Publ. Math. Debrecen
44 / 1–2 (1994), 5–30

Weighted Nikolskiĭ-type inequalities

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To the memory of Professor András Rapcsák

Weighted polynomial inequalities play a very important role in the theory of weighted approximations and they are interesting in themselves. Moreover, they can be useful in other areas, e.g. concerning orthogonal polynomials or convergence of interpolation processes. Nikolskiĭ-type inequalities seek relationship between different finite dimensional metric spaces of polynomials. The first such inequality was found by S. M. Nikolskiĭ [15], and it deals with estimating L_q norms of trigonometric polynomials in terms of their L_p norms for $p < q$ (for $p > q$ this is trivially done by Hölder's inequality). In this paper we shall consider similar questions concerning exponential weights. Our paper extends earlier results to the case $\alpha \neq 0$, i.e. when the weight has a zero at the origin.

Let

$$w(x) = w_\alpha(x) = |x|^{\frac{\alpha}{2}} \cdot \exp(-|x|^m), \quad m \in \mathbb{R},\ m > 0.$$

Given p, q and m such that $0 < p,\ q \leq \infty$, $m > 0$ define the Nikolskiĭ constant, $N_n = N_n(m, p, q)$ $n = 1, 2, \dots$ by

$$N_n(m,p,q) = \begin{cases} n^{1/m(1/p-1/q)} & \text{if } p \leq q, \\ n^{(1-1/m)(1/q-1/p)} & \text{if } p > q \text{ and } m > 1, \\ (\log(n+1))^{1/q-1/p} & \text{if } p > q \text{ and } m = 1, \\ 1 & \text{if } p > q \text{ and } 0 < m < 1. \end{cases} \tag{1}$$

In what follows, for $0 < p \leq \infty$ the expression $\|f\|_p$ is defined by

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(t)|^p dt \right)^{\frac{1}{p}}.$$

For $\alpha = 0$ the Theorem 1 (see below) was proved by P. Névai – V. Totik [6], for $m = 2$, $\alpha \geq 0$ it was proved by S. Szabó [8] and for Hermite-weight and Laguerre-weight ($1 \leq p,\ q \leq \infty,\ \alpha \geq 0$) it was proved by C. Markett [1]. As for Theorem 2, for the weights $\alpha = 0$, $m \geq 2$ G. Freud [16] found the correct analogue of the classical Markov–Bernstein inequality. It was extended by A.L. Levin and D.S. Lubinsky [10] which is the $\alpha = 0$, $1 < m \leq 2$ case of Theorem 2. The case $\alpha = 0$, $0 < m \leq 1$ was proved by P. Névai and V. Totik [5] which paper also contains Theorems 3 and 4 with $\alpha = 0$. Actually, in [16], [10], [5] more general weights were considered than $w(x)$ with $\alpha = 0$, but the parameter range of m was essentially the same as in our case.

The aim of the present paper is to prove the following theorems.

Theorem 1. *Suppose $0 < p, q \leq \infty$, $\alpha \geq 0$, $m > 0$. Then for any polynomial $p_n \in \Pi_n$ of degree $\leq n$ we have*

$$\|p_n w_\alpha\|_p \leq c N_n(m,p,q) \cdot \|p_n w_\alpha\|_q, \tag{2}$$

where $c = c(m,p,q)$ is a positive constant independent of n, p_n. The estimate (2) is sharp, i.e. given m, p, q, with $0 < m < \infty$, $0 < p$, $q \leq \infty$ there exists $c^* > 0$ and polynomials $\{R_n^*\}_{n=0}^\infty \deg R_n^* \leq n$ such that

$$\|R_n^* w_\alpha\|_p \geq c^* N_n(m,p,q) \cdot \|R_n^* w_\alpha\|_q \tag{3}$$

for $n = 1, 2, \ldots$.

Theorem 2. *Suppose $0 < p < \infty$, $\alpha/2 > -1/p$ or $p = \infty$, $\alpha \geq 0$ then for any polynomials p_n of degree $\leq n$ we have*

$$\|p_n' w_\alpha\|_p \leq c \cdot n^{1-\frac{1}{m}} \|p_n w_\alpha\|_p, \quad \text{if } m > 1 \tag{4}$$

$$\|p_n' w_\alpha\|_p \leq c \cdot \log n \|p_n w_\alpha\|_p, \quad \text{if } m = 1 \tag{5}$$

$$\|p_n' w_\alpha\|_p \leq c \cdot \|p_n w_\alpha\|_p, \quad \text{if } m < 1, \tag{6}$$

where $c = c(p, m, \alpha) > 0$ is a constant independent of n, p_n.

Theorem 3. *Suppose Φ is a non-negative even function which is concave and increases on $[0, \infty)$ and such that*

$$\int_{-\infty}^{\infty} \frac{\Phi(x)}{1+x^2} dx < +\infty, \quad \frac{\Phi(x)}{\log x} \to \infty \quad (x \to +\infty).$$

Let $0 < q \le p \le \infty$, then

$$\left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \exp(-\Phi(t))\right)^p dt\right)^{\frac{1}{p}} \le \tag{7}$$

$$\le c \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \exp(-\Phi(t))\right)^q dt\right)^{\frac{1}{q}}$$

with a constant $c = c(\alpha, p, q, \Phi) > 0$ is independent of n, p_n.

Theorem 4. *Suppose Φ is a non-negative even function which is concave and increasing on $[0, \infty)$ and*

$$\int_{-\infty}^{\infty} \frac{\Phi(x)}{1+x^2} dx < +\infty.$$

Then

$$\max_{x\in\mathbb{R}} |p_n'(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x)) \le c \max_{x\in\mathbb{R}} |p_n| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x)) \tag{8}$$

here $\alpha \ge 0$ and $c = c(\alpha, \Phi)$; and if in addition

$$(2\Phi(x) - \Phi(2x))/\log(x) \to +\infty \quad (x \to \infty),\ 0 < p < \infty,\ \frac{\alpha}{2} > -\frac{1}{p} \tag{8*}$$

is filfilled, then

$$\left(\int_{-\infty}^{\infty} \left(|p_n'(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt\right)^{\frac{1}{p}} \le \tag{9}$$

$$\le c \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt\right)^{\frac{1}{p}}, \quad c = c(\alpha, p, \Phi).$$

Our results are not refinable. The proof of this fact will be given in a subsequent paper of the same journal.

PROOF of Theorem 1. We need some lemmas.

Lemma 1.1 ([9], Theorem 4.16.2)**.** *Suppose $\alpha/2 > -1$, $m > 0$, $p > 0$. Then for any polynomial p_n of degree $\le n$ we have*

$$\int_{\mathbb{R}} |p_n(t)|^p w_\alpha(t) dt \le 2 \int_{-c_1 n^{\frac{1}{m}}}^{c_1 n^{\frac{1}{m}}} |p_n(t)|^p w_\alpha(t) dt \tag{1.1}$$

where the constant $c_1 = c_1(m, p, \alpha) > 0$ *does not depend of* n, p_n.

Lemma 1.2. *Suppose* $\alpha/2 > -1/p$, $m > 0$, $p > 0$. *Then for any polynomial* p_n *of degree* $\le n$ *we have*

$$\int_{-\infty}^{\infty} (|p_n(t)|w_\alpha(t))^p dt \le 2 \int_{-\frac{c_1}{\sqrt[m]{p}}n^{\frac{1}{m}}}^{\frac{c_1}{\sqrt[m]{p}}n^{\frac{1}{m}}} (|p_n(t)|w_\alpha(t))^p dt \tag{1.2}$$

further for any $\alpha \in \mathbb{R}$, $m > 0$

$$\max_{t\in\mathbb{R}} |p_n(t)|w_\alpha(t) \le c \max_{|t|<c_2 n^{\frac{1}{m}}} |p_n(t)|w_\alpha(t). \tag{1.2$'$}$$

Proof. It is an easy consequence of Lemma 1.1.

Lemma 1.3 ([8], Lemma 2). *Denote* $\chi_{[a,b]}(t)$ *the characteristic function of the interval* $[a, b]$ *and let* $0 < q < p \le \infty$, $v(t) = |t|^{\alpha/2}$, $\alpha \ge 2$. *Then*

$$\begin{aligned}\left(\int_{-\infty}^{\infty} \left|p_n(t)\chi_{[-1,1]}(t)v(t)\right|^p dt\right)^{\frac{1}{p}} &\le \\ \le cn^{\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot \left(\int_{-\infty}^{\infty} \left|p_n(t)\chi_{[-2,2]}(t)v(t)\right|^q dt\right)^{\frac{1}{q}}&\end{aligned} \tag{1.3}$$

for any algebraic polynomial of degree $\le n$. *The constant* c *is independent of* n *and* p_n.

For the sake of completeness we write here the

Proof. Using the ideas and results of [13], p. 118 we have

$$\frac{c}{n+1}\left(|x| + \frac{1}{n+1}\right)^\Gamma \le \inf_{p_n\in\Pi_n} \frac{1}{|p_n(x)|^q} \int_{-\infty}^{\infty} |p_n(t)|^q \cdot |t|^\Gamma \cdot \chi_{[-2,2]}(t)dt,$$
$$-1 \le x \le 1,\ \Gamma > -1,\ 0 < q < \infty.$$

Hence for every $p_n \in \Pi_n$ and $\Gamma \geq 0$

$$\int_{-\infty}^{\infty} |p_n(t)|^q \cdot |t|^\Gamma \cdot \chi_{[-2,2]}(t)dt \geq$$

$$(1.4) \qquad \geq \frac{c}{n}|p_n(x)|^q \left(|x| + \frac{1}{n+1}\right)^\Gamma \geq \frac{c}{n}|p_n(x)|^q \cdot |x|^\Gamma,$$

$$-1 \leq x \leq 1,\ 0 < q < \infty.$$

Let $\Gamma = \frac{\alpha}{2}q$. Then we obtain from (1.4)

$$(1.5) \qquad \int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}}\right)^q \cdot \chi_{[-2,2]}(t)dt \geq \frac{c}{n}\left(|p_n(x)| \cdot |x|^{\frac{\alpha}{2}}\right)^q,$$

$$-1 \leq x \leq 1,\ 0 < q < \infty.$$

Hence

$$(1.6) \qquad \sup_{x\in\mathbb{R}} |p_n(x)v(x)\chi_{[-1,1]}(x)| \leq$$

$$\leq cn^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} |p_n(x)v(x)\chi_{[-2,2]}(x)|^q dx\right)^{\frac{1}{q}},$$

i.e

$$\left(\int_{-\infty}^{\infty} |p_n(x)v(x)\chi_{-[1,1]}(x)|^p dx\right)^{\frac{1}{p}} =$$

$$= \left(\int_{-\infty}^{\infty} |p_n(x)v(x)\chi_{-[1,1]}(x)|^{p-q+q} dx\right)^{\frac{1}{p}} \leq$$

$$\leq \sup_{x\in\mathbb{R}} |p_n(x)v(x)\chi_{[-1,1]}(x)|^{\frac{p-q}{p}} \cdot \left(\int_{-\infty}^{\infty} |p_n(x)v(x)\chi_{[-1,1]}(x)|^q dx\right)^{\frac{1}{p}} \leq$$

$$\leq cn^{\frac{p-q}{pq}} \left(\int_{-\infty}^{\infty} |p_n(x)v(x)\chi_{[-2,2]}(x)|^q dx\right)^{\frac{p-q}{pq}}.$$

$$\cdot\left(\int_{-\infty}^{\infty}|p_n(x)v(x)\chi_{-[1,1]}(x)|^q dx\right)^{\frac{1}{p}}\le$$

$$\le cn^{\left(\frac{1}{q}-\frac{1}{p}\right)}\cdot\left(\int_{-\infty}^{\infty}|p_n(x)v(x)\chi_{[-2,2]}(x)|^q dx\right)^{\frac{1}{q}}.$$

Lemma 1.3 is proved.

Proof of the Nikolskiĭ-type inequality (2).

Case (i): $0<p\le q\le\infty$. By Lemma 1.2 we can estimate $\|p_n w\|_p$ as an integral over a finite interval. Applying Hölder's inequality to the latter integral we obtain (2).

Case (ii): $p>q$ and $m>1$. We can prove this case similarly to [8].

Proof. First consider the case $0<q<p<\infty$. From Lemma 1.2 (1.2) follows

$$\left(\int_{-\infty}^{\infty}|p_n(t)w_\alpha(t)|^p dt\right)^{\frac{1}{p}}\le c\left(\int_{-c_0 n^{\frac{1}{m}}}^{c_0 n^{\frac{1}{m}}}|p_n(t)w_\alpha(t)|^p dt\right)^{\frac{1}{p}}.$$

According to [10], Theorem 1 (see also three pages later at "Case a) $m>1$") there exists a polynomial R_n of degree $\le c\cdot n$ such that

$$R_n(x)\asymp e^{-|x|^m},\quad |x|\le 2c_0 n^{\frac{1}{m}}.$$

Using the substitution $t=c_0 n^{\frac{1}{m}}x$ and applying Lemma 1.3 we obtain

$$\left(\int_{-c_0 n^{\frac{1}{m}}}^{c_0 n^{\frac{1}{m}}}|p_n(t)w_\alpha(t)|^p dt\right)^{\frac{1}{p}}\asymp\left(\int_{-c_0 n^{\frac{1}{m}}}^{c_0 n^{\frac{1}{m}}}|p_n(t)v(t)R_n(t)|^p dt\right)^{\frac{1}{p}}\le$$

$$\le cn^{\frac{1}{mp}}\left(\int_{-\infty}^{\infty}|p_n(c_0 n^{\frac{1}{m}}x)v(c_0 n^{\frac{1}{m}}x)R_n(c_0 n^{\frac{1}{m}}x)\chi_{[-1,1]}(x)|^p dx\right)^{\frac{1}{p}}\le$$

$$\le cn^{\frac{1}{mp}}\cdot n^{\left(\frac{1}{q}-\frac{1}{p}\right)}\cdot$$

$$\cdot\left(\int_{-\infty}^{\infty}|p_n(c_0 n^{\frac{1}{m}}x)v(c_0 n^{\frac{1}{m}}x)R_n(c_0 n^{\frac{1}{m}}x)\chi_{[-2,2]}(x)|^q dx\right)^{\frac{1}{q}}.$$

Using the substitution $x = \dfrac{t}{c_0 n^{\frac{1}{m}}}$ we obtain

$$cn^{\frac{1}{mp}} \cdot n^{\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} |p_n(c_0 n^{\frac{1}{m}} x) v(c_0 n^{\frac{1}{m}} x) R_n(c_0 n^{\frac{1}{m}} x) \chi_{-[2,2]}(x)|^q dx \right)^{\frac{1}{q}} \leq$$

$$\leq cn^{\frac{1}{mp}} \cdot n^{\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot n^{-\frac{1}{mq}} \cdot \left(\int_{-2c_0 n^{\frac{1}{m}}}^{2c_0 n^{\frac{1}{m}}} |p_n(t) v(t) R_n(t)|^q dt \right)^{\frac{1}{q}} \leq$$

$$\leq cn^{\left(1-\frac{1}{m}\right)\left(\frac{1}{q}-\frac{1}{p}\right)} \cdot \left(\int_{-2c_0 n^{\frac{1}{m}}}^{2c_0 n^{\frac{1}{m}}} |p_n(t) w_\alpha(t)|^q dt \right)^{\frac{1}{q}} .$$

The case $0 < q < p = \infty$ follows using Lemma 1.2 (1.2') and (1.6). Hence the case (ii) is proved.

Case (iii): $p > q$ and $m = 1$. The proof of this case is the same as the one for case (ii) but we have to use [5], pp. 125–126 instead of [10], Theorem 1.

Case (iv): $p > q$ and $0 < m < 1$. It follows from Theorem 3 at $\Phi(x) = |x|^m$.

The estimate (2) is proved.

The proof of (3) will be given later.

PROOF of Theorem 2. First we prove (4). For this we need some lemmas.

Lemma 2.1 ([12], Theorem 5). *Let $0 < p < \infty$, $\Gamma_1, \ldots, \Gamma_N$ be arbitrary real numbers, $1 = x_1 > x_2 > \ldots > x_N = -1$, $\Gamma_i > -1$ and*

$$w(t) = \prod_{i=1}^{N} |t - x_i|\Gamma_i,$$

$$w_n(t) = \left((1-t)^{1/2} + n^{-1}\right)^{2\Gamma_1} \prod_{i=2}^{N-1} \left(|t-x_i| + n^{-1}\right)^{\Gamma_i} \left((1+t)^{1/2} + n^{-1}\right)^{2\Gamma_N}.$$

Then any algebraic polynomial p of degree $\leq n$ satisfies

$$\int_{-1}^{1} \left| p_n'(1-t^2)^{1/2} \right|^p w_n(t)w(t)dt \leq cn^p \int_{-1}^{1} |p_n(t)|^p w(t)w(t)dt. \tag{2.1}$$

Corollary 2.1. *Let $0 < p < \infty$, $u(t) = (1-t)^a(1+t)^b$, $a, b > -1$, then for any algebraic polynomial p_n of degree $\leq n$ we have*

$$\int_{-1}^{1} \left| p_n'(t)(1-t^2)^{1/2} \right|^p u(t)dt \leq cn^p \int_{-1}^{1} |p_n(t)|^p u(t)dt. \tag{2.2}$$

Lemma 2.2 [13], p. 119, Corollary 26)**.** *Let $0 < p < \infty$, $\Gamma > -1$, $\varepsilon > 0$, then for any polynomial p_n of degree $\leq n$ we have*

$$\int_{-1}^{1} |p_n(t)|^p \cdot |t|^\Gamma dt \leq cn^\varepsilon \int_{-1}^{1} |p_n(t)|^p \cdot |t|^{\Gamma+\varepsilon} dt \tag{2.3}$$

where $c = c(p, \Gamma, \varepsilon)$ is a constant not depending on n, p_n.

Lemma 2.3 ([13], p. 163, Lemma 17)**.** *Let $a \in \mathbb{R}$. Then there exists a number $\varepsilon = \varepsilon(a) > 0$ such that for any algebraic polynomial p_n of degree $\leq n$ we have*

$$\max_{|x| \leq \varepsilon/n} |p_n(x)| \leq cn^a \max_{\varepsilon/n \leq |x| \leq 1} |p_n(x)| \cdot |x|^a \tag{2.4}$$

where $c = c(a)$ is a constant independent of n, p_n.

Lemma 2.4 ([13], p. 163, Lemma 18)**.** *Let $a \in \mathbb{R}$, then*

$$\max_{|x| \leq 1} |p_n'(x)| \cdot (|x| + n^{-1})^a \leq cn \cdot \max_{|x| \leq 2} |p_n(x)|(|x| + n^{-1})^a. \tag{2.5}$$

Lemma 2.5. *Let $a > -1$, $0 < p < \infty$. Then for any algebraic polynomial p_n of degree at most n we have*

$$\left(\int_{-1}^{1} |p_n'(t)|^p \cdot |t|^a dt \right)^{\frac{1}{p}} \leq cn \left(\int_{-2}^{2} |p_n(t)|^p \cdot |t|^a dt \right)^{\frac{1}{p}}, \tag{2.6}$$

where $c = c(p, a)$ is a constant not depending on n and p_n. Let $a \geq 0$, $p = \infty$, then

$$\max_{|x| \leq 1} |p_n'(x)| \cdot |x|^a \leq cn \cdot \max_{|x| \leq 2} |p_n(x)| \cdot |x|^a. \tag{2.6'}$$

Proof. First we prove the case $0 < p < \infty$. We are following the proof of Lemma 15 in [13], p. 162 where the case $1 \le p < \infty$ was considered.

Let first p_n be an even algebraic polynomial, i.e. of the form $p_n(x) = G_n(x^2)$. Then $p_n'(x) = 2xG_n'(x^2)$ and we have to show that

$$\int_{-1}^{1} |xG_n'(x^2)|^p \cdot |x|^a dx \le cn^p \cdot \int_{-2}^{2} |G_n(x^2)|^2 \cdot |x|^a dx$$

or

$$\int_{0}^{1} |G_n'(x)|^p \cdot |x|^{\frac{p+a-1}{2}} dx \le cn^p \int_{0}^{4} |G_n(x)|^p |x|^{\frac{a-1}{x}} dx.$$

But

$$\int_{0}^{1} |G_n'(x)|^p \cdot |x|^{\frac{p+a-1}{2}} dx \le c \int_{0}^{4} |G_n'(x)\sqrt{x}\sqrt{4-x}\,|^p \cdot |x|^{\frac{a-1}{2}} dx.$$

Hence (2.6) follows from Corollary (2.1) when p_n is even. let now p_n be odd: $p_n(x) = xG_n(x^2)$. In this case we have $p_n'(x) = G_n(x^2) + 2x^2G_n'(x^2)$ and we will prove that

$$\int_{-1}^{1} |x^2G_n'(x^2)|^p \cdot |x|^a dx \le cn^p \int_{-2}^{2} |xG_n(x^2)|^p \cdot |x|^a dx$$

and

$$\int_{-1}^{1} |G_n(x^2)|^p \cdot |x|^a dx \le cn^p \int_{-2}^{2} |xG_n(x^2)|^p \cdot |x|^a dx.$$

The first inequality here follows from the first part of the proof by putting there $p + a$ instead of a. The second inequality follows from Lemma 2.2 choosing $\varepsilon = p$. hence Lemma 2.5 is proved in the case of $0 < p < \infty$.

Now consider the case $p = \infty$. We have to prove that

$$\max_{|x| \le 1} |p_n'(x)| \cdot |x|^a \le n \max_{|x| \le 2} |p_n(x)| \cdot |x|^a.$$

Using (2.5) in Lemma 2.4 we obtain

$$\max_{\frac{\varepsilon}{n} \le |x| \le 1} |p_n'(x)| \cdot |x|^a \le \max_{\frac{\varepsilon}{n} \le |x| \le 1} |p_n'(x)| \cdot \left(|x| + \frac{1}{n}\right)^a \le$$

$$\le c \max_{|x|\le 1} |p_n'(x)| \cdot \left(|x| + \frac{1}{n}\right)^a \le cn \cdot \max_{|x|\le 2} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a ,$$

i.e.

$$\max_{\frac{\varepsilon}{n}\le|x|\le 1} |p_n'(x)| \cdot |x|^a \le cn \cdot \max_{|x|\le 2} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a . \tag{2.7}$$

Here

$$\max_{|x|\le 2} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a \le$$

$$\le \max_{|x|\le \frac{\varepsilon}{2}} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a + \max_{\frac{\varepsilon}{2}\le|x|\le 2} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a .$$

But

$$\begin{aligned} &\max_{\frac{\varepsilon}{2}\le|x|\le 2} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a \le \\ &\le c \max_{\frac{\varepsilon}{n}\le|x|\le 2} |p_n(x)| \cdot |x|^a \le c \max_{|x|\le 2} |p_n(x)| \cdot |x|^a \end{aligned} \tag{2.8}$$

and using (2.4) in Lemma 2.3 we have

$$\begin{aligned} &\max_{|x|\le\frac{\varepsilon}{n}} |p_n(x)| \cdot \left(|x| + \frac{1}{n}\right)^a \le c \max_{|x|\le\frac{\varepsilon}{n}} |p_n(x)| \frac{1}{n^a} \le \\ &\le c \max_{\frac{\varepsilon}{n}\le|x|\le 1} |p_n(x)| \cdot |x|^a \le c \max_{|x|\le 2} |p_n(x)| \cdot |x|^a . \end{aligned} \tag{2.9}$$

Hence from (2.7), (2.8) and (2.9) we get

$$\max_{\frac{\varepsilon}{n}|x|\le 1} |p_n'(x)| \cdot |x|^a \le c \cdot n \cdot \max_{|x|\le 2} |p_n(x)| \cdot |x|^a . \tag{2.10}$$

On the other hand, using (2.4) in Lemma 2.3 we have

$$\begin{aligned} \max_{|x|\le\frac{\varepsilon}{n}} |p_n'(x)| \cdot |x|^a &\le c \cdot n^{-a} \cdot \max_{|x|\le\frac{\varepsilon}{n}} |p_n'(x)| \le \\ &\le c \cdot \max_{\frac{\varepsilon}{n}\le|x|\le 1} |p_n'(x)| \cdot |x|^a , \end{aligned} \tag{2.11}$$

so using (2.10) we obtain

$$\max_{|x|\le\frac{\varepsilon}{n}} |p_n'(x)| \cdot |x|^a \le cn \cdot \max_{|x|\le 2} |p_n(x)| \cdot |x|^a . \tag{2.12}$$

From (2.10) and (2.12) we obtain (2.6′) so Lemma 2.5 is proved completely.

PROOF of the Markov-Berstein type inequality (4), (5), (6). The idea of the proof goes back to [2].

Case a) $m > 1$. For the proof we need a suitable polynomial $S_n(x)$ for which

(i) $S_n(x)$ is even and its degree is at most n,

(ii) $S_n(x) \asymp w_0(x)$ if $|x| \le c_1 n^{1/m}$,[1]

(iii) $|S_n'(x)| \le c_2 |x|^{m-1} \cdot w_0(x)$ if $|x| \le c_1 n^{1/m}$

and in particular

$$|S_n'(x)| \le c_3 n^{1-\frac{1}{m}} \cdot w_0(x) \quad \text{ if } |x| \le c_1 n^{1/m}.$$

Here the constants c_1, c_2, c_3 are independent of n and x. Such a polynomial $S_n(x)$ was constructed in the paper of LEVIN and LUBINSKY [10], Theorem 1.

By the "infinite–finite range inequality" (Lemma 1.2) there exists a constant c_4 such that

$$\|p_n' w\|_p \le c \left\| p_n' w \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p . \tag{2.13}$$

By (ii)

$$\begin{aligned} \|p_n' w\|_p &\le c \left\| p_n' v S_n \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p = \\ &= c \left\| v[(p_n S_n)' - p_n S_n'] \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p \le \\ &\le c \left\| v(p_n S_n)' \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p + c \left\| v p_n S_n' \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p , \end{aligned} \tag{2.14}$$

where $v(x) = |x|^{\alpha/2}$.

By Lemma 2.5 we get: for every $0 < p \le \infty$ there is a constant c such that

$$\|r_n' v \chi_{[-1,1]}\|_p \le c \cdot n \cdot \|r_n v \chi_{[-2,2]}\|_p \tag{2.15}$$

for every algebraic polynomial r_n of degree at most n. Hence

$$\left\| r_n' v \chi_{\left[-\frac{c_4}{2} n^{\frac{1}{m}}, \frac{c_4}{2} n^{\frac{1}{m}}\right]} \right\|_p \le c n^{1-\frac{1}{m}} \left\| r_n v \chi_{\left[-c_4 n^{\frac{1}{m}}, c_4 n^{\frac{1}{m}}\right]} \right\|_p . \tag{2.16}$$

[1] In this paper $f(x) \asymp g(x)$, $x \in I \subset \mathbb{R}$ denotes that there exist positive constants d_1, $d_2 \in \mathbb{R}$ such that $d_1|f(x)| \le |g(x)| \le d_2|f(x)|$ for $x \in I$.

Now we can apply (2.16) to the first term on the right-hand side of (2.14) and we obtain

$$\|p_n'w\|_p \le cn^{1-\frac{1}{m}} \left\| vp_nS_n\chi_{\left[-c_4n^{\frac{1}{m}},c_4n^{\frac{1}{m}}\right]}\right\| + \tag{2.17}$$
$$+c\left\| vp_nS_n'\chi_{\left[-\frac{c_4}{2}n^{\frac{1}{m}},\frac{c_4}{2}n^{\frac{1}{m}}\right]}\right\|_p .$$

Finally, by (ii) and (iii) we can estimate S_n and S_n' with $w(x)$, and thus (4) follows from (2.17).

Case b) $m=1$. For $p=\infty$ see Lemma 4.4 below. If $0<p<\infty$, then the proof is the same as in the case a) using instead of $S_n(x)$ the polynomial $R_{Ln[\log(n+1)]}(x)$ (L is an integer), of degree at most $Ln[\log(n+1)]$ such that $R_{Ln[\log(n+1)]}(x)$ is even and $R_{Ln[\log(n+1)]}(x) \asymp \exp(-|x|)$, if $|x|\le c_2n$, and

$$\left|R'_{Ln[\log(n+1)]}(x)\right| \le c\log n\cdot\exp(-|x|), \quad \text{ if } |x|\le c_2n.$$

The existence of such a polynomial $R_{Ln[\log(n+1)]}(x)$ was proved in [5], Theorem 3.

Case c) $0<m<1$. This case will be dealt with in the proof of Theorem 4. Thus Theorem 2 is proved.

Proof of Theorem 3. We need some lemmas. In what follows denote by Φ any fixed non-negative even function which increases on $[0,\infty)$.

Lemma 3.1 ([5] Theorem 1)**.** *There exists a sequence of polynomials* $\{p_n\}_{n=1}^{\infty}$ *such that*

$$p_n(0)=1, \quad |p_n(x)|\le K\exp(-\Phi(x)), \quad |x|\le 1 \tag{3.1}$$

if and only if

$$\int\limits_{-\infty}^{\infty}\frac{\Phi(x)}{1+x^2}dx<\infty. \tag{3.2}$$

In what follows we assume also that Φ satisfies $\Phi(x)/\log x\to\infty$, $x\to\infty$. This condition ensures that every polynomial belongs to all of the space $L^p(\exp(-\Phi))$, $0<p\le\infty$.

Lemma 3.2. *Suppose* $0<p<\infty$, $\alpha\ge 0$ *and that* (3.2) *holds. Then there is a constant* $K=K(p,\alpha,\Phi)$ *such that for every polynomial* p_n *of degree at most* n

$$|p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}} \le$$
$$\le K\sup_u\left(\int\limits_{u-1}^{u+1}\left(|p_n(t)|\cdot|t|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(t-x_0))\right)^p dt\right)^{\frac{1}{p}}, \tag{3.3}$$

where $x_0 \in \mathbb{R}$ is arbitrary. If we suppose also that Φ is concave on $[0,\infty)$ then we have

$$\max_{x\in\mathbb{R}} |p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x)) \le$$

$$\le K \sup_u \left(\int_{u-1}^{u+1} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt \right)^{\frac{1}{p}}. \tag{3.4}$$

Proof. The idea of the proof is similar to that of Corollary 3 in [5], but we need some new ideas too. Let $Q_n(x)$ be a polynomial of degree at most n such that Q_n is even, $Q_n(0) = 1$ and

$$|Q_n(x)| \le K \exp(-2\Phi(x)), \quad |x| \le n.$$

Such a polynomial was constructed in [5] Theorem 1. Now we consider

$$Q_n^*(x) := Q_n(x - x_0), \quad \text{where } x_0 \ne 0 \text{ is arbitrary fixed.}$$

We know, see [8], (6) above, that

$$|p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \le cn^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}}\right)^p \chi_{\left[\frac{-1}{2},\frac{1}{2}\right]}(t) dt \right)^{\frac{1}{p}}, \tag{3.5}$$

where $|x| \ge 1/4$, $\alpha \le 0$, $0 < p < \infty$.

In case $|x_0|/n \le 1/4$ we use (3.5) for the polynomial $p_n(n\cdot)Q_n^*(n\cdot)$ at the point x_0/n. Then we have

$$\left|p_n\left(n\frac{x_0}{n}\right)\right| \cdot \left|Q_n^*\left(n\frac{x_0}{n}\right)\right| \cdot \left|\frac{x_0}{n}\right|^{\frac{\alpha}{2}} \le$$

$$\le cn^{1/p} \left(\int_{-\infty}^{\infty} \left|p_n(nt)Q_n^*(nt) \cdot |t|^{\frac{\alpha}{2}}\right|^p \chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(t) dt \right)^{\frac{1}{p}} \le$$

$$\le c\left(\frac{1}{n}\right)^{\frac{\alpha}{2}})^p \left(\int_{-\infty}^{\infty} |p_n(s)Q_n^*(s) \cdot |s|^{\frac{\alpha}{2}})^p \chi_{\left[-\frac{n}{2},\frac{n}{2}\right]}(s) ds \right)^{\frac{1}{p}} \le$$

$$\le c\left(\frac{1}{n}\right)^{\frac{\alpha}{2}} \left(\int_{-\infty}^{\infty} |p_n(s) \cdot \exp(-2\Phi(s - x_0)) \cdot |s|^{\frac{\alpha}{2}})^p \chi_{\left[-\frac{n}{2},\frac{n}{2}\right]}(s) ds \right)^{\frac{1}{p}} \le$$

$$\le c\left(\frac{1}{n}\right)^{\frac{\alpha}{2}} \left(\int_{-\infty}^{\infty} |p_n(s) \cdot \exp(-2\Phi(s - x_0)) \cdot |s|^{\frac{\alpha}{2}})^p ds \right)^{\frac{1}{p}}$$

i.e.

$$(3.6)\quad |p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\le c\left(\int\limits_{-\infty}^{\infty}(|p_n(s)|\cdot\exp(-2\Phi(s-x_0)|\cdot|s|^{\frac{\alpha}{2}})^p ds\right)^{\frac{1}{p}}.$$

Here we have for the case $2k-1\le s_k\le 2k+1$

$$\left(\int\limits_{-\infty}^{\infty}(|p_n(s)|\cdot\exp(-2\Phi(s-x_0))\cdot|s|^{\frac{\alpha}{2}})^p ds\right)^{\frac{1}{p}}\le$$

$$\le c\left\{\sum_{k=-\infty}^{\infty}\left(\exp(-\Phi(s_k-x_0)))^p\cdot\right.\right.$$

$$\left.\left.\cdot\int\limits_{2k-1}^{2k+1}(|p_n(s)|\cdot\exp(-\Phi(s-x_0))\cdot|s|^{\frac{\alpha}{2}}\right)^p ds\right\}^{\frac{1}{p}}\le$$

$$\le c\sup_u\left(\int\limits_{u-1}^{u+1}(|p_n(s)|\cdot\exp(-\Phi(s-x_0))\cdot|s|^{\frac{\alpha}{2}})^p ds\right)^{\frac{1}{p}}.$$

Using (3.6) we obtain in the case $|x_0|/n\le 1/4$

(3.7)

$$|p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\le c\sup_u\left(\int\limits_{u-1}^{u+1}(|p_n(s)|\cdot\exp(-\Phi(s-x_0))\cdot|s|^{\frac{\alpha}{2}})^p ds\right)^{\frac{1}{p}}.$$

In the case $|x_0|/n<1/4$ there exists $A_n>1$ such that $|x_0|/nA_n\le 1/4$. Using again (3.5) for $p_n(nAn\cdot)Q^*_{nA_n}(nA_n\cdot)$ at the point $|x_0|/nA_n$ we obtain

$$\left|p_n\left(nA_n\frac{x_0}{nA_n}\right)\right|\cdot\left|Q^*_{nA_n}\left(nA_n\frac{x_0}{nA_n}\right)\right|\cdot\left|\frac{x_0}{nA_n}\right|^{\frac{\alpha}{2}}\le$$

$$\le c(nA_n)^{1/p}\left(\int\limits_{-\infty}^{\infty}\left(|p_n(nA_nt)Q^*_{nA_n}(nA_nt)|\cdot|t|^{\frac{\alpha}{2}}\right)^p\chi_{\left[-\frac{1}{2},\frac{1}{2}\right]}(t)dt\right)^{\frac{1}{p}}\le$$

$$\le c\left(\frac{1}{nA_n}\right)^{\frac{\alpha}{2}}\left(\int\limits_{-\infty}^{\infty}\left(|p_n(s)\cdot Q^*_{nA_n}(s)|\cdot|s|^{\frac{\alpha}{2}}\right)^p\chi_{\left[-\frac{nA_n}{2},\frac{nA_n}{2}\right]}(s)ds\right)^{\frac{1}{p}}\le$$

$$\leq c\left(\frac{1}{nA_n}\right)^{\frac{\alpha}{2}}\left(\int\limits_{-\infty}^{\infty}\left(|p_n(s)|\cdot\exp(-2\Phi(s-x_0))\cdot|s|^{\frac{\alpha}{2}}\right)^p ds\right)^{\frac{1}{p}},$$

i.e.

$$(3.8)\quad |p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\leq c\left(\int\limits_{-\infty}^{\infty}\left(|p_n(s)|\cdot\exp(-2\Phi(s-x_0)|\cdot|s|^{\frac{\alpha}{2}}\right)^p ds\right)^{\frac{1}{p}}.$$

From (3.8) we obtain that (3.7) is true in case $|x_0|/n>1/4$. Hence (3.3) is proved.

Now suppose that Φ is also concave. From (3.3) we get

$$|p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\leq K_1\sup_u\left(\int\limits_{u-1}^{u+1}\left(|p_n(t)|\cdot|t|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(t-x_0))\right)^p dt\right)^{\frac{1}{p}}=$$

$$=K_1\sup_u\left(\int\limits_{u-1}^{u+1}\left(|p_n(t-x_0)|\cdot|t-x_0|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(t-2x_0))\right)^p dt\right)^{\frac{1}{p}}.$$

Multiplying this by $\exp(-\Phi(x_0))$ we obtain

$$|p_n(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(x_0))\leq K_1\sup_u\left(\int\limits_{u-1}^{u+1}(|p_n(t-x_0)|\cdot\right.$$

$$\left.\cdot|t-x_0|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(x_0))\cdot\exp(-\Phi(t-2x_0))\right)^p dt\Bigg)^{\frac{1}{p}}\leq$$

$$\leq K_1\sup_u\left(\int\limits_{u-1}^{u+1}\left(|p_n(t-x_0)|\cdot\exp(-\Phi(t-x_0))\cdot|t-x_0|^{\frac{\alpha}{2}}\right)^p dt\right)^{\frac{1}{p}}=$$

$$=K_1\sup_u\left(\int\limits_{u-1}^{u+1}\left(|p_n(t)|\cdot|t|^{\frac{\alpha}{2}}\cdot\exp(-\Phi(t))\right)^p dt\right)^{\frac{1}{p}}$$

where we have used that by the concavity of Φ we have

$$\exp(-\Phi(y))\exp(-\Phi(x))=\exp(-\Phi(|y|)-\Phi(|x|)\leq\exp(-\Phi(|y|+|x|))\leq$$
$$\leq\exp(-\Phi(|y+x|))=\exp(-\Phi(y+x)).$$

Thus (3.4) holds.

Lemma 3.3. *Suppose* (3.2) *is fulfilled and* Φ *is concave on* $[0,\infty)$. *Then for* $0 < p \le \infty$ *all the "norms"*

$$\sup_u \left(\int_{u-1}^{u+1} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt \right)^{\frac{1}{p}}, \quad \alpha \ge 0$$

are equivalent (uniformly in the polynomial p_n*). For* $0 < p_1 < p_2 \le \infty$ *and any polynomial* p_n *we have*

$$\left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^{p_2} dt \right)^{\frac{1}{p_2}} \le \tag{3.9}$$

$$\le K \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^{p_1} dt \right)^{\frac{1}{p_1}} \qquad K = K(p_1, p_2, \Phi).$$

Proof. The fact that all the "norms" $\sup\limits_u \left(\int\limits_{u-1}^{u+1} (\cdot)^p \right)^{\frac{1}{p}}$ are equivalent follows from (3.4). By (3.4) we have

$$\max_{x \in \mathbb{R}} |p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x)) \le$$

$$\le K \sup_u \left(\int_{u-1}^{u+1} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt \right)^{\frac{1}{p}} \le$$

$$\le K \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^p dt \right)^{\frac{1}{p}},$$

and in the case $q \ge p$ this yields

$$\left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^q dt \right)^{\frac{1}{q}} =$$

$$= \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t))\right)^{q-p+p} dt \right)^{\frac{1}{q}} \le$$

$$\leq \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t)) \right)^p dt \right)^{\frac{p}{q}} \cdot$$

$$\cdot \left(\max_{x\in\mathbb{R}} |p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x)) \right)^{\frac{q-p}{p}} \leq$$

$$\leq K \left(\int_{-\infty}^{\infty} \left(|p_n(t)| \cdot |t|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(t)) \right)^p dt \right)^{\frac{1}{p}}.$$

Lemma 3.3 is proved, and so Theorem 3 is also proved.

Proof of Theorem 4.

Lemma 4.1. *Let $a \geq 0$, $b > 0$. Then for every polynomial p_n of degree at most n we have*

$$\max_{|t|\leq \frac{b}{2}} |p_n'(t)| \cdot |t|^a \leq c\frac{n}{b} \max_{|t|\leq b} |p_n(t)| \cdot |t|^a.$$

Proof. Using the substitution $x = 2t/b$ in $(2.6')$ we have

$$\max_{\substack{t \\ \left|\frac{2t}{b}\right|\leq 1}} \left| p_n'\left(\frac{2t}{b}\right)\right| \cdot \left|\frac{2t}{pb}^a\right| \leq c\frac{n}{b} \max_{\substack{t \\ \left|\frac{2t}{b}\right|\leq 2}} \left| p_n\left(\frac{2t}{b}\right)\right| \cdot \left|\frac{2t}{pb}\right|^a,$$

i.e.

$$\max_{|t|\leq \frac{b}{2}} |p_n'(t)| \cdot |t|^a \leq c\frac{n}{b} \max_{|t|\leq b} |p_n(t)| \cdot |t|^a.$$

Lemma 4.2 ([13], p. 114, Corollary 15)**.** *Let u be the Jacobi weight $u(t) = (1-t)^a(1+t)^b$; $a, b > -1$, $0 < p < \infty$, $\varepsilon > 0$. Then for every polynomial p_n of degree at most n we have*

$$\int_{-1}^{1} |p_n(t)|^p \cdot u(t)dt \leq cn^{2\varepsilon} \int_{-1}^{1} |p_n(t)|^p \cdot u(t) \cdot (1-t^2)^{\varepsilon} dt$$

where the constant c depends only on p, a, b, ε.

Lemma 4.3. *Let u be the Jacobi weight, $0 < p < \infty$, then for every polynomial T_n of degree at most n we have*

$$\int_{-1}^{1} |T_n'(t)|^p \cdot u(t)dt \leq cn^{2p} \cdot \int_{-1}^{1} |T_n(t)|^p \cdot u(t)dt.$$

Remark. Goetgheluck proved in [14] a similar statement with $u_1^p(x)$ instead of $u(x)$, where

$$u_1(x) = (1-x)^{\alpha_1}(1+x)^{\beta_1}\prod_{i=1}^{s}|x-x_i|^{c_i}f_1(x), \alpha_1, \beta_1, c_i \geq 0.$$

Here f_1 is a measurable function that is bounded away from 0 and from infinity, and $1 \leq p \leq \infty$. In our case the weight function is of a more special kind but we allow $0 < p < \infty$.

PROOF. Using Lemma 4.2 with $p_n = T_n'$, $\varepsilon = p/2$ we get

$$\int_{-1}^{1} |T_n'(s)|^p \cdot u(s)ds \leq cn^p \int_{-1}^{1} |T_n'(s)|^p (1-s^2)^{\frac{p}{2}} u(s)ds$$

and hence, taking into account Corollary 2.1 we obtain the desired statement.

Now we turn to the proof of Theorem 4. To this end, let $Q_n(x)$ be an algebraic polynomial of degree at most n and such that Q_n is even, $Q_n(0) = 1$, further the estimate

$$|Q_n(x)| \leq K\exp(-\Phi(x)|, \quad |x| \leq n$$

is fulfilled (sse Lemma 3.1).

Let $x_0 \neq 0$ be an arbitrary fixed number and

$$Q_n^*(x) := Q_n(x - x_0).$$

In the case of $\frac{|x_0|}{n} \leq 1/4$ use Lemma 4.1 with $b = n/2$. We get

$$|p_n'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} = |(p_n Q_n^*)'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} \leq K \max_{|t|\leq \frac{n}{2}} |(p_n Q_n^*)(t)| \cdot |t|^{\frac{\alpha}{2}} \leq$$
$$\leq K \max_{|t|\leq \frac{n}{2}} |p_n(t)| \cdot \exp(-\Phi(t-x_0)) \cdot |t|^{\frac{\alpha}{2}}.$$

Multiplying this inequality by $\exp(-\Phi(x_0))$ and using concavity of Φ, we obtain

$$|p_n'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} \cdot \exp(-\Phi(x_0)) \leq K \max_{t\in\mathbb{R}} |p_n(t)| \cdot \exp(-\Phi(t)) \cdot |t|^{\frac{\alpha}{2}} \tag{4.1}$$

where $\frac{|x_0|}{n} \leq 1/4$. In the case $\frac{|x_0|}{n} > 1/4$ there exists $A_n > 1$ such that $\frac{|x_0|}{nA_n} < 1/4$. Use again Lemma 4.1 with $b = \frac{nA_n}{2}$. We obtain

$$|p_n'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} = |(p_n Q_{nA_n}^*)'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} \leq$$

$$\leq c\frac{nA_n}{nA_n}\max_{|t|\leq\frac{nA_n}{2}}|(p_nQ^*_{nA_n})(t)|\cdot|t|^{\frac{\alpha}{2}}\leq$$

$$\leq K\cdot\max_{|t|\leq\frac{nA_n}{2}}|p_n(t)|\cdot\exp(-\Phi(t-x_0))\cdot|t|^{\frac{\alpha}{2}}.$$

From this we obtain that (4.1) holds in the case $|x_0|/n>1/4$. Thus (8) is proved.

Now let us asume that (8^*) holds and let $0<p<\infty$. For the polynomial Q_n described at the beginning of the proof, we can define a constant c_4 such that

$$Q_n(x)\geq\frac{1}{2},\quad |Q_n'(x)|\leq 2\quad\text{ on }\quad[-c_4,c_4]\quad n=1,2,\dots$$

For $\tau\in[-c_4,c_4]$

$$|p_n'(\tau)|\leq 2\cdot|p_n'(\tau)Q_n(\tau)|=2|(p_nQ_n)'(\tau)-p_n\cdot Q_n'(\tau)|, \tag{4.2}$$

is fulfilled therefore by Lemma 2.5 (2.6) we obtain

$$\int_{c_4}^{c_4}|p_n'(\tau)|^p\cdot\tau|^{\frac{\alpha}{2}p}d\tau\leq K\left(\int_{-n}^{n}|(p_nQ_n)(\tau)|^p\cdot|\tau|^{\frac{\alpha}{2}p}d\tau+\right.$$

$$\left.+\int_{c_4}^{c_4}|p_n(\tau)|^p\cdot\tau|^{\frac{\alpha}{2}p}d\tau\right)\leq K\int_{-\infty}^{\infty}\left(|p_n(\tau)|\cdot\exp(-\Phi(\tau))\cdot|\tau|^{\frac{\alpha}{2}}\right)^p d\tau.$$

Now we estimate the integral $\int_{-\frac{c_4}{4}}^{\frac{c_4}{4}}|p_n'(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau$. In the case of $|y|\leq c_4/2$ we have

$$\int_{-\frac{c_4}{4}}^{\frac{c_4}{4}}|p_n'(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau=\int_{-\frac{c_4}{4}+y}^{\frac{c_4}{4}+y}|p_n'(\tau)|^p\cdot|\tau|^{\frac{\alpha}{2}p}d\tau\leq$$

$$\leq\int_{-\frac{c_4}{4}-|y|}^{\frac{c_4}{4}+y}|p_n'(\tau)|^p\cdot|\tau|^{\frac{\alpha}{2}p}d\tau\leq\int_{-c_4}^{c_4}|p_n'(\tau)|^p\cdot|\tau|^{\frac{\alpha}{2}p}dt\leq$$

$$\leq K\int_{-\infty}^{\infty}\left(|p_n(\tau)|\cdot\exp(-\Phi(\tau))\cdot|\tau|^{\frac{\alpha}{2}}\right)^p dt,$$

i.e.

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau \leq \tag{4.3}$$

$$K \int\limits_{-\infty}^{\infty} (|p_n(\tau)| \leq \cdot \exp(-\Phi(\tau)) \cdot |\tau|^{\frac{\alpha}{2}})^p dt.$$

Now we investigate the case $|y| > c_4/2$. Obviously

$$\frac{c_4}{2} \int\limits_{-\infty}^{\infty} \left(|p_n'(t)| \cdot \exp(-\Phi(t)) \cdot |t|^{\frac{\alpha}{2}}\right)^p dt = \tag{4.4}$$

$$= \lim_{N\to+\infty} \int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} \int\limits_{-N}^{N} e^{-p\Phi(y+\tau)} \cdot |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} dy\, d\tau \leq$$

$$\leq c \lim_{N\to+\infty} \int\limits_{-N}^{N} \int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} e^{-p\Phi(y)} \cdot |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy =$$

$$= c \lim_{N\to+\infty} \left(\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{2}} \int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} + \int\limits_{\frac{c_4}{2}<|y|\leq N} \int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} \right).$$

The first term in the bracket has already been estimated (see (4.3)). Now we estimate the other term. With Q_n being the polynomial described at the beginning of the proof, define

$$Q^*_{(n+2)N}(x) := Q_{(n+2)N}(x-y).$$

Use (4.2) with $p_n(y+\tau)$ instead of $p_n(\tau)$ and $Q^*_{(n+2)N}(\tau+y)$ instead of $Q_n(\tau)$. We get

$$|p_n'(y+\tau)| \leq 2|p_n'(y+\tau)Q^*_{(n+2)N}(y+\tau)| = \tag{4.5}$$

$$= 2|(p_n Q^*_{(n+2)N}),(y+\tau) - p_n(y+\tau){Q^*}'_{(n+2)N}(y+\tau)|,$$

where $t \in [-c_4, c_4]$.
Hence

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau \leq \tag{4.6}$$

$$\leq K\left(\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |(p_nQ^*_{(n+2)N})'(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau+\right.$$

$$\left.+\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot|Q^{*'}_{(n+2)N}(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau\right).$$

Here

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot|Q^{*'}_{(n+2)N}(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau\leq$$

$$\leq c\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau\leq$$

$$\leq c\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot|Q^{*}_{(n+2)N}(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau\leq$$

$$\leq c\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot\exp(-p\Phi(\tau))\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau,$$

i.e.

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot|Q^{*'}_{(n+2)N}(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}\leq \tag{4.7}$$

$$\leq c\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n(y+\tau)|^p\cdot\exp(-p\Phi(\tau))\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau.$$

On the other hand,

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |(p_nQ^*_{(n+2)N})'(y+\tau)|^p\cdot|y+\tau|^{\frac{\alpha}{2}p}d\tau=$$

$$= \int\limits_{-\frac{c_4}{4}+y}^{\frac{c_4}{4}+y} |(p_n Q^*_{(n+2)N})'(\tau)|^p \cdot |\tau|^{\frac{\alpha}{2}p} d\tau \leq$$

$$\leq \int\limits_{-\frac{3}{2}|y|}^{\frac{3}{2}|y|} |(p_n Q^*_{(n+2)N})'(\tau)|^p \cdot |\tau|^{\frac{\alpha}{2}p} d\tau \leq \int\limits_{-(n+1)N}^{(n+1)N} |(p_n Q^*_{(n+2)N})'(\tau)|^p \cdot |\tau|^{\frac{\alpha}{2}p} d\tau.$$

Using Lemma 2.5 with $a = \frac{\alpha}{2}p$, we obtain

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |(p_n Q^*_{(n+2)N})'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau \leq \tag{4.8}$$

$$\leq c \int\limits_{-(n+1)N}^{(n+1)N} |(p_n Q^*_{(n+2)N})(\tau)|^p \cdot |\tau|^{\frac{\alpha}{2}p} d\tau.$$

Here

$$\int\limits_{-(n+1)N}^{(n+1)N} |(p_n Q^*_{(n+2)N})(\tau)|^p \cdot |\tau|^{\frac{\alpha}{2}p} d\tau \leq \tag{4.9}$$

$$\leq \int\limits_{-(n+2)N}^{(n+2)N} |(p_n Q^*_{(n+2)N})(\tau+y)|^p \cdot |\tau+y|^{\frac{\alpha}{2}p} d\tau \leq$$

$$\leq \int\limits_{-(n+2)N}^{(n+2)N} |(p_n(\tau+y)|^p \cdot \exp(-p\Phi(\tau)) \cdot |\tau+y|^{\frac{\alpha}{2}p} d\tau.$$

From (4.7) and (4.9) we obtain in the case $\dfrac{c_4}{2} < |y| \leq N$, that

$$\int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau \leq \tag{4.10}$$

$$\leq c \int\limits_{-\frac{c_4}{4}}^{\frac{c_4}{4}} |(p_n(y+\tau)|^p \cdot \exp(-p\Phi(\tau)) \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau +$$

$$+\int_{-(n+2)N}^{(n+2)N} |(p_n(y+\tau)|^p \cdot \exp(-p\Phi(\tau)) \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau \le$$

$$\le c \int_{-\infty}^{\infty} |p_n(y+\tau)|^p \cdot \exp(-p\Phi(\tau)) \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau.$$

Using (4.3) and (4.10) we have (remember (4.4))

(4.11)

$$\left(\int_{-\frac{c_4}{2}}^{\frac{c_4}{2}} \int_{-\frac{c_4}{4}}^{\frac{c_4}{4}} + \int_{\frac{c_4}{2}<|y|\le N} \int_{-\frac{c_4}{4}}^{\frac{c_4}{4}} \right) e^{-p\Phi(y)} \cdot |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy \le$$

$$\le c \int_{-\frac{c_4}{4}}^{\frac{c_4}{2}} e^{-p\Phi(y)} \int_{-\infty}^{\infty} \left(|p_n(\tau) \cdot \exp(-\Phi(\tau)) \cdot |\tau|^{\frac{\alpha}{2}} \right)^p d\tau dy +$$

$$+c \int_{\frac{c_4}{2}<|y|\le N} e^{-p\Phi(y)} \int_{-\infty}^{\infty} |p_n(y+\tau)|^p \cdot \exp(-p\Phi(\tau)) \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy.$$

Here

$$\int_{\frac{c_4}{2}<|y|\le N} e^{-p\Phi(y)} \int_{-\infty}^{\infty} |p_n(y+\tau)|^p \cdot \exp(-p\Phi(\tau)) \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy \le$$

$$\le \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-p\Phi(y)-p\Phi(\tau)} \cdot |p_n(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy =$$

$$= \int_{-\infty}^{\infty} e^{-p\Phi(u)} \cdot |p_n(u)|^p \cdot |u|^{\frac{\alpha}{2}p} \cdot \int_{-\infty}^{\infty} \exp(p(\Phi(u)-\Phi(x)-\Phi(u-x)))dx\, du.$$

We shall show that

$$\int_{-\infty}^{\infty} \exp(p(\Phi(u)-\Phi(x)-\Phi(u-x)))dx \le c = c(p,\Phi),$$

independently of u.

Next we use the idea of [5], p. 123. We may suppose without loss of generality that $u \geq 0$. Then

$$\int_{-\infty}^{0} \exp(p(\Phi(u) - \Phi(x) - \Phi(u-x)))dx \leq \int_{-\infty}^{0} \exp(-(p\Phi(x))dx \leq c,$$

and similarly

$$\int_{u}^{+\infty} \exp(p(\Phi(u) - \Phi(x) - \Phi(u-x))) \leq \int_{0}^{+\infty} \exp(-(p\Phi(x))dx \leq c.$$

By the concavity of Φ, for any fixed x the difference $\Phi(u) - \Phi(u-x)$ decreases as $u > x$ increases. Hence $\Phi(u) - \Phi(u-x) - \Phi(x) \leq \Phi(2x) - 2\Phi(x)$, $(0 \leq x \leq \frac{u}{2})$, and using also (8^*) we obtain

$$\int_{0}^{\infty} \exp(p(\Phi(u) - \Phi(x) - \Phi(u-x)))dx =$$

$$= 2\int_{0}^{\frac{u}{2}} \exp(p(\Phi(u) - \Phi(x) - \Phi(u-x)))dx \leq 2\int_{0}^{\frac{u}{2}} e^{p(\Phi(2x) - 2\Phi(x))}dx \leq$$

$$\leq 2\int_{0}^{\infty} e^{p(\Phi(2x) - 2\Phi(x))}dx < +\infty.$$

Hence from (4.4) and (4.11)

$$\frac{c_4}{2}\int_{-\infty}^{\infty} \left(|p_n'(t)| \cdot \exp(-\Phi(t)) \cdot |t|^{\frac{\alpha}{2}}\right)^p dt =$$

$$= c \lim_{N\to+\infty} \left(\int_{-\frac{c_4}{2}}^{\frac{c_4}{2}} \int_{-\frac{c_4}{4}}^{\frac{c_4}{4}} + \int_{\frac{c_4}{2} < |y| \leq N} \int_{-\frac{c_4}{4}}^{\frac{c_4}{4}} \right)$$

$$e^{-p\Phi(y)} \cdot |p_n'(y+\tau)|^p \cdot |y+\tau|^{\frac{\alpha}{2}p} d\tau dy \leq$$

$$\leq c \lim_{N\to+\infty} \int_{-\infty}^{\infty} e^{-p\Phi(u)} \cdot |p_n(u)|^p \cdot |u|^{\frac{\alpha}{2}p} du =$$

$$= c \int_{-\infty}^{\infty} \left(|p_n(t)| \cdot \exp(-\Phi(t)) \cdot |t|^{\frac{\alpha}{2}}\right)^p dt.$$

Theorem 4 is proved.

Proof of (5) of Theorem 2.

Lemma 4.4. *Let $\alpha \geq 0$ be any fixed number. Then for any polynomial p_n of degree at most n we have*

$$\max_{x\in\mathbb{R}} \exp(-|x|)\cdot|p_n'(x)|\cdot|x|^{\frac{\alpha}{2}} \leq c\log n\cdot\max_{x\in\mathbb{R}} \exp(-|x|)\cdot|p_n(x)|\cdot|x|^{\frac{\alpha}{2}}. \tag{4.12}$$

Proof. It is similar to that of Lemma 3.2. Let $Q_N(x)$ be a polynomial such that its degree is at most n, $N = [10\pi n\log n]$ and $Q_N(0)$=1, $|Q_n(x)| \leq ce^{-|x|}$, $(|x| \leq (3/2)n)$, (see [5], p. 124). Now we choose $Q_N^*(x) := Q_N(x - x_0)$, where $x_0 \neq 0$ is an arbitrary fixed number.

In the case $|x_0| \leq 1$ we use Lemma 2.5 (2.6′) with $a = \frac{\alpha}{2}$. We get

$$\begin{aligned} |p_n'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} &= |(p_nQ_N^*)'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} \leq \\ &\leq c\frac{n+N}{n} \max_{|x|\leq n} |(p_nQ_N^*)(x)| \cdot |x|^{\frac{\alpha}{2}} \leq \\ &\leq c\log n \cdot \max_{x\in\mathbb{R}} |p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-|x-x_0|) = \\ &= c\log \cdot \max_{x\in\mathbb{R}} |p_n(x-x_0)| \cdot |x-x_0|^{\frac{\alpha}{2}} \cdot \exp(-|x-2x_0|). \end{aligned} \tag{4.12}$$

Multiplying this inequality by $\exp(-|x_0|)$ we obtain

$$\begin{aligned} &|p_n'(x_0)| \cdot |x_0|^{\frac{\alpha}{2}} \cdot \exp(-|x_0|) \leq \\ &\leq c\log n \cdot \max_{x\in\mathbb{R}} |p_n(x-x_0)| \cdot |x-x_0|^{\frac{\alpha}{2}} \cdot \exp(-|x-2x_0|) \cdot \exp(-|x_0|) \leq \\ &\leq c\log n \cdot \max_{x\in\mathbb{R}} |p_n(x-x_0)| \cdot |x-x_0|^{\frac{\alpha}{2}} \cdot \exp(-|x-x_0|) = \\ &= c\log n \max_{x\in\mathbb{R}} |p_n(x)| \cdot |x|^{\frac{\alpha}{2}} \cdot \exp(-|x|), \end{aligned}$$

i.e. if $|x_0| \leq 1$ then

$$|p_n'(x_0)|\cdot|x_0|^{\frac{\alpha}{2}}\cdot\exp(-|x_0|) \leq c\log n\cdot\max_{x\in\mathbb{R}} |p_n(x)|\cdot|x|^{\frac{\alpha}{2}}\cdot\exp(-|x|). \tag{4.13}$$

In the case of $|x_0| > 1$ there exists $A > 1$ such that $\frac{|x_0|}{A} \leq 1$. Repeating the idea of the case $|x_0| \leq 1$ we obtain that (4.13) is true for every $x_0 \neq 0$, thus Lemma 4.4 is proved.

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(Received February 3, 1992; revised April 13, 1993)