theta\rho^*(X,Y) - \frac{FP^2\rho(X,Y)}{\sin^2\theta} = \sin\theta\rho^*(X,Y) + \frac{\rho^*(X,Y)\cos^2\theta}{\sin\theta}$$

which gives that

$$h(X,Y) = -\csc\theta\rho^*(X,Y). \tag{1.23}$$

From (1.19) and (1.23), we have

$$h(X,Y)=\csc^2\theta(P\rho(X,Y)-\varphi\rho(X,Y)). \tag{1.24}$$

On the other hand, from (1.14), we have

$$g((\nabla_X P)Y,Z)=g(\rho(X,Y),Z)-g(\rho(X,Z),Y). \tag{1.25}$$

Next, from (1.3), we get

$$\begin{aligned}\overline{R}(X,Y,Z,W)&=\frac{c}{4}\big(g(\varphi Y,\varphi Z)g(X,W)-g(\varphi X,\varphi Z)g(Y,W)\\&\quad+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)+g(\varphi Y,Z)g(\varphi X,W)\\&\quad-g(\varphi X,Z)g(\varphi Y,W)+2g(X,\varphi Y)g(\varphi Z,W)\big)\end{aligned} \tag{1.26}$$

for all $X,Y,Z,W\in TM$. Using (1.1), (1.4) and (1.8) in (1.26), we find

$$\begin{aligned}&R(X,Y,Z,W)-g(h(X,W),h(Y,Z))+g((X,Z),h(Y,W))\\&=\frac{c}{4}(g(Y,Z)g(X,W)-g(X,W)\eta(Y)\eta(Z)-g(X,Z)g(Y,W)\\&\quad+g(Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\\&\quad+g(PY,Z)g(PX,W)-g(PX,Z)g(PY,W)\\&\quad+2g(X,PY)g(PZ,W))\end{aligned} \tag{1.27}$$

which, in view of (1.23), gives

$$\begin{aligned}R(X,Y,Z,W)&=\csc^2\theta(g(\rho(X,W),\rho(Y,Z))-g(\rho(X,Z),\rho(Y,W)))\\&\quad+\frac{c}{4}(g(,Y,Z)g(X,W)-g(X,W)\eta(Y)\eta(Z)-g(X,Z)g(Y,W)\\&\quad+g(Y,W)\eta(X)\eta(Z)+g(X,Z)\eta(Y)\eta(W)-g(Y,Z)\eta(X)\eta(W)\\&\quad+g(PY,Z)g(PX,W)-g(PX,Z)g(PY,W)\\&\quad+2g(X,PY)g(PZ,W)).\end{aligned} \tag{1.28}$$

Taking normal part of equation (1.3), we get

$$\big(\overline{R}(X,Y)Z\big)^{\perp}=\frac{c}{4}(g(PY,Z)FX-g(PX,Z)FY+2g(X,PY)FZ). \tag{1.29}$$

Moreover,

$$\begin{aligned}\big(\overline{\nabla}_X h\big)(Y,Z)&=-\csc^2\theta(\csc^2\theta FP\rho(X,\rho(Y,Z))\\&\quad+\csc^2\theta F\rho(X,P\rho(Y,Z))+F((\nabla_X\rho)(Y,Z)).\end{aligned} \tag{1.30}$$

Using (1.29) and (1.30) in Codazzi equation, we get

$$\begin{aligned}(\nabla_X\rho)(Y,Z) &+ \csc^2\theta\{P\rho(X,\rho(Y,Z)+\rho(X,P\rho(Y,Z))\} \\ &+ \frac{c}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi)+g(X,PY)(Z-\eta(Z)\xi\} \\ &= (\nabla_Y\rho)(X,Z)+\csc^2\theta\{P\rho(Y,\rho(X,Z))+\rho(Y,P\rho(X,Z))\} \\ &+ \frac{c}{4}\sin^2\theta\{g(Y,PZ)(X-\eta(X)\xi)+g(Y,PX)(Z-\eta(Z)\xi)\}.\end{aligned} \tag{1.31}$$

2. Existence theorem for slant immersions into cosymplectic space form

In this section we establish the existence theorem for slant immersions into cosymplectic space form. We need the following:

Theorem A ([5]). *Consider a manifold S with complete connection $\overline{D}$ having parallel torsion and curvature tensors. Let M be a simply connected manifold and E be a vector bundle with connection $\overline{D}$ over M having the algebraic structure $(\overline{R},\overline{T})$ of S. Let $F: TM \to E$ be a vector bundle homomorphism satisfying the equations*

$$\overline{D}_V(F(W)) - \overline{D}_W(F(V)) - F([V,W]) = \overline{T}(F(V),F(W))$$

$$\overline{D}_V\overline{D}_W U - \overline{D}_W\overline{D}_V U - \overline{D}_{[V,W]}U = \overline{R}(F(V),F(W))U$$

for any sections V, W of TM and U of E. Then there exists a smooth map $f: M \to S$ and a parallel bundle isomorphism $\overline{\Phi}: E \to f^*TS$ preserving T and R such that $df = \overline{\Phi} oF$. If S is simply connected, then f is unique up to affine diffeomorphisms of S.

Now, we prove:

Theorem 2.1 (Existence). *Let c and θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M be a simply connected $(m+1)$-dimensional Riemannian manifold with metric tensor g. Suppose that there exist a unit global vector field ξ on M, an endomorphism P of the tangent bundle TM and a symmetric bilinear TM-valued form ρ on M such that*

$$P(\xi) = 0, \quad g(\rho(X,Y),\xi)) = 0, \quad \nabla_X\xi = 0 \tag{2.1}$$

$$P^2 = -\cos^2\theta(X - \eta(X)\xi) \tag{2.2}$$

$$g(PX, Y) + g(X, PY) = 0 \tag{2.3}$$

$$\rho(X, \xi) = 0 \tag{2.4}$$

$$g((\nabla_X P)Y, Z) = g(\rho(X, Y), Z) - g(\rho(X, Z), Y) \tag{2.5}$$

$$\begin{aligned} R(X,Y,Z,W) &= \cos^2\theta(g(\rho(X,W),\rho(Y,Z)) - g(\rho(X,Z),\rho(Y,W))) \\ &+ \frac{c}{4}\{g(Y,Z)g(X,W) - g(X,W)\eta(Y)\eta(Z) - g(X,Z)g(Y,W) \\ &+ g(Y,W)\eta(X)\eta(Z) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W) \\ &+ g(PY,Z)g(PX,W) - g(PX,Z)g(PY,W) \\ &+ 2g(X,PY)g(PZ,W)\} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &(\nabla_X\rho)(Y,Z) + \csc^2\theta\{P\rho(X,\rho(Y,Z)) + \rho(X,P\rho(Y,Z))\} \\ &\quad + \frac{c}{4}\sin^2\theta\{g(X,PZ)(Y-\eta(Y)\xi) + g(X,PY)(Z-\eta(Z)\xi)\} \\ &= (\nabla_Y\rho)(X,Z) + \csc^2\theta\{P\rho(Y,\rho(X,Z)) + \rho(Y,P\rho(X,Z))\} \\ &\quad + \frac{c}{4}\sin^2\theta\{g(Y,PZ)(X-\eta(X)\xi) + g(Y,PX)(Z-\eta(Z)\xi)\} \end{aligned} \tag{2.7}$$

for all $X, Y, Z \in TM$, where η is a dual 1-form of ξ. Then, there exists a θ-slant immersion from M into $\overline{M}^{2m+1}(c)$ whose second fundamental form h is given by

$$h(X,Y) = \csc^2\theta(P\rho(X,Y) - \varphi\rho(X,Y)). \tag{2.8}$$

Proof. Let all the conditions hold. Consider the Whitney sum $TM \oplus D$ and identify $(X, 0)$ with X for each $X \in TM$. We also identify $(0, Z)$ by Z^* for each Z in D and let us denote $\hat{\xi} = (\xi, 0)$. Let $\hat{g}$ be the product metric on $TM \oplus D$. Hence, if we denote by $\hat{\eta}$ the dual 1-form of $\hat{\xi}$, then we can write $\hat{\eta}(X, Z) = \eta(X)$, for all $X \in TM$ and $Z \in D$.

We denote the endomorphism on $TM \oplus D$ by $\hat{\varphi}$, which is defined as

$$\hat{\varphi}(X,0) = (PX, \sin\theta(X - \eta(X)\xi)), \quad \hat{\varphi}(0,Z) = (-\sin\theta Z, -PZ). \tag{2.9}$$

Then, $\hat{\varphi}^2(X,0) = -(X,0) + \hat{\eta}(X,0)\hat{\xi}$, $\hat{\varphi}^2(0,Z) = -(0,Z)$ and $\hat{\varphi}^2(X,Z) = -(X,Z) + \hat{\eta}(X,Z)\hat{\xi}$, for all $X \in TM$ and $Z \in D$. Clearly, $(\hat{\varphi}, \hat{g}, \hat{\xi}, \hat{\eta})$ is an almost contact structure on $TM \oplus D$.

Now, for $X \in TM$ and $Z \in D$, we define A, h, $\nabla^{\perp}$ as

$$A_{Z^*}X = \csc\theta((\nabla_X P)Z - \rho(X,Z)) \tag{2.10}$$

$$h(X,Y) = -\csc\theta\rho^*(X,Y) \tag{2.11}$$

$$\begin{aligned}\nabla^{\perp}_X Z^* = {} & (\nabla_X Z - \eta(\nabla_X Z)\xi)^* \\ & + \csc^2\theta((P\rho(X,Z)^* + \rho^*(X,PZ)).\end{aligned} \tag{2.12}$$

We can check that each A is an endomorphism on TM; h is a $(D)^*$-valued symmetric bilinear form on TM and $\nabla^{\perp}$ is a metric connection of the vector bundle $(D)^*$ over M.

Let $\hat{\nabla}$ denote the connection on $TM \oplus D$ induced from equations (2.10)–(2.12). Then, from (2.1), (2.2) and (2.9), we have $(\hat{\nabla}_{(X,0)}\varphi)(Y,0){=}0$, $(\hat{\nabla}_{(X,0)}\varphi)(0,Y) = 0$ and $\hat{\nabla}_{(X,0)}(\xi,0) = 0$, for all X, Y tangent to M.

Let $R^{\perp}$ denote the curvature tensor associated with the connection $\nabla^{\perp}$ on $(D)^*$, i.e, $R^{\perp}(X,Y)Z^* = \nabla^{\perp}_X\nabla^{\perp}_Y Z^* - \nabla^{\perp}_Y\nabla^{\perp}_X Z^* - \nabla^{\perp}_{[X,Y]}Z^*$ for $X, Y \in TM$ and $Z \in D$.

Then, using (1.28), (2.1), (2.5) and (2.12), we get

$$\begin{aligned}R^{\perp}(X,Y)Z^* = {} & (R(X,Y)Z - \eta(R(X,Y)Z)\xi)^* \\ & + \Big[\frac{c}{4}P\big\{g(Y,PZ)X + 2g(Y,PX)Z - g(X,PZ)Y\big\} \\ & + \frac{c}{4}\big\{g(Y,P^2Z)(X - \eta(X)\xi) + 2g(Y,PX)PZ - g(X,P^2Z)(Y - \eta(Y)\xi)\big\} \\ & + \csc^2\theta\big\{(\nabla_X P)\rho(Y,Z) - (\nabla_Y P)\rho(X,Z) - \eta(\nabla_X(P\rho(Y,Z)))\xi \\ & + \eta(\nabla_Y(P\rho(X,Z)))\xi + \rho(Y,(\nabla_X P)Z) - \rho(X,(\nabla_Y P)Z) \\ & - \eta((\nabla_X\rho)(Y,PZ))\xi + \eta((\nabla_Y\rho)(X,PZ))\xi\big\}\Big]^*.\end{aligned} \tag{2.13}$$

On the other hand

$$\begin{aligned}\sin^2\theta g\big([A_{Z^*}, A_{W^*}]X,Y\big) = {} & g((\nabla_Y P)Z,(\nabla_X P)W) - g(\rho(Y,Z),(\nabla_X P)W) \\ & - g((\nabla_Y P)Z,\rho(X,W)) + g(\rho(Y,Z),\rho(X,W)\big) \\ & - g\big((\nabla_Y P)W,(\nabla_X P)Z) + g(\rho(Y,W),(\nabla_X P)Z) \\ & + g((\nabla_Y P)W,\rho(X,Z)) - g(\rho(Y,W),\rho(X,Z)\big).\end{aligned} \tag{2.14}$$

From (1.11), we have

$$g(\rho(Y,Z),PW) + g(P\rho(Y,Z),W) = 0. \tag{2.15}$$

The covariant derivative of the above equation with respect to X gives

$$g(\rho(Y,Z),(\nabla_X P)W) + g((\nabla_X P)\rho(Y,Z),W) = 0. \tag{2.16}$$

Moreover, by virtue of (1.25), we have

$$\begin{aligned} g((\nabla_Y P)W,(\nabla_X P)Z) &= g(\rho(Y,W),(\nabla_X P)Z) \\ &\quad - g(\rho(Y,(\nabla_X P)Z),W). \end{aligned} \tag{2.17}$$

Using (2.17), (2.16), (2.14) and (2.13), we get

$$\begin{aligned} g(R^{\perp}(X,Y)Z^*,W^*) - g([A_{Z^*},A_{W^*}]X,Y) &= \frac{c}{4}\{\sin^2\theta g(Y,Z), g(X,W) \\ - \sin^2\theta g(X,Z)g(Y,W) + 2g(Y,PX)g(PZ,W)\}. \end{aligned} \tag{2.18}$$

Equations (1.3), (2.2), (2.3) and (2.18) imply that $(M, A, \nabla^{\perp})$ satisfies the equation of Ricci for an $(m+1)$-dimensional θ-slant submanifold in $\overline{M}^{2m+1}(c)$. Also, (1.28) and (1.31) imply that (M,h) satisfies the equations of Gauss and Codazzi for a θ-slant submanifold. Hence, the vector bundle $TM \oplus D$ over M equipped with the product metric $\hat{g}$, the shape operator A, the second fundamental form h and the connections $\nabla^{\perp}$ and $\hat{\nabla}$ satisfy the structure equations of $(m+1)$-dimensional θ-slant submanifold in $\overline{M}^{2m+1}(c)$. Therefore, from Theorem A, we know that there exists a θ-slant isometric immersion of M in $\overline{M}^{2m+1}(c)$ with h as its second fundamental form, A as its shape operator and $\nabla^{\perp}$ as its normal connection. $\square$

3. Uniqueness theorem for slant immersions into cosymplectic space form

In this section we establish uniqueness theorem for slant immersions into cosymplectic space form. We prove:

Theorem 3.1 (Uniqueness). *Let $x^1, x^2 : M \to \overline{M}(c)$ be two slant immersions with slant angle θ $(0 < \theta \leq \frac{\pi}{2})$, of a connected Riemannian manifold M of dimension $(m+1)$ into the cosymplectic space-form $\overline{M}^{2m+1}(c)$. Let h^1, h^2 denote the second fundamental forms of x^1 and x^2*

respectively. Let there be a vector field $\overline{\xi}$ *on* M *such that* $x^1_{*p}(\overline{\xi_p}) = \xi_{x^i(p)}$, *for* $i = 1, 2$ *and* $p \in M$, *and*

$$g(h^1(X, Y), \varphi x^1_* Z) = g(h^2(X, Y), \varphi x^2_* Z) \tag{3.1}$$

for all vector fields X, Y, Z *tangent to* M. *Suppose also that we have one of the following conditions:*

(i) $\theta = \frac{\pi}{2}$

(ii) *there exists a point* p *of* M *such that* $P_1 = P_2$

(iii) $c \neq 0$

Then there exists an isometry Ψ *of* $\overline{M}^{2m+1}(c)$ *such that* $x^1 = \Psi o x^2$.

Proof. Let p be any point of M. Assume that $x^1(p) = x^2(p)$ and $x^1_*(p) = x^2_*(p)$. We can take a geodesic γ through the point $p = \gamma(0)$ and let us define $\gamma_1 = x^1(\gamma)$ and $\gamma_2 = x^2(\gamma)$. To prove the theorem it is sufficient to show that γ_1 and γ_2 coincide. We already know that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1'(0) = \gamma_2'(0)$. Let $E_1, E_2, \ldots, E_m, \overline{\xi}$ be any orthonormal frame along γ. We can define frames along γ_1 and γ_2 as follows:

Take $A_i = x^1_*(E_i)$, $B_i = x^2_*(E_i)$, $A_{n+i} = (x^1_*(E_i))^*$, $B_{n+i} = (x^2_*(E_i))^*$, where $X^* = \frac{FX}{\sin\theta}$ for $X \in D$.

From (2.11), we have $h^i = -\csc\theta(\rho^i)^*$, for $i = 1, 2$ and from (3.1), we have $g(\rho^1(X, Y), x^1_* Z) = g(\rho^2(X, Y), x^2_* Z)$. Since $x^1_*(p) = x^2_*(p)$ and Z is arbitrary, we have $\rho^1 = \rho^2$. Now, we show that $P_1 = P_2$.

If (i) is satisfied, it is obvious that $P_1 = 0$, $P_2 = 0$ and hence $P_1 = P_2$.

If (ii) is satisfied, it follows from (2.5) that $(\nabla_X(P_1 - P_2))Y = 0$. Since we have $P_1 = P_2$ at a point p, therefore we have $P_1 = P_2$ everywhere.

Now, suppose that (iii) is satisfied and assume that $P_1 \neq P_2$ and (i) and (ii) are not satisfied. Then in this case we show that $P_1 = -P_2$. From (2.6), we find

$$\begin{aligned} g(P_1X, W)g(P_1Y, Z) - g(P_1X, Z)g(P_1Y, W) + 2g(P_1Z, W)g(P_1Y, X) \\ = g(P_2X, W)g(P_2Y, Z) - g(P_2X, Z)g(P_2Y, W) \\ + 2g(P_2Z, W)g(P_2Y, X). \end{aligned} \tag{3.2}$$

Putting $X = W$, $Y = Z$ and using the skew symmetric property of P_1 and P_2, equation (3.2) reduces to

$$g(P_1Y, X)^2 = g(P_2Y, X)^2. \tag{3.3}$$

Next, put $e_1 = X$ and $e_2 = P_1X$ and suppose that P_2e_1 has a component in the direction of a vector e_3 which is orthogonal to both e_1 and e_2. Then a contradiction follows from (3.3) which states that $g(P_2e_1, e_3)^2 = g(P_1e_1, e_3)^2 = g(e_2, e_3)^2 = 0$. Thus, by applying (2.2) and (2.3), we get $P_1\nu = \pm P_2\nu$ for every tangent vector ν.

Now choose a basis $\{e_1, \ldots, e_m, e_{m+1}\}$ of the tangent space T_pM at a point p. Then there exists a number $\varepsilon_i \in \{-1, 1\}$ such that $P_1e_i = \varepsilon_i P_2 e_i$. So, we also have $\pm P_1(e_i + e_j) = P_2(e_i + e_j) = \varepsilon_i P_1 e_i + \varepsilon_j P_1 e_j$. Hence, the above formula shows that all ε_i have to be equal. Thus, either $P_1\nu = P_2\nu$ or $P_1\nu = -P_2\nu$, for all $\nu \in T_pM$. Since M is connected, this implies that we have either $P_1 = P_2$ or $P_1 = -P_2$ in case (iii).

Let us now assume that we have two immersions with $P_1 = -P_2$. From (2.5), it follows that

$$g((\nabla_X P_1)Y, Z) = g(\rho^1(X, Y), Z) - g(\rho^1(X, Z), Y)$$

and

$$g((\nabla_X P_2)Y, Z) = -g((\nabla_X P_1)Y, Z) = g(\rho^2(X, Y), Z) - g(\rho^2(X, Z), Y).$$

Since $\rho^1 = \rho^2 = \rho$, we get

$$g(\rho(X, Y), Z) = g(\rho(X, Z), Y). \tag{3.4}$$

Writing the equation (2.7) for both the immersions and using the fact that $P_1 = -P_2 = P$, we find

$$\begin{aligned} &P\rho(X, \rho(Y, Z)) + \rho(X, P\rho(Y, Z)) - P\rho(Y, \rho(X, Z)) - \rho(Y, P\rho(X, Z)) \\ &\quad + \frac{c}{4}\sin^4\theta\{g(X, PZ)(Y - \eta(Y)\xi) - g(Y, PZ)(X - \eta(X)\xi) \\ &\quad + 2g(X, PY)(Z - \eta(Z)\xi)\} = 0. \end{aligned} \tag{3.5}$$

Taking inner product with a vector W in (3.5) and using (3.4), we get

$$- g(\rho(X, PW), \rho(Y, Z)) + g(\rho(Y, PW), \rho(X, Z)) + g(\rho(X, W), P\rho(Y, Z))$$

$$
\begin{aligned}
&- g(\rho(Y,W), P\rho(X,Z)) + \frac{c}{4}\sin^4\theta\{g(X,PZ)g(Y,W) \\
&- g(X,PZ)\eta(Y)\eta(W) - g(Y,PZ)g(X,W) + g(Y,PZ)\eta(X)\eta(W) \\
&+ 2g(X,PY)g(Z,W) - 2g(X,PY)\eta(Z)\eta(W)\} = 0. \qquad (3.6)
\end{aligned}
$$

If ρ vanishes identically at a point, then a contradiction follows from (3.6) since $c \neq 0$.

Now, we take a fixed point p of M and look at the function f defined on the set of all unit tangent vectors UM_p at the point p by $f(\nu) = g(\rho(\nu,\nu),\nu)$. Since UM_p is compact there exists a vector u such that f attains an absolute maximum at the vector u. Let w be a unit vector orthogonal to u. Then the function $f(t) = f(g(t))$, where $g(t) = (\cos t)u + (\sin t)w$, satisfies $f'(0) = 0$ and $f''(0) \leq 0$. The first condition implies that $g(\rho(u,u),w) = 0$ whereas the second condition implies $g(\rho(u,w),w) \leq \frac{1}{2}g(\rho(u,u),u)$.

Now, using the total symmetry of ρ, it follows that we can choose an orthonormal basis $e_1 = u, \dots, e_m, e_{m+1}$ such that

$$\rho(e_1,e_1) = \lambda_1 e_1, \qquad \rho(e_1,e_i) = \lambda_i e_i \qquad (3.7)$$

with $i > 1$ and $\lambda_i \leq \frac{1}{2}\lambda_1$.

Since ρ is not identically zero, it follows from total symmetry of (3.4) that $\lambda_l > 0$. Using (3.4) and (3.7) in (3.6) with $X = Z = W = e_1$ and $Y = e_i$, we find

$$\left(\lambda_i^2 + \lambda_i\lambda_1 + 3\frac{c}{4}\sin^4\theta\right) g(Pe_1, e_i) = 0. \qquad (3.8)$$

Now, we show that Pe_1 is an eigen vector of $\rho(e_1, .)$. For this we put $X = Z = e_1$, $W = e_j$ and $Y = e_i$ in (3.6) for i, $j > 1$. Then, we get

$$(\lambda_i^2 - \lambda_i\lambda_1 + \lambda_i\lambda_j)g(Pe_j, e_i) + \lambda_1 g(\rho(e_i,e_j), Pe_1) = 0. \qquad (3.9)$$

Interchanging the indices i and j in (4.9), we obtain

$$(\lambda_j^2 - \lambda_j\lambda_1 + \lambda_i\lambda_j)g(Pe_i, e_j) + \lambda_1 g(\rho(e_i,e_j), Pe_1) = 0. \qquad (3.10)$$

Combining (3.9) and (3.10), we find

$$(\lambda_i + \lambda_j)(\lambda_1 - \lambda_i - \lambda_j)g(Pe_j, e_i) = 0. \qquad (3.11)$$

Since $\lambda_1 \geq 2\lambda_i$, therefore $\lambda_1 - \lambda_i - \lambda_j = 0$ only if $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$. Now, if we put $X = W = e_1$, $Z = e_j$ and $Y = e_i$ for i, $j > 1$ in (3.6), we find that

$$\begin{aligned} g(\rho(e_1, Pe_1), \rho(e_i, e_j)) - \lambda_j g(\rho(e_i, e_j), Pe_1) + \lambda_i\lambda_j g(e_i, Pe_j) \\ +\lambda_1 g(\rho(e_i, e_j), Pe_1) + \frac{c}{4}\sin^4\theta g(e_i, Pe_j) = 0. \end{aligned} \tag{3.12}$$

Interchanging the indices i and j in (3.12), we get

$$\begin{aligned} g(\rho(e_1, Pe_1), \rho(e_i, e_j)) - \lambda_i g(\rho(e_i, e_j), Pe_1) + \lambda_i\lambda_j g(e_j, Pe_i) \\ +\lambda_1 g(\rho(e_i, e_j), Pe_1) + \frac{c}{4}\sin^4\theta g(e_j, Pe_i) = 0. \end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13), we find

$$(\lambda_i - \lambda_j) g(\rho(e_i, e_j), Pe_1) + 2\lambda_i\lambda_j g(e_i, Pe_j) + \frac{c}{2}\sin^4\theta\, g(e_i, Pe_j) = 0. \tag{3.14}$$

Now, we summarise the previous equations in the following way. First, taking $i = j$ in (4.9), we get

$$g(\rho(e_i, e_i), Pe_1) = 0. \tag{3.15}$$

Hence, we have $g(\rho(\nu, \nu), Pe_1) = 0$ if ν is an eigenvector of $\rho(e_1, .)$. Moreover, the symmetry of ρ implies that $g(\rho(e_i, e_j), Pe_1) = 0$, whenever $\lambda_i = \lambda_j$.

We consider four different cases:

(1) $\lambda_i + \lambda_j \neq 0$, but not $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$. In this case (3.11) implies $g(Pe_i, e_j) = 0$.

(2) $\lambda_i + \lambda_j = 0$, and $\lambda_i \neq 0$. In this case, (3.9) implies $g(\rho(e_i, e_i), Pe_1) = \lambda_i g(Pe_j, e_i)$. Substituting this into (3.14), we obtain $\frac{c}{2}\sin^4\theta g(e_i, Pe_j) = 0$ which yields $g(Pe_j, e_i) = 0$.

(3) $\lambda_i + \lambda_j = 0$, and $\lambda_i = 0$ or equivalently $\lambda_i = \lambda_j = 0$. In this case it follows from (3.14) that $g(e_i, Pe_j) = 0$.

(4) $\lambda_i = \lambda_j = \frac{1}{2}\lambda_1$.

Therefore, if $e_{i_1}, \ldots, e_{i_k}$ are eigenvectors belonging to an eigenvalue different from $\frac{1}{2}\lambda_1$, then each Pe_{i_l}, $l = 1, \ldots, k$, can only have a component in the direction of e_1, say $Pe_{i_l} = \mu_l e_1$. Thus, $\mu_l Pe_1 = -\cos^2\theta e_{i_l}$. Consequently, either $k = 1$ or there does not exist an eigenvector with eigenvalue different from $\frac{1}{2}\lambda_1$. If $k = 1$, then clearly Pe_1 is an eigenvector.

In the latter case $\rho(e_1,.)$ restricted to the space $e_1^{\perp}$ only has one eigenvalue, namely $\frac{1}{2}\lambda_1$. Since Pe_1 is always orthogonal to e_1, Pe_1 is also an eigenvector in this case. Hence Pe_1 is always an eigenvector of $\rho(e_1,.)$.

We may assume that e_2 is in the direction of Pe_1. Then it follows that $\rho(e_1, Pe_1) = \lambda_2 Pe_1$, where λ_2 satisfies the equation

$$\lambda_2^2 + \lambda_2\lambda_1 + \frac{3c}{4}\sin^4\theta = 0 \tag{3.16}$$

by virtue of (3.8).

If we choose $X = Z = e_1$, $W = Pe_1$ and $Y = e_i$ for $i > 2$ in (3.6), then

$$\lambda_1 g(\rho(e_i, Pe_1), Pe_1) = \lambda_1 g(\rho(Pe_1, Pe_1), e_i) = 0.$$

Thus, $\rho(Pe_1, Pe_1) = \lambda_2\cos^2\theta e_1$. Putting $X = Z = Pe_1$ and $Y = e_1$ in (3.6), we find

$$-\lambda_2^2 - \lambda_2\lambda_1 + \frac{3c}{4}\sin^4\theta = 0. \tag{3.17}$$

From (3.16) and (3.17), we get $\frac{3c}{4}\sin^4\theta = 0$, which is a contradiction since $c \neq 0$. Therefore $P_1 = P_2$. Now it is easy to check from (2.10)–(2.12) that $g(\gamma_1', A_k) = g(\gamma_2', B_k)$ and $g(\hat{\nabla}_\gamma A_k, A_l) = g(\hat{\nabla}_\gamma B_k, B_l)$ for $k, l = 1, \ldots 2m$, such that by [9, Proposition 3], $\gamma_1 = \gamma_2$. $\square$

4. Applications and examples

Let $\phi = \phi(x)$ and $\phi_i = \phi_i(x)$, $i = 1, 2, 3$, be four functions defined on an open interval containing 0. Let c and θ be two constants with $0 < \theta \leq \frac{\pi}{2}$ and M be a simply connected open neighbourhood of the origin $(0,0,0) \in \Re^3$. Suppose

$$f(x) = \exp\int \phi_3(x)dx \tag{4.1}$$

$$\eta = dz \tag{4.2}$$

$$g = \eta\otimes\eta + dx\otimes dx + f^2(x)dy\otimes dy \tag{4.3}$$

and

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{f(x)}\frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z}. \tag{4.4}$$

Then, we can verify that $\{e_1, e_2, \xi\}$ is a local orthonormal frame field of TM and η is the dual 1-form of structure vector field ξ. Also, we can obtain

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = 0, && \nabla_{e_1} e_3 = 0,\\
&\nabla_{e_2} e_1 = \phi_3 e_2, && \nabla_{e_2} e_2 = -\phi_3 e_1, && \nabla_{e_2} e_3 = 0,\\
&\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0.
\end{aligned} \tag{4.5}$$

We define the tensor φ as

$$\varphi e_1 = e_2, \quad \varphi e_2 = -e_1 \quad \text{and} \quad \varphi e_3 = \varphi\xi = 0,$$

and a symmetric bilinear TM-valued form ρ on M as follows:

$$\begin{gathered}
\rho(e_1, e_1) = \phi e_1 + \phi_1 e_2, \quad \rho(e_1, e_2) = \phi_1 e_1 + \phi_2 e_2,\\
\rho(e_2, e_2) = \phi_2 e_1 - \phi_1 e_2
\end{gathered} \tag{4.6}$$

$$\rho(e_1, \xi) = 0, \quad \rho(e_2, \xi) = 0, \quad \rho(\xi, \xi) = 0. \tag{4.7}$$

It is easy to check that $(M, \varphi, \xi, \eta, g)$ is an almost contact metric manifold. Now, to calculate the value of $(\nabla_X \varphi)Y$, we choose vector fields $X = a_1 e_1 + a_2 e_2 + a_3 e_3$ and $Y = b_1 e_1 + b_2 e_2 + b_3 e_3$, where a_1, a_2, a_3, b_1, b_2, b_3, are real valued functions. Then, $(\nabla_X \varphi)Y = 0$, for any $X, Y \in TM$.

If we take $P = \cos\theta\varphi$, then it will satisfy equation (2.3). Similarly, we can show that $(M, \varphi, \xi, \eta, g, \rho)$ satisfy the equations (2.1)–(2.4) and (2.5).

On the other hand, it can be proved that M satisfy the conditions (2.6) and (2.7) if we have the following equations:

$$\phi_3' + \phi_3^2 = -\csc^2\theta(\phi\phi_2 - 2\phi_1^2 - \phi_2^2) - \frac{c}{4}(1 + 3\cos^2\theta) \tag{4.8}$$

$$\phi_1' = -3\phi_1\phi_3 + \cot\theta\csc\theta(\phi_2^2 + \phi_2\phi) + 3\frac{c}{4}\sin^2\theta\cos\theta \tag{4.9}$$

$$\phi_1' = -3\phi_1\phi_3 + \cot\theta\csc\theta(\phi_2^2 + \phi_2\phi) - 3\frac{c}{4}\sin^2\theta\cos\theta \tag{4.10}$$

and

$$\phi_2' = \phi\phi_3 - 2\phi_3\phi_2 - \cot\theta\csc\theta(\phi_1\phi + \phi_2\phi_1). \tag{4.11}$$

Equations (4.9) and (4.10) are satisfied together if and only if $3\frac{c}{4}\sin^2\theta \cdot \cos\theta = 0$. Since $0 < \theta \leq \frac{\pi}{2}$ implies $\sin^2\theta \neq 0$, therefore either $c = 0$ or $\theta = \frac{\pi}{2}$.

Using Theorem 2.1, we obtain:

Theorem 4.1. *Let $\phi = \phi(x)$ be a function defined on an open interval containing 0 and d_1, d_2, d_3, c, θ be the five constants with $0 < \theta \leq \frac{\pi}{2}$. Consider the system of first order ordinary differential equations*

$$\begin{aligned}
y_1' &= -3y_1y_3 + \cot\theta\csc\theta(y_2^2 + y_2\phi)\\
y_2' &= \phi y_3 - 2y_3y_2 - \cot\theta\csc\theta(y_1\phi + y_2y_1)\\
y_3' &= -\csc^2\theta(\phi y_2 - 2y_1^2 - y_2^2) - \frac{c}{4}(1 + 3\cos^2\theta) - y_3^2,
\end{aligned}$$

with the initial conditions $y_1(0) = d_1$, $y_2(0) = d_2$, $y_3(0) = d_3$. Let ϕ_1, ϕ_2 and ϕ_3 be the components of the unique solution of this differentiable system on some open interval containing 0. Let M be a simply connected open neighbourhood of the origin $(0,0,0) \in \Re^3$ endowed with the metric given by (4.1)–(4.4) and let ρ be the TM-valued form defined by (4.6)–(4.7). Then,

(i) *if $c = 0$, there exists a θ-slant isometric immersion of M in $\overline{M}^5(c)$ whose second fundamental form is given by*

$$h(X,Y) = \cos^2\theta(P\rho(X,Y) - \varphi\rho(X,Y)),$$

(ii) *if $\theta = \frac{\pi}{2}$, then there exists an anti-invariant immersion whose second fundamental form is given by*

$$h(X,Y) = -\varphi\rho(X,Y).$$

We can obtain from Theorem 4.1 the following existence result for three dimensional slant submanifolds with prescribed scalar curvature or mean curvature.

Corollary 4.2. *For a given constant θ with $0 < \theta < \frac{\pi}{2}$ and a given function $F_1 = F_1(x)$ (resp. $F_2 = F_2(x)$), there exist infinitely many three-dimensional θ slant submanifolds in $\overline{M}^5(c)$ with F_1 (resp. F_2) as the prescribed scalar curvature (resp. mean curvature) function for $c = 0$.*

Corollary 4.2 follows from Theorem 4.1 by putting $d_2 \neq 0$ and choosing ϕ to be a function satisfying $F_1 \sin^2\theta = 2(2\phi_1^2 + \phi_2^2 - \phi\phi_2)$. On the other hand, it is enough to put $\phi = 3F_2 \sin\theta - \phi_2$ in order to obtain F_2 as the prescribed mean curvature function.

Clearly, we can obtain a similar result for anti-invariant submanifolds in $\overline{M}^5(c)$ for a given constant c.

We prove:

Proposition 4.3. *For each given constant θ with $0 < \theta < \frac{\pi}{2}$, there exist three-dimensional θ slant submanifolds in $\overline{M}^5(-4)$ with non zero constant mean curvature and constant negative scalar curvature.*

Proof. For a given constant θ with $0 < \theta < \frac{\pi}{2}$, we can choose two nonzero constants β and γ such that

$$\beta^2 + \gamma^2 = 4\cos^2\theta. \tag{4.12}$$

Let a, b, c be constants defined by

$$a = -\sin^2\theta\sec^3\theta\left(\frac{\beta^3}{4} - \frac{3}{2}\beta\cos^2\theta + \frac{6}{\beta}\cos^4\theta\right), \tag{4.13}$$

$$b = \gamma\sin^2\theta\sec^3\theta\left(\frac{\beta^2}{4} - \cos^2\theta\right), \tag{4.14}$$

$$c = -\beta\sin^2\theta\sec^3\theta\left(\frac{\beta^2}{4} - \frac{1}{2}\cos^2\theta + \frac{1}{2}\gamma^2\right). \tag{4.15}$$

Let M be $\Re^3$ and define the 1-form $\eta = dz$. We consider on M the metric g given by

$$\begin{aligned} g = \eta\otimes\eta + (dx\otimes dx - \beta e^{-\gamma x}(dx\otimes dy + dy\otimes dx)) \\ + (\beta^2+\gamma^2)e^{-2\gamma x} dy\otimes dy. \end{aligned} \tag{4.16}$$

If we take

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{1}{\gamma}\left(\beta\frac{\partial}{\partial x} + e^{\gamma x}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \xi = \frac{\partial}{\partial z} \tag{4.17}$$

then $\{e_1, e_2, \xi\}$ form an orthonormal frame field for (M, g) and η is the dual 1-form of ξ. It is easy to see that

$$\begin{array}{lll} \nabla_{e_1} e_1 = \beta e_2, & \nabla_{e_1} e_2 = -\beta e_1, & \nabla_{e_1} \xi = 0, \\ \nabla_{e_2} e_1 = -\gamma e_2, & \nabla_{e_2} e_2 = -\gamma e_1, & \nabla_{e_2} \xi = 0, \\ \nabla_{\xi} e_1 = 0, & \nabla_{\xi} e_2 = 0, & \nabla_{\xi} \xi = 0. \end{array} \tag{4.18}$$

Equations (4.12) and (4.18) imply that the scalar curvature of M is given by

$$\tau = -2(\beta^2 + \gamma^2) < 0.$$

We define a TM-valued symmetric bilinear form ρ on M by:

$$\begin{aligned} \rho(e_1, e_1) = a e_1 + b e_2, \quad \rho(e_1, e_2) = b e_1 + c e_2, \\ \rho(e_2, e_2) = c e_1 - b e_2, \end{aligned} \tag{4.19}$$

$$\rho(e_1, \xi) = 0, \quad \rho(e_2, \xi) = 0, \quad \rho(\xi, \xi) = 0. \tag{4.20}$$

Let P be the endomorphism on TM defined by $Pe_1 = \cos\theta e_2$, $Pe_2 = -\cos\theta e_1$ and $P\xi = 0$. Then using (4.12)–(4.20) and after a long computation, we find that $(M, \xi, \eta, g, P, \rho)$ satisfies the equations (2.1)–(2.7) stated in Theorem 2.1 for $c = -4$. Therefore, Theorem 2.1 implies that there exists a θ-slant immersion of (M, g) into $\overline{M}^5(-4)$ whose second fundamental form is given by $h(X, Y) = \csc^2\theta(P\rho(X, Y) - \varphi\rho(X, Y))$. Since θ, a, b and c are constants such that $0 < \theta < \frac{\pi}{2}$, and $\beta \neq 0$ the proper slant submanifolds have nonzero constant mean curvature and constant negative scalar curvature. $\square$

References

[1] A. Cabra, A. Ianus and Gh. Pitis, Extrinsic spheres and parallel submanifolds in cosymplectic manifolds, *Math. J. Toyama Univ.* **17** (1994), 31–53.

[2] A. Lotta, Slant submanifolds in contact geometry, *Bull. Math. Soc. Roumanie* (1996), 183–198.

[3] A. Lotta, Three dimensional slant submanifolds of K-contact manifolds, *Balkan J. Geom. Appl.* **3**(1) (1998), 37–51.

[4] B. Y. Chen, Geometry of slant submanifolds, *Katholieke Universiteit Leuven*, 1990.

[5] B. Y. Chen and L. Vrancken, Existence and uniqueness theorem for slant immersions and its applications, *Result. Math.* **31** (1997), 28–39.

[6] B. Y. Chen and L. Vrancken, Addendum to: Existence and uniqueness theorem for slant immersions and its applications, *Result. Math.* **39** (2001), 18–22.

[7] B. Y. Chen and Y. Tazawa, Slant submanifolds in complex Euclidean spaces, *Tokyo J. Math.* **14** (1991), 101–120.

[8] D. E. Blair, Contact Manifolds in Riemannian Geometry, Lect. Notes in Math., Vol. 509, *Springer Verlag, Berlin – New York*, 1976.

[9] H. Reckziege, On the problem whether the image of a given differentiable map into a Riemannian manifold is contained in a submanifold with parallel second fundamental form, *J. Reine. Angew. Math.* **325** (1981), 87–104.

[10] J. H. Eschenburg and R. Tribuzy, Existence and uniqueness of maps into affine homogeneous spaces, *Rend. Sem. Mat. Univ. Padova* **89** (1993), 11–18.

[11] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Existence and uniqueness theorem for slant immersion in Sasakian manifolds, *Publicationes Mathematicae Debrecen* **58** (2001), 559–574.

[12] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Slant submanifolds in Sasakian manifolds, *Glasgow Math. J.* **42** (2000), 125–138.

[13] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez, Structure on a slant submanifold of a contact manifold, *Indian J. Pure and Appl. Math.* **31**(7) (2000), 857–864.

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