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Compatible mappings and a common fixed point theorem of Chang type

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In this paper, using a condition of "compatibility" between the mappings under discussion, due to G. Jungck [5], we generalize a common fixed point theorem of S. S. Chang [1] in complete metric spaces. This theorem extends well-known results of Lj.B. Ciric [2], K.M. Das and K.V. Naik [3], G. Jungck [4] and S. Sessa [8].

1. Two equivalent conditions

Following S. S. Chang [1], let $A : [0, +\infty) \to [0, +\infty)$ be a real-valued function such that the following conditions $(A_1), (A_2)$ or $(A_1), (A_3)$ hold:

(A_1) $A(t)$ is nondecreasing and right-continuous,

(A_2) for any real number $q \geq 0$, there exists a suitable real number $t(q)$ such that

(a) $t(q)$ is the "upper bound" of the set $A_q = \{t \geq 0 : t \leq q + A(t)\}$,

(b) $\lim\limits_{n\to\infty} A^n(t(q)) = 0$,

(A_3) for any $t > 0$,

(c) $A(t) < t$,

(d) $\lim\limits_{t\to\infty} (t - A(t)) = \infty$.

Remark 1. S. S. Chang [1] says that $t(q)$ is the "upper bound" of the set A_q. Here we assume that $t(q)$ stands for the "least upper bound" of the set A_q, i.e., $t(q) = \sup A_q$. Presumably, S. S. Chang [1] intended to assume this and, of course, we have $t > q + A(t)$ for any $t > t(q)$.

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In accordance with J. Matkowski [6], B. A. Meade and S. P. Singh [7] and Lemma 2 (i) of S. S. Chang [1], we point out the following simple results:

Lemma 1. *If $A(t)$ is nondecreasing, then for any $t > 0$ we have $A(t)<t$ if $\lim_{n\to\infty} A^n(t) = 0$.*

Lemma 2. *If A is right-continuous and has the property (c), then we have $\lim_{n\to\infty} A^n(t) = 0$ for any $t > 0$.*

Remark 2. We note that $A_q \neq \emptyset$ since q lies in A_q for any $q \geq 0$. If $q > 0$, then $t(q) > 0$ since $t(q) \geq q > 0$. If $q = 0$ and $A(t) < t$ for any $t > 0$, then we have $A_0 = \{0\}$.

Now we give the following result:

Theorem 1. *If A satisfies the condition (A_1), then the conditions (A_2) and (A_3) are equivalent.*

Proof. Suppose that (A_2) holds. By property (a), then for any $q > 0$ there exists a real number $t(q) > 0$ such that $t > q + A(t)$ for any $t > t(q)$, which means that property (d) of (A_3) holds.

Since $t(q) \geq q$ for any $q > 0$ and A is nondecreasing and so is A^n, using property (b), we have

$$0 \leq \lim_{n\to\infty} A^n(q) \leq \lim_{n\to\infty} A^n(t(q)) = 0\,,$$

i.e., $\lim_{n\to\infty} A^n(q) = 0$. This implies $A(q) < q$ for any $q > 0$ by Lemma 1. Therefore, the property (c) of (A_3) holds.

Conversely, we must show that the properties (c) and (d) of (A_3) imply the properties (a) and (b) of (A_2). Indeed, it suffices to assume that $t(q) = 0$ if $q = 0$, and in this case the property (b) is clearly satisfied. Since the property (d) holds, if $q > 0$ then there exists certainly a real number q^* such that $t - A(t) > q$ for any $t > q^*$. Assume that $t(q)$ is the infimum of such q^*'s. If there exists some $\bar{t} \in A_q$ such that $\bar{t} > t(q)$, let q^* be such that $t(q) \leq q^* < \bar{t}$, which implies that $\bar{t} > q + A(\bar{t})$. This is a contradiction since $\bar{t} \in A_q$. Hence $t(q)$ is an upper bound of A_q. Let $\bar{q} \geq t$ for any $t \in A_q$. We must show that $\bar{q} \geq t(q)$. In fact, if there exists some $\bar{t} > \bar{q}$ such that $\bar{t} \leq q + A(\bar{t})$, then $\bar{t}$ is an A_q and hence $\bar{t} \leq \bar{q}$, which is a contradiction. This means that $t > q + A(t)$ for any $t > \bar{q}$, i.e., $\bar{q} \geq t(q)$ by the definition of $t(q)$. Then $t(q)$ is the least upper bound of A_q, i.e., the property (a) of (A_2) holds. The property (b) of (A_2) is also satisfied by Lemma 2 since $t(q) \geq q > 0$. This completes the proof.

2. Basic preliminaries

Let (X, d) be a complete metric space and $\mathbb{N}$ be the set of the positive integers. Adopting the same notations of S. S. Chang [1], let $f : X \to X$ be a mapping such that f^m is continuous for some $m \in \mathbb{N}$ and let $\{g_i\}_{i=1}^{\infty}$ be a sequence of mappings $g_i : f^{m-1}(X) \to X,\ i = 1, 2, \dots$, such that

$$g_i(f^{m-1}(X)) \subseteq f^m(X) \tag{1}$$

for any $i \in \mathbb{N}$ (if $m = 1$, assume $f^{m-1} =$ identity on X). Further, assume that a sequence $\{m_i\}_{i=1}^{\infty}$ of elements of $\mathbb{N}$ exists and is such that the following inequality holds:

$$d(g_i^{m_i}(x),\, g_j^{m_j}(y)) \leq A(M(i, j, x, y, f)) \tag{2}$$

for any $i, j \in \mathbb{N}$ and $x, y \in f^{m-1}(X)$, where

$$\begin{aligned} M(i, j, x, y, f) = \max\big\{&d(fx, fy), d(fx, g_i^{m_i}(x)), d(fy, g_j^{m_j}(y)),\\ &d(fy, g_i^{m_i}(x)), d(fx, g_j^{m_j}(y))\big\}\end{aligned}$$

and $A : [0, +\infty) \to [0, +\infty)$ is a real-valued function satisfying the conditions (A_1) and (A_3) (or equivalently (A_2)).

As in [1], we observe that the condition (1) implies that

$$g_i^{m_i} : f^{m-1}(X) \to f^m(X) = f(f^{m-1}(X)) \tag{3}$$

for any $i \in \mathbb{N}$. Let x_1 be a point of $f^{m-1}(X)$ and, in view of the condition (3), let $x_2 \in f^{m-1}(X)$ be such that $g_1^{m_1}(x_1) = f(x_2)$. Iterating this process, we can define a sequence $\{x_n\}$ of elements of $f^{m-1}(X)$ such that

$$y_n = g_n^{m_n}(x_n) = f(x_{n+1}) \tag{4}$$

for $n = 1, 2, \dots$.

S. S. Chang [1] proved the following result, which generalizes the results of Lj.B. Ciric [2], K.M. Das and K.V. Naik [3], G. Jungck [4]:

Theorem 2. *Let $f : X \to X$ be a mapping such that f^m is continuous for some $m \in \mathbb{N}$ and let $\{g_i\}_{i=1}^{\infty}$ be a sequence of mappings $g_i : f^{m-1}(X) \to X$, $i = 1, 2, \dots$, such that the condition* (1) *holds. Suppose that g_i commutes with f for any $i \in \mathbb{N}$ and further there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of elements of $\mathbb{N}$ such that the inequality* (2) *holds for any $i, j \in \mathbb{N}$ and $x, y \in f^{m-1}(X)$, where $A : [0, +\infty) \to [0, +\infty)$ is a real-valued function satisfying the conditions $(A_1), (A_2)$ or $(A_1), (A_3)$. Then f and g_i, $i = 1, 2, \dots$, have a unique common fixed point $f^m(z)$, where z is the limit of the sequence defined by (4).*

Remark 3. In view of Theorem 1, we can say that the function A in Theorem 2 satisfies the conditions (A_1) and (A_3) (or equivalently, (A_2)). On the other hand, the proof of S.S. Chang [1] works only under the conditions (A_1) and (A_3).

Remark 4. Lemmas 1 and 3 of S.S. Chang [1] are identical.

We now denote by $\delta(O(y_k, n))$ and $\delta(O(y_1, \infty))$ the diameters of the sets

$$O(y_k, n) = \{y_k, y_{k+1}, \dots, y_{k+n}\}, \quad k \in \mathbb{N},$$

and

$$O(y_1, \infty) = \{y_1, y_2, \dots, y_n, \dots\},$$

respectively.

Slightly modifying in some details Lemma 2 of S. Sessa [8] (cf. also Remark 6 below), it is not hard to prove the following basic lemma:

Lemma 3. *Let $f : X \to X$ be a mapping and $\{g_i\}_{i=1}^{\infty}$ be a sequence of mappings $g_i : f^{m-1}(X) \to X$, $i = 1, 2, \dots$, such that the condition* (1) *folds for some $m \in \mathbb{N}$. Further, there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of elements of $\mathbb{N}$ such that the inequality* (2) *holds for any $i, j \in \mathbb{N}$ and $x, y \in f^{m-1}(X)$, where $A : [0, +\infty) \to [0, +\infty)$ is a real-valued function satisfying the conditions (A_1) and (A_3). If $\delta(O(y_k, n)) > 0$ for any $k, n \in \mathbb{N}$, then we have $\delta(O(y_1, \infty)) < \infty$ and $\delta(O(y_k, n)) \leq A^{k-1}(\delta(O(y, \infty)))$.*

Remark 5. Note that the continuity of f^m in Lemma 3 is not used. For the same reason the hypothesis that f is continuous can be removed from Lemma 2 of [8].

In this work, motivated by a recent paper of G. Jungck [5], we generalize Theorem 2 using the following condition of "compatibility":

Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of mappings $g_n : X \to X$, $n = 1, 2, \dots$, and $f : X \to X$.

We define $\{g_n\}_{n=1}^{\infty}$ and f to be compatible with respect to a sequence $\{m_n\}_{n=1}^{\infty}$ of elements of $\mathbb{N}$ and $m \in \mathbb{N}$, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that if $g_n^{m_n}(x_n)$, $f(x_n) \to t$ for some $t \in X$, then $d(f^h g_n^{m_n}(x_n), g_n^{m_n} f^h(x_n))$, $d(f g_n(x_n), g_n f(x_n)) \to 0$, where $h = 1, m$.

Note that if $g_n = g$ and $m_n = m = 1$, then we obtain Definition 2.1. of G. Jungck [5], which in turn extends the concept of weak commutativity introduced in [8]. Of course, if f commutes with g_n for any $n \in \mathbb{N}$, then they are compatible with respect to any sequence in $\mathbb{N}$ and any $m \in \mathbb{N}$. But the converse is not necessarily true as is shown in the following example:

Example 1. Let $X = [0, 1]$ with the Euclidean metric d and define

$$g_n(x) = g(x) = \frac{x}{a + x} \text{ and } f(x) = \frac{x}{a}$$

for any $n \in \mathbb{N}$ and $x \in X$, where $a > 1$. Assuming that $m_n = 1$ for any $n \in \mathbb{N}$, we have for any $m \in \mathbb{N}$,

$$\begin{aligned} d(gf^m(x), f^m g(x)) &= \frac{x}{a^{m+1}+x} - \frac{x}{a^{m+1}+a^m x} \\ &\le \frac{x^2}{a+x} = \frac{x}{a} - \frac{x}{a+x} = d(gx, fx) \end{aligned}$$

for all $x \in X$. Then it is easily seen that the mappings f and g are compatible with respect to the constant sequence $\{1\}$ and any $m \in \mathbb{N}$, but $fgx \neq gfx$ for all $x \in X - \{0\}$.

We shall use the following lemma for our main theorem. The proof of this lemma is identical to that of Proposition 2.2 of G. Jungck [5]:

Lemma 4. *Let $\{g_n\}_{n=1}^{\infty}$ and f be compatible with respect to a sequence $\{m_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ and $m \in \mathbb{N}$. Then we have the following:*

(a) *If $g_n^{m_n}(t) = f(t)$ for any $n \in \mathbb{N}$, then $fg_n^{m_n}(t) = g_n^{m_n} f(t)$ and $fg_n(t) = g_n f(t)$.*

(b) *If $g_n^{m_n}(x_n)$, $f(x_n) \to t$ for some $t \in X$, then $g_n^{m_n} f^m(x_n) \to f^m(t)$ if f^m is continuous at t.*

3. Main theorem

The proof of Theorem 2 by S. S. Chang [1] must be modified in the details where compatibility is used in place of commutativity. However, we will exhibit another technical proof along the same lines of [8] in order to prove the following theorem:

Theorem 3. *Let $f : X \to X$ be a mapping such that f^m is continuous for some $m \in \mathbb{N}$ and $\{g_i\}_{i=1}^{\infty}$ be a sequence of mappings $g_i : f^{m-1}(X) \to X$, $i = 1, 2, \ldots$, such that condition (1) holds. Suppose that there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of elements of $\mathbb{N}$ such that the inequality (2) holds for any $i, j \in \mathbb{N}$ and $x, y \in f^{m-1}(X)$, where $A : [0, +\infty) \to [0, +\infty)$ is a real-valued function satisfying the conditions (A_1) and (A_3).*

If $\{g_i\}_{i=1}^{\infty}$ and f are compatible with respect to the above sequence $\{m_i\}_{i=1}^{\infty}$ and m, then the conclusion of Theorem 2 still holds.

Proof. We suppose two cases. Firstly, assume that $\delta(O(y_k, n)) = 0$ for some $k, n \in \mathbb{N}$. Then we have

$$f(x_{k+1}) = y_k = y_{k+1} = g_{k+1}^{m_{k+1}}(x_{k+1}),$$

where x_{k+1} is in $f^{m-1}(X)$. Using the inequality (2), we have

$$\begin{aligned} d(g_i^{m_i}(x_{k+1}), y_k) &= d(g_i^{m_i}(x_{k+1}), g_{k+1}^{m_{k+1}}(x_{k+1})) \\ &\le A(\max\{d(y_k, y_k), d(y_k, g_i^{m_i}(x_{k+1}))\}) \\ &= A(d(g_i^{m_i}(x_{k+1}), y_k)) \end{aligned}$$

for any $i \in \mathbb{N}$, which implies that

$$g_i^{m_i}(x_{k+1}) = f(x_{k+1})$$

for any $i \in \mathbb{N}$ by the property (c) of (A_3).

Secondly, assume that $\delta(O(y_k, n)) > 0$ for any $k, n \in \mathbb{N}$. By Lemma 3, $\delta(O(y_1, \infty))$ is finite. It follows from Lemmas 2 and 3 that, for $p, q \in \mathbb{N}$ with $1 < p < q$,

$$\lim_{p\to\infty} d(y_p, y_q) \le \lim_{p\to\infty} \delta(O(y_p, q-p)) \le \lim_{p\to\infty} A^{p-1}(\delta(O(y_1, \infty))) = 0\,.$$

This means that the sequence, defined by (4), is a Cauchy sequence in X and hence it converges to some point $z \in X$ since X is complete. Since f^m is continuous, we deduce that, by Lemma 4(b),

$$g_n^{m_n} f^{m-1}(y_{n-1}) = g_n^{m_n} f^m(x_n) \to f^m(z)\,.$$

It is easily seen that for any $i \in \mathbb{N}$,

$$\lim_{n\to\infty} M(n, i, f^{m-1}(y_{n-1}), f^{m-1}(z), f) = d(f^m(z), g_i^{m_i} f^{m-1}(z))\,.$$

Using the inequality (2) and the right continuity of A, then we obtain

$$\begin{aligned} d(f^m(z), g_i^{m_i} f^{m-1}(z)) &= \lim_{n\to\infty} d(g_n^{m_n} f^{m-1}(y_{n-1}), g_i^{m_i} f^{m-1}(z)) \\ &\le \lim_{n\to\infty} A(M(n, i, f^{m-1}(y_{n-1}), f^{m-1}(z), f)) \\ &= A(d(f^m(z), g_i^{m_i} f^{m-1}(z))) \end{aligned}$$

for any $i \in \mathbb{N}$ and hence, by the property (c) of (A_3),

$$g_i^{m_i} f^{m-1}(z) = f^m(z) = f f^{m-1}(z)$$

for any $i \in \mathbb{N}$. In both cases, we have proved the existence of a point $w \in f^{m-1}(X)$ such that

$$g_i^{m_i}(w) = fw$$

for any $i \in \mathbb{N}$ and so, by Lemma 4(a), we have

$$g_i^{m_i}(fw) = f g_i^{m_i}(w) = f^2 w \quad \text{and} \quad g_i(fw) = f g_i(w)$$

for any $i \in \mathbb{N}$. Since $fw \in f^m(X) \subseteq f^{m-1}(X)$, using again the inequality (2), we have also for any $i \in \mathbb{N}$

$$\begin{aligned} d(f^2w, fw) &= d(g_i^{m_i}(fw), g_i^{m_i}(w)) \\ &\leq A(\max\{d(f^2w, fw), d(f^2w, f^2w), d(fw, fw)\}) \\ &= A(d(f^2w, fw))\,, \end{aligned}$$

which means that $f^2w = fw$ by the property (c) of (A_3). We also deduce, from the inequality (2),

$$\begin{aligned} d(fw, g_i(fw)) &= d(g_i^{m_i}(w), g_i g_i^{m_i}(w)) = d(g_i^{m_i}(w), g_i^{m_i}(g_i(w))) \\ &\leq A(\max\{d(fw, fg_i(w), d(fw, fw), d(fg_i(w), g_i(fw))\}) \\ &= A(d(fw, g_i(fw)))\,, \end{aligned}$$

for any $i \in \mathbb{N}$, which means that $g_i(fw) = fw$ for any $i \in \mathbb{N}$. Therefore, we have proved that fw is a fixed point of f and g_i for any $i \in \mathbb{N}$. The uniqueness of the fixed point is easily proved. This completes the proof.

The following example shows that Theorem 3 is a stronger generalization of Theorem 2.

Example 2. Let $X, f, g_i{=}g$ and $m_i{=}1$ for any $i \in \mathbb{N}$ be as in Example 1 and define $A(t) = t/(t+1)$ for any $t \geq 0$. We have for any $m \in \mathbb{N}$,

$$f^m(X) = [0, 1/a^m] \supseteq [0, 1/(a^m + 1)] = g(f^{m-1}(X))\,.$$

Of course, A satisfies the conditions (A_1) and (A_3). Further, we have

$$\begin{aligned} d(gx, gy) &= \frac{a|x-y|}{(a+x)(a+y)} \leq \frac{|x-y|}{a+|x-y|} = A\left(\frac{|x-y|}{a}\right) \\ &= A(d(fx, fy)) \leq A(M(i, j, x, y, f)) \end{aligned}$$

for any $i, j, m \in \mathbb{N}$ and $x, y \in f^{m-1}(X)$. Since $\{g_i\}_{i=1}^{\infty}$ and f are compatible with respect to the constant sequence $\{1\}$ and any $m \in \mathbb{N}$ (cf. Example 1), all the conditions of Theorem 3 are satisfied, but Theorem 2 is not applicable since $fgx \neq gfx$ for all $x \in X - \{0\}$.

Remark 6. In Lemma 3 of S.S. Chang [1], it is proved that the sequence defined by (4) has finite diameter as well as in Lemma 3. This is a consequence of the fact that the function A has the property (d), but it is evident that, omitting this condition, Theorem 3 still holds if one assumes the existence of the sequence, defined by (4), with finite diameter in X. For instance, see Lemma 2 of S. Sessa [8]. In this case, assuming $g_i = g$ and $m_i = m = 1$ for any $i \in \mathbb{N}$, Theorem 3 generalizes Theorem 4 of [8].

References

[1] S. S. Chang, A common fixed point theorem for commuting mapping, *Proc. Amer. Math. Soc.* **83** (1981), 645–652.

[2] Lj. B. Ciric, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* **45** (1974), 267–273.

[3] K. M. Das and K. V. Naik, Common fixed point theorems for commuting maps on a metric space, *Proc. Amer. Math. Soc.* **77** (1979), 369–373.

[4] G. Jungck, Commuting mappings and fixed points, *Amer. Math. Monthly* **83** (1976), 261–263.

[5] G. Jungck, Compatible mappings and common fixed points, *Internat, J. Math. & Math. Sci.* **9** (1986), 771–779.

[6] J. Matkowski, Fixed point theorems for the mappings with a contractive iterate at a point, *Proc. Amer. Math. Soc.* **72** (1977), 344–348.

[7] B. A. Meade and S. P. Singh, On common fixed point theorems, *Bull. Austral. Math. Soc* **16** (1977), 49–53.

[8] S. Sessa, On a week commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd)* **32** (46) (1982), 149–153.

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