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On a family of connections in Finsler geometry

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Abstract. In this paper, we introduce a new family of linear torsion-free connections for Finsler metrics. This family of connections defines a Riemannian curvature tensor R and a non-Riemannian quantity P. We show that P contains all the non-Riemannian information, namely, $P = 0$ if and only if the Finsler metric is Riemannian. In fact, this family of connections makes a systematical analysis of connections that characterize Riemannian metrics.

1. Introduction

A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms; one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner-product. After Einstein's formulation of general relativity, Riemannian geometry became fashionable and one of the connections, namely that due to Christoffel (Levi–Civita), came to the forefront. This connection is both torsion-free and metric-compatible. Likewise, connections in Finsler geometry can be prescribed on π^*TM and its tensor products. Examples of such connections were proposed by J. L. Synge (1925), J. H. Taylor (1925), L. Berwald (1928) [9], but most important of all is Elie Cartan's connection (1934) [10]. There is also such a connection given by Chern [11] in 1948. It is torsion-free but not completely compatible with the inner product (on π^*TM) defined by the g_{ij}'s. Incidentally, in the generic Finslerian setting, it is not possible to have a connection on π^*TM which is both torsion-free and

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compatible with the said inner product. The Chern connection, like many other connections, solves the equivalence problem for Finsler structures [7]. Namely, it gives rise to a list of criteria which decide when two such structures differ only by a change of coordinates. For a treatment of this connection using moving frames, see Chern's article [12]. The Chern connection coincides with the Rund connection, as pointed out by Anastasiei [4]. Asanov [5], Miron and Anastasiei [17], Bejancu [6], Ikeda [14], Kozma [25] and Tamássy [23], [24] have worked on connection theory. Recently Z. Shen [20] has found a new torsion-free connection in Finsler geometry. He proved that $P = 0$ if and only if F is Riemannian.

In this paper we will give a new family of torsion-free linear connections in π^*TM, which are torsion-free and compatible with the Finsler structure in a certain sense, where as torsion-free connections, in our connection we define two curvature tensors R and P. The R-term is the so-called Riemannian curvature tensor which is a natural extension of the usual Riemannian curvature tensor of Riemannian metrics, while the P-term is a purely non-Riemannian quantity. The main result of this paper states that $P = 0$ if and only if the Finsler metric is Riemannian. This is the second torsion-free linear connection with such property ever discovered since Shen‘s work [20]. We know there are already several well-known linear connections in Finsler geometry which are introduced from various points of view, in particular the connection constructed by Chern and Bao [7], that shows its extraordinary usefulness in treating global problems in Finsler geometry. However, the non-Riemannian quantity of our connections as well as the Shen connection seems to capture all non-Riemannian information on the Finsler metric.

2. Preliminaries

Let M be nn n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) := x$.

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π^*_vTM at $v \in TM_0$ is T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v) \mid y \in T_xM_0,\ v \in T_xM\}.$$

Some authors prefer to define connections in the pull-back tangent bundle π^*TM. From a geometrical point of view the construction of these connections on π^*TM seems to be simple because here the fibers are n-dimensional (i.e., $\pi^*(TM)_u = T_{\pi(u)}M, \forall u \in TM$) thus torsions and curvatures are obtained quickly from the structure equations. When the construction is done on $T(TM)$ many geometrical objects appear twice and one needs to split $T(TM)$ into the vertical and horizontal parts where the later is called horizontal distribution or non-linear connection. Nevertheless we do not need to split π^*TM. Indeed, the connection on $\pi^*(TM)$ is the most natural connection for Physicists.[1] In order to define curvatures, it is more convenient to consider the pull-back tangent bundle than the tangent bundle, because our geometric quantities depend on directions.

For the sake of simplicity, we denote by $\big\{\partial_i|_v := \big(v, \frac{\partial}{\partial x^i}\big)|_x)\big\}_{i=1}^n$ the natural basis for π_v^*TM. In Finsler geometry, we study connections and curvatures in (π^*TM, g), rather than in (TM, F). The pull-back tangent bundle π^*TM is a very special tangent bundle.

Throughout this paper, we use the *Einstein summation convention* for expressions with indices.[2]

Finsler structure

A (globally defined) Finsler structure on a manifold M is a function

$$F : TM \to [0, \infty)$$

with the following properties:

(i) F is a differentiable function on the manifold TM_0 and F is continuous on the null section of the projection $\pi : TM \to M$.

(ii) F is a positive function on TM_0.

(iii) F is positively 1-homogeneous on the fibers of the tangent bundle TM.

(iv) The Hessian of F^2 with elements

$$(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j} \right)$$

is positive definite on TM_0.

Given a manifold M and a Finsler structure F on M, the pair (M, F) is called a Finsler manifold. F is called Riemannian if $g_{ij}(x, y)$ are independent of $y \neq 0$.

Every Finsler metric on a manifold defines a length structure on oriented piecewise C^∞ curves. Let C be an oriented piecewise C^∞ curve from p to q in

[1] For more details on the structure of $\pi^*(TM)$ see [19] and [8].

[2] That is, if an index appears twice, namely as a subscript as well as a superscript, then that term is assumed to be summed over all values of that index.

a Finsler manifold (M, F). Let $C : [a, b] \to M$ be a parameterization of C with $C(a) = p$ and $C(b) = q$. Then the length of C is defined by

$$\mathbf{L}_F(C) := \int_a^b F\left(C(t), \frac{d\,C(t)}{d\,t}\right) dt. \qquad (*)$$

The homogeneity of F implies that $\mathbf{L}_F(C)$ is independent of the choice of the particular parameterization of C.

The Finsler structure F defines a fundamental tensor $g : \pi^*TM \otimes \pi^*TM \to [0, \infty)$ by the formula $g(\partial_i|_v, \partial_j|_v) = g_{ij}(x, y)$, where $v = y^i \frac{\partial}{\partial x^i}|_x$. Let

$$g_{ij}(x, y) := FF_{y^i y^j} + F_{y^i} F_{y^j},$$

where $F_{y^i} = \frac{\partial F}{\partial y^i}$. Then (π^*TM, g) becomes a Riemannian vector bundle over TM_0. Let

$$A_{ijk}(x, y) = \frac{1}{2} F(x, y) \frac{\partial g_{ij}}{\partial y^k}(x, y).$$

Clearly, A_{ijk} is symmetric with respect to i, j, k. The Cartan[3] tensor $A : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to R$ is defined by

$$A(\partial_i|_v, \partial_j|_v, \partial_k|_v) = A_{ijk}(x, y),$$

where $v = y^i \frac{\partial}{\partial x^i}|_x$. The homogeneity condition (iii) holds in particular for positive λ. Therefore, by Euler's theorem we see that

$$y^i \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^j \frac{\partial g_{ij}}{\partial y^k}(x, y) = y^k \frac{\partial g_{ij}}{\partial y^k}(x, y) = 0.$$

We recall that the canonical section ℓ is defined by

$$\ell = \ell(x, y) = \frac{y^i}{F(x, y)} \frac{\partial}{\partial x^i} = \frac{y^i}{F} \frac{\partial}{\partial x^i} := \ell^i \frac{\partial}{\partial x^i}.$$

Put $\ell_i := g_{ij}\ell^j = F_{y^i}$. Thus the canonical section ℓ satisfies

$$g(\ell, \ell) = g_{ij} \frac{y^i}{F} \frac{y^j}{F} = 1$$

and

$$\ell^i A_{ijk} = \ell^j A_{ijk} = \ell^k A_{ijk} = 0.$$

Thus $A(X, Y, \ell) = 0$.

[3]In some literature $C_{ijk} = \frac{A_{ijk}}{F}$ is called Cartan tensor. Riemannian manifolds are characterized by $A \equiv 0$.

3. Existence and uniqueness of a new Finsler connection on π^*TM

In this section we introduce a new Finsler connection which is torsion-free and almost compatible with Finsler metric.

Bundle Maps μ and ρ.
The bundle map $\rho : T(TM_0) \to \pi^*TM$ is defined by

$$\rho\left(\frac{\partial}{\partial x^i}\right) = \partial_i, \qquad \rho\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{1}$$

Put $VTM := \ker\rho = \mathrm{span}\{\frac{\partial}{\partial y^i}\}_{i=1}^n$. VTM is an n-dimensional subbundle of $T(TM_0)$, whose fiber V_vTM at v is just the tangent space $T_v(T_xM) \subset T_v(TM_0)$. VTM is called the *vertical tangent bundle* of TM_0.

The bundle map $\mu : T(TM_0) \to \pi^*TM$ is defined by $\mu\big(\frac{\partial}{\partial y^i}\big) = \partial_i$.

Put $HTM := \mathrm{Ker}\,\mu$. HTM is called the *horizontal tangent bundle* of TM_0.

We have the direct decomposition $T(TM_0) = HTM \oplus VTM$. Tangent vectors in HTM are called *horizontal* and vectors in VTM are called *vertical*. We summarize: $\mathrm{Ker}\,\rho = VTM$, $\mathrm{Ker}\,\mu = HTM$, ρ restricted to HTM is an isomorphism onto π^*TM, and μ restricted to VTM is the bundle isomorphism onto π^*TM.

Definition 3.1. A tensor $T : \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to R$ is called compatible if it has the following properties:

(i) $T(X, Y, Z)$ is symmetric with respect to X, Y, Z.

(ii) $T(X, Y, \ell) = 0$.

(iii) T is homogeneous, i.e., $T_{ijk}(x, ty) = T_{ijk}(x, y)$, $\forall t \in R$, where $T_{ijk}(x, y) = T(\partial_i, \partial_j, \partial_k)$.

Let (M, F) be a Finsler n-manifold. Let g, A and T denote the fundamental tensor, the Cartan tensor and a compatible tensor in π^*TM, respectively.

Definition 3.2. Let D be a Finsler connection on M. Then we say that

(i) D is *torsion-free*, if

$$\mathbf{T}_D(\hat{X}, \hat{Y}) := D_{\hat{X}}\rho(\hat{Y}) - D_{\hat{Y}}\rho(\hat{X}) - \rho([\hat{X}, \hat{Y}]) = 0, \quad \forall \hat{X}, \hat{Y} \in C^\infty(T(TM_0)). \tag{2}$$

(ii) D is *almost compatible* with the Finsler structure in the following sense: for all $X, Y \in C^\infty(\pi^*TM)$ and $\hat{Z} \in T_v(TM_0)$,

$$\begin{aligned}(D_{\hat{Z}}g)(X, Y) &:= \hat{Z}g(X, Y) - g(D_{\hat{Z}}X, Y) - g(X, D_{\hat{Z}}Y)\\ &= A(\rho(\hat{Z}), X, Y) - 2T(\rho(\hat{Z}), X, Y) + 2F^{-1}A(\mu(\hat{Z}), X, Y),\end{aligned} \tag{3}$$

where $\rho(\hat{Z}) := (v, \pi_*(\hat{Z}))$, $\mu(\hat{Z}) := D_{\hat{Z}}F\ell$, and T is the given compatible tensor.

Theorem 3.1. *Let (M, F) be a Finsler n-manifold and T an arbitrary compatible tensor in π^*TM. Then there is a unique linear torsion-free connection D in π^*TM, which is almost compatible with the above Finsler structure.*[4]

We define the *Landsberg* tensor $\dot{A} = \pi^*TM \otimes \pi^*TM \otimes \pi^*TM \to R$ by

$$\dot{A}(X,Y,Z) := \bar{\ell}A(X,Y,Z) - A(D_{\bar{\ell}}X,Y,Z) - A(X,D_{\bar{\ell}}Y,Z) - A(X,Y,D_{\bar{\ell}}Z).$$

It is obvious that

$$\ell^i \dot{A}_{ijk} = \ell^j \dot{A}_{ijk} = \ell^k \dot{A}_{ijk} = 0.$$

Then $\dot{A}(X,Y,\ell) = 0$. It is easy to check that $T = \alpha A + \beta \dot{A}$ is a compatible tensor $\forall \alpha, \beta \in R$.

4. Nonlinear connections and Finsler connections

Let M be a real n-dimensional connected manifold of C^∞-class and (TM, π, M) its tangent bundle with the zero section removed. Every local chart $(U, \varphi = (x^i))$ on M induces a local chart $(\varphi^{-1}(U), \varphi = (x^i, y^i))$ on TM. The kernel of the linear map $\pi_* : TTM \to TM$ is called the *vertical distribution* and is denoted by VTM. For every $u \in TM$, $\mathrm{Ker}\, \pi_{*,u} = V_uTM$ is spanned by $\{\frac{\partial}{\partial y^i}|_u\}$. By a *nonlinear connection* on TM we mean a regular n-dimensional distribution $H : u \in TM \to H_uTM$ which is supplementary to the vertical distribution i.e.

$$T_u(TM) = H_uTM \oplus V_uTM, \quad \forall u \in TM.$$

A basis for T_uTM adapted to the above direct sum is $\left(\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u\right)$, where N^i_j are the coefficients of the nonlinear connection and $\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i} - N^j_i(u)\frac{\partial}{\partial y^j}|_u$. The dual basis of $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ is given by $(dx^i, \delta y^i + N^i_j dx^j)$. These are the *Berwald bases.*

Let M be a real n-dimensional C^∞ manifold and $VTM = \cup_{v \in TM} V_vTM$ its vertical vector bundle. Suppose that $HTM = \cup_{v\in TM} H_vTM$ is a nonlinear connection on TM and ∇ a linear connection on VTM; then the pair (HTM, ∇) is called a *Finsler connection* on the manifold M.

[4]In the sequel we will refer to this connection as *"New connection"*.

Proof of Theorem 3.1. In a standard local coordinate system (x^i, y^i) in TM_0, we write

$$D_{\frac{\partial}{\partial x^i}}\partial_j = \Gamma^k_{ij}\partial_k, \qquad D_{\frac{\partial}{\partial y^i}}\partial_j = F^k_{ij}\partial_k.$$

Clearly, (2) and (3) are equivalent to the following:

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{4}$$

$$F^k_{ij} = 0 \tag{5}$$

$$\frac{\partial}{\partial x^k}(g_{ij}) = -\Gamma^l_{ki}g_{lj} + \Gamma^l_{kj}g_{li} + 2A_{ijk} - 2T_{ijk} + 2A_{ijl}\Gamma^l_{km}\ell^m \tag{6}$$

$$\frac{\partial}{\partial y^k}(g_{ij}) = -F^l_{kj}g_{li} + F^l_{ik}g_{jl} + 2C_{ijk} - 2T_{ijk}F^l_{mk}\ell^m + 2A_{ijk}F^l_{mk}\ell^m \tag{7}$$

where $g_{ij} = g_{ij}(x,y)$, $A_{ijk} = A_{ijk}(x,y)$ and $T_{ijk} = T_{ijk}(x,y)$ is the given compatible tensor. Notice that (5) and (7) are just the definition of A_{ijk}. We must compute Γ^k_{ij} from (4) and (6). Then permuting i, j, k in (6), and using (4), one obtains

$$\Gamma^k_{ij} = \gamma^k_{ij} - A^k_{ij} + T^k_{ij} + g^{kl}\left\{A_{ijm}\Gamma^m_{lb} - A_{jlm}\Gamma^m_{ib} - A_{lim}\Gamma^m_{jb}\right\}\ell^b, \tag{8}$$

where we have put

$$\gamma^k_{ij} = \frac{1}{2}g^{kl}\left\{\frac{\partial}{\partial x^i}(g_{jl}) + \frac{\partial}{\partial x^j}(g_{il}) - \frac{\partial}{\partial x^l}(g_{ij})\right\}$$

and $A^k_{ij} = g^{kl}A_{ijl}$. Multiplying (8) by ℓ^i, yields

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - A^k_{im}\Gamma^m_{lb}\ell^l\ell^b. \tag{9}$$

Multiplying (9) by ℓ^i gives

$$\Gamma^k_{ab}\ell^a\ell^b = \gamma^k_{ab}\ell^a\ell^b. \tag{10}$$

By substituting (10) into (9) one obtains

$$\Gamma^k_{ib}\ell^b = \gamma^k_{ib}\ell^b - A^k_{im}\gamma^m_{ab}\ell^a\ell^b. \tag{11}$$

By substituting (11) into (8) one obtains

$$\begin{aligned}\Gamma^k_{ij} = {} & \gamma^k_{ij} - A^k_{ij} + T^k_{ij} + g^{kl}\left\{A_{ijm}\gamma^m_{lb} - A_{jlm}\gamma^m_{ib} - A_{lim}\gamma^m_{jb}\right\}\ell^b \\ & + \left\{A^k_{jm}A^m_{is} + A^k_{im}A^m_{js} - A^k_{sm}A^m_{ij}\right\}\gamma^s_{ab}\ell^b\ell^a.\end{aligned} \tag{12}$$

This proves the uniqueness of D. The set $\{\Gamma^k_{ij}, F^k_{ij} = 0\}$, where $\{\Gamma^k_{ij}\}$ are given by (12), satisfy a linear connection D with properties (2) and (3). $\square$

The bundle map $\mu : T(TM_0) \to \pi^*TM$ defined in Section 3 can be expressed in the following form:

$$\mu\left(\frac{\partial}{\partial x^i}\right) = N_i^k \partial_k, \quad \mu\left(\frac{\partial}{\partial y^i}\right) = \partial_i, \tag{13}$$

where

$$N_i^k = y^j \Gamma_{ij}^k = y^j \gamma_{ij}^k - \frac{1}{F} g^{ks} A_{sil} \gamma_{ab}^l y^a y^b.$$

The above N_j^i are known in the literature as the *nonlinear connection coefficients* on TM_0. The Berwald connection is most directly related to the nonlinear connection N_j^i, and is most amenable to the study of path geometry.

Defining $G^i := \gamma_{jk}^i y^j y^k$, one can prove that $\frac{\partial G^i}{\partial y^j} = N_j^i$. Finslerian geodesics are curves in M which obey the equation $\dot{y}^i + G^i = 0$. Thus, if the geodesic equation is once known, the nonlinear connection N_j^i can be computed without having to calculate first the Cartan tensor A_{ijk} and the formal Christoffel symbols γ_{ijk}. The formula (12) in terms of the coefficients N_j^i is given by

$$\Gamma_{ij}^k = \gamma_{ij}^k - A_{ij}^k + T_{ij}^k - g^{kl}\{N_j^s C_{ski} + N_i^s C_{sjk} - N_k^s C_{sij}\}. \tag{14}$$

It is obvious that

$$\Gamma_{ij}^k \ell^i = \Gamma_{ji}^k \ell^i = \frac{N_j^k}{F}. \tag{15}$$

Let us express the Christoffel coefficients of the Berwald, Chern and Shen connections and of the New connection, by ${}^b\Gamma_{ij}^k$, ${}^c\Gamma_{ij}^k$, ${}^s\Gamma_{ij}^k$ and Γ_{ij}^k respectively; then we see that:

$$\Gamma_{ij}^k := {}^b\Gamma_{ij}^k - A_{ij}^k - \dot{A}_{ij}^k + T_{ij}^k, \tag{16}$$

$$\Gamma_{ij}^k := {}^c\Gamma_{ij}^k - A_{ij}^k + T_{ij}^k, \tag{17}$$

$$\Gamma_{ij}^k := {}^s\Gamma_{ij}^k + T_{ij}^k. \tag{18}$$

It is clear that in a locally Minkowski space, $\Gamma_{ij}^k = -A_{ij}^k + T_{ij}^k$ and $N_j^i = 0$. The reader can consult [24] and [15].

5. Curvatures of the New connection

In this section we study the curvature tensor of the "new Finsler connection" introduced in the above section, which is torsion-free and almost compatible with the Finsler metric. As a torsion-free connection, it defines two curvatures R and P. The R-term is the so-called Riemannian curvature tensor, which is a natural extension of the usual Riemannian curvature tensor of a Riemannian metric, while the P-term is a purely non-Riemannian quantity. We prove also that the hv-curvature P of this connection vanishes if and only if the Finsler structure is Riemannian. The curvature tensor Ω of D is defined by

$$\Omega(\hat{X},\hat{Y})Z = D_{\hat{X}}D_{\hat{Y}}Z - D_{\hat{Y}}D_{\hat{X}}Z - D_{[\hat{X},\hat{Y}]}Z, \tag{19}$$

where $\hat{X},\hat{Y} \in C^\infty(T(TM_0))$ and $Z \in C^\infty(\pi^*TM)$.

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field (with respect to g) for the vector bundle π^*TM, such that $g(e_i, e_n) = 0$, $i = 1,\dots,n-1$ and

$$e_n := \frac{y}{F} = \frac{y^i}{F(x,y)}\frac{\partial}{\partial x^i} = \ell.$$

Let $\{\omega^i\}_{i=1}^n$ be its dual co-frame field. These are local sections of the dual bundle π^*TM. One readily finds that

$$\omega^n := \frac{\partial F}{\partial y^i}dx^i = \ell_i dx^i = \omega,$$

which is the *Hilbert form*. It is obvious that

$$\omega(\ell) = 1.$$

We observe that for a curve $x^i = x^i(t)$ with $y^i = \frac{dx^i}{dt}$, Euler's theorem allows us to rewrite the integral $(*)$ as $\int_a^b \omega^n$.

Put

$$\rho = \omega^i \otimes e_i, \quad De_i = {\omega_i}^j \otimes e_j, \quad \Omega e_i = 2{\Omega_i}^j \otimes e_j.$$

$\{{\Omega_i}^j\}$ and $\{{\omega_i}^j\}$ are called the *curvature forms* and *connection forms* of D with respect to $\{e_i\}$. We have $\mu := DF\ell = F\{{\omega_n}^i + d(\log F){\delta_n}^i\} \otimes e_i$. Put $\omega^{n+i} := {\omega_n}^i + d(\log F)\delta^i n$. It is easy to see that $\{\omega^i, \omega^{n+i}\}_{i=1}^n$ is a local basis for $T^*(TM_0)$. By definition

$$\rho = \omega^i \otimes e_i, \quad \mu = F\omega^{n+i} \otimes e_i.$$

Use the above formula for Theorem 3.1, then it will re-express the structure equation of the New connection

$$d\omega^i = \omega^j \wedge {\omega_j}^i \tag{20}$$

$$dg_{ij} = g_{kj}{\omega_i}^k + g_{ik}{\omega_j}^k + 2A_{ijk}\omega^k - 2T_{ijk}\omega^k + 2A_{ijk}\omega^{n+k}. \tag{21}$$

Define $g_{ij.k}$ and $g_{ij|k}$ by

$$dg_{ij} - g_{kj}{\omega_i}^k - g_{ik}{\omega_j}^k = g_{ij|k}\omega^k + g_{ij.k}\omega^{n+k}, \tag{22}$$

where $g_{ij.k}$ and $g_{ij|k}$ are the vertical and horizontal covariant derivative respectively of g_{ij} with respect to the New connection. This gives

$$g_{ij|k} = 2(A_{ijk} - T_{ijk}), \tag{23}$$

$$g_{ij.k} = 2A_{ijk}. \tag{24}$$

It can be shown that $\delta^i_{j|s} = 0$ and $\delta^i_{j.s} = 0$, thus $(g^{ij}g_{jk})_{|s} = 0$ and $(g^{ij}g_{jk})_{.s} = 0$. So

$$g^{ij}_{|s} = 2(T^{ij}_s - A^{ij}_s) \tag{25}$$

and

$$g^{ij}_{.s} = -2A^{ij}_s. \tag{26}$$

Moreover, torsion freeness is equivalent to the absence of dy^k in $\{{\omega_j}^i\}$, namely

$${\omega_j}^i = \Gamma^i_{jk}(x,y)dx^k. \tag{27}$$

(19) is equivalent to

$$d{\omega_i}^j - {\omega_i}^k \wedge {\omega_k}^j = {\Omega_i}^j. \tag{28}$$

Since the ${\Omega_j}^i$ are 2-forms on the manifold TM_0, they can be expanded as

$${\Omega_i}^j = \frac{1}{2}{R_i}^j{}_{kl}\omega^k \wedge \omega^l + {P_i}^j{}_{kl}\omega^k \wedge \omega^{n+l} + \frac{1}{2}{Q_i}^j{}_{kl}\omega^{n+k} \wedge \omega^{n+l}. \tag{29}$$

The objects R, P and Q are the hh-, hv- and vv-curvature tensors respectively of the connection D. Let $\{\bar{e}_i, \dot{e}_i\}_{i=1}^n$ be a local basis for $T(TM_0)$, which is dual to $\{\omega^i, \omega^{n+i}\}_{i=1}^n$, i.e., $\bar{e}_i \in HTM$, $\dot{e}_i \in VTM$ such that $\rho(\bar{e}_i) = e_i$, $\mu(\dot{e}_i) = Fe_i$. We have put $R(\bar{e}_k, \bar{e}_l)e_i = {R_i}^j{}_{kl}e_j$, $P(\bar{e}_k, \dot{e}_l)e_i = {P_i}^j{}_{kl}e_j$ and $Q(\dot{e}_k, \dot{e}_l)e_i = {Q_i}^j{}_{kl}e_j$. Since the New connection is torsion-free we have ([21] and [22]) that

$$Q = 0.$$

The first Bianchi identities for R are

$${R_i}^j{}_{kl} + {R_k}^j{}_{li} + {R_l}^j{}_{ik} = 0 \tag{30}$$

and

$${P_i}^j{}_{kl} = {P_k}^j{}_{il}. \tag{31}$$

Exterior differentiation of (28) gives the Second Bianchi identities:

$$d{\Omega_i}^j - {\omega_i}^k \wedge {\Omega_k}^j + {\omega_k}^j \wedge {\Omega_i}^k = 0. \tag{32}$$

We decompose the covariant derivative of the Cartan tensor on TM

$$dA_{ijk} - A_{ljk}\omega_i{}^l - A_{ilk}\omega_j{}^l - A_{ijl}\omega_k{}^l = A_{ijk|l}\omega^l + A_{ijk.l}\omega^{n+l}, \tag{33}$$

and so for $\dot{A}_{ijk}$ we have

$$d\dot{A}_{ijk} - \dot{A}_{ljk}\omega_i{}^l - \dot{A}_{ilk}\omega_j{}^l - \dot{A}_{ijl}\omega_k{}^l = \dot{A}_{ijk|l}\omega^l + \dot{A}_{ijk.l}\omega^{n+l}. \tag{34}$$

Clearly, in the above relations the tensors $A_{ijk|l}$, $A_{ijk.l}$, $\dot{A}_{ijk|l}$ and $\dot{A}_{ijk.l}$ are symmetric with respect to the indices i, j, k.

Put $\dot{A}_{ijk} = \dot{A}(e_i, e_j, e_k)$, $\dot{A}^k_{ij} = g^{kl}\dot{A}_{ijl}$. The quantity $A_{ijk|n}$ plays a somewhat privileged role in Finsler geometry so much that it deserves perhaps a special notation:

$$A_{ijk|n} = \dot{A}_{ijk}. \tag{35}$$

It follows from (33) and (34) that

$$A_{njk|l} = 0, \quad \text{and} \quad A_{njk.l} = -A_{jkl}. \tag{36}$$

$$\dot{A}_{njk|l} = 0, \quad \text{and} \quad \dot{A}_{njk.l} = -\dot{A}_{jkl}. \tag{37}$$

Theorem 5.1. *Let (M, F) be a Finsler manifold and D be a torsion-free connection defined in theorem (1) with the condition $T_{ijk} := k_1 A^{(1)}{}_{jk}{}^i + \cdots + k_m A^{(m)}_{jk}{}^i$, where $A^{(m)}_{ijk} = A_{ijk}\overbrace{{}_{|n|n}\cdots{}_{|n}}^{m\ times}$, $m \in N$. Then F is Riemannian if and only if $P = 0$.*

Proof. Let (M, F) be a Finsler manifold. Differentiating (21) and using (20), (21),(33), (36) and (37) leads to

$$\begin{aligned} g_{kj}\Omega_i{}^k + g_{ik}\Omega_j{}^k = {} & -2A_{ijk}\Omega^k_n - 2A_{ijk|l}\omega^k \wedge \omega^l + 2A_{ijk.l}\omega^{n+k} \wedge \omega^{n+l} \\ & - 2\{A_{ijk.l} - A_{ijk|l}\}\omega^k \wedge \omega^{n+l} \\ & + k_1(A^{(1)}_{ijk|l}\omega^l + A^{(1)}_{ijk.l}\omega^{n+l}) \wedge \omega^k + \cdots \\ & + k_m(A^{(m)}_{ijk|l}\omega^l + A^{(m)}_{ijk.l}\omega^{n+l}) \wedge \omega^k. \end{aligned} \tag{38}$$

Using (29), we get

$$\begin{aligned} R_{ijkl} + R_{jikl} = {} & 2k_1\{A^{(1)}_{ijl|k} - A^{(1)}_{ijk|l}\} + \cdots + 2k_m\{A^{(m)}_{ijl|k} - A^{(m)}_{ijk|l}\} \\ & - 2A_{ijs}R_n{}^s{}_{kl} + 2\{A_{ijk|l} - A_{ijl|k}\}, \end{aligned} \tag{39}$$

$$\begin{aligned} P_{ijkl} + P_{jikl} = {} & -2\{k_1 A^{(1)}_{ijk.l} + \cdots + k_m A^{(m)}_{ijk.l}\} \\ & + 2\{A_{ijk.l} - A_{ijl|k}\} - 2A_{ijs}P_n{}^s{}_{kl}, \end{aligned} \tag{40}$$

$$A_{ijk.l} = A_{ijl.k}. \tag{41}$$

Permuting i, j, k in (40) yields

$$\begin{aligned} P_{ijkl} = &-\{k_1 A^{(1)}_{ijk.l} + \cdots + k_m A^{(m)}_{ijk.l}\} + A_{ijk.l} - (A_{ijl|k} + A_{jkl|i} - A_{kil|j}) \\ &+ A_{kis} P_n{}^s{}_{jl} - A_{jks} P_n{}^s{}_{il} - A_{ijs} P_n{}^s{}_{kl} \end{aligned} \tag{42}$$

and

$$P_{njkl} = \{k_1 A^{(1)}_{jkl} + \cdots + k_m A^{(m)}_{jkl}\} - A_{jkl} - \dot{A}_{jkl}, \tag{43}$$

because of $P_{njnl} = 0$.
Now if F is Riemannian, then from (42) and (43) we conclude that $P = 0$.
Conversely let $P = 0$. It follows from (43) that

$$k_1 A^{(1)}_{jkl} + \cdots + k_m A^{(m)}_{jkl} = \dot{A}_{jkl} + A_{jkl}. \tag{44}$$

By (42) one has

$$k_1 A^{(1)}_{ijk.l} + \cdots + k_m A^{(m)}_{ijk.l} = A_{ijk.l} + A_{kil|j} - A_{ijl|k} - A_{jkl|i}.$$

Permuting i, j, k in the above identity leads to

$$k_1 A^{(1)}_{ijk.l} + \cdots + k_m A^{(m)}_{ijk.l} = A_{ijk.l} + A_{jkl|i} - A_{kil|j} - A_{ijl|k},$$

and then

$$A_{ijl|k} = A_{jkl|i}.$$

Letting $k = n$, we can conclude

$$\dot{A}_{ijk} = 0. \tag{45}$$

It is obvious that

$$A^{(m)}_{ijk} = 0, \quad \forall m \in N. \tag{46}$$

Therefore we conclude that $A_{ijk} = 0$, and thus F is Riemannian. $\square$

6. Complete Finsler manifolds

Let $\bar{\ell}$ denote the unique vector field in HTM such that $\rho(\bar{\ell}) = \ell$. We call $\bar{\ell}$ the *geodesic field* on TM_0, because it determines all geodesics and it is called a *spray*.

Let $c : [a,b] \to (M,F)$ be a unit speed C^∞ curve. The canonical lift of c to TM_0 is defined by $\hat{c} := \frac{dc}{dt} \in TM_0$. It is easy to see that

$$\rho\left(\frac{d\hat{c}}{dt}\right) = \ell_{\hat{c}}.$$

The curve c is called a *geodesic* if its canonical lift $\hat{c}$ satisfies

$$\frac{d\hat{c}}{dt} = \bar{\ell}_{\hat{c}},$$

where $\bar{\ell}$ is the *geodesic field* on TM_0, i.e., $\ell \in HTM$, $\rho(\bar{\ell}) = \ell$.

Let $I_xM = \{v \in T_xM, F(v) = 1\}$ and $IM = \bigcup_{p\in M} I_xM$. The I_xM is called *indicatrix*, and it is a compact set. We can show that the projection of the integral curve $\varphi(t)$ of $\bar{\ell}$ with $\varphi(0) \in IM$ is a unit speed geodesics c, whose canonical lift is $\hat{c}(t) = \varphi(t)$. A Finsler manifold (M,F) is called *complete* if any unit speed geodesic $c : [a,b] \to M$ can be extended to a geodesic defined on R. This is equivalent to requiring that the geodesic field $\bar{\ell}$ restricted to IM is complete.

If we put $k_1 = k_3 = \cdots = k_m = 0$ and $k_2 = -1$, then we have a connection and we obtain

$$A + \dot{A} + \ddot{A} = 0,$$

where $\ddot{A} := A^{(2)}$ is defined in Theorem 5.1.

Let (M,F) be a Finsler manifold and c a unit speed geodesic in M. A section $X = X(t)$ of π^*TM along $\hat{c}$ is said to be parallel if $D_{\frac{d\hat{c}}{dt}}X = 0$. For $v \in TM_0$ we define $\|A\|_v := \sup A(X,Y,Z)$. Then we put $\|A\| = \sup_{v\in IM} \|A\|_v$ where the supremum is taken over all unit vectors of π_v^*TM.

Theorem 6.1. *Let (M,F) be complete with bounded $\|A\|$. If $k_1{=}k_3{=}\cdots = k_m = 0$ and $k_2 = -1$, then F is Riemannian, whenever*

$$A + \dot{A} + \ddot{A} = 0. \tag{47}$$

Proof. If F is Riemannian, then (47) is true. Conversely, let the above condition be true. Fix any $X, Y, Z \in \pi^*TM$ at $v \in I_xM$. Let $c : M \to R$ be the unit speed geodesic with $\frac{dc}{dt}(0) = v$. Let $X(t)$, $Y(t)$ and $Z(t)$ denote the parallel sections along $\hat{c}$ with $X(0) = X$, $Y(0) = Y$, $Z(0) = Z$. Putting $A(t) = A(X(t), Y(t), Z(t))$, $\dot{A}(t) = \dot{A}(X(t), Y(t), Z(t))$, and $\ddot{A}(t) = \ddot{A}(X(t), Y(t), Z(t))$, one has

$$\frac{dA}{dt} = \dot{A} \quad \textit{and} \quad \frac{d\dot{A}}{dt} = \ddot{A}. \tag{48}$$

Therefore by (47) and (48) we have

$$\frac{d^2A}{dt^2}+\frac{dA}{dt}+A=0. \tag{49}$$

Now

$$A(t)=e^{\frac{-t}{2}}\left(c_1\cos\frac{\sqrt{3}}{2}t+c_2\sin\frac{\sqrt{3}}{2}t\right).$$

Using $\|A\|<\infty$ and letting $t\to-\infty$, we get $c_1=c_2=0$, and $A(0)=A(X,Y,Z)=0$, which completes the proof. □

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