

Small perturbation of normally solvable relations

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Abstract. In the present paper we investigate the stability of the index, the nullity and the deficiency of normally solvable linear relations in paracomplete spaces under small perturbation. Results of Kato and of Goldberg about operators in complete spaces are covered.

1. Introduction

A theorem due to KATO [15] proves that if X and Y are Banach spaces, S and T are operators with domain in X and range in Y such that $D(T) \subset D(S)$, T is normally solvable with an index and $\|S\| < \gamma(T)$, then $T + S$ is normally solvable and the index of $T + S$ coincides with the index of T . Subsequently, this result was partially generalized by CROSS [7] for the case when T is a multivalued linear operator and S is an operator. In [11], GOLDBERG proves that if X , Y , T and S satisfy the hypothesis in Kato's theorem, then there exists $\eta > 0$ such that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \eta$.

The purpose of this paper is to extend the results of the type mentioned above to multivalued linear relations in paracomplete spaces.

To make the paper easily accessible some results from the theory of linear relations in normed spaces due to CROSS [7] are recalled in Section 2. In particular, results concerning the adjoint and the norm of a linear relation and some small perturbation results are presented. Section 3 is devoted to the stability of

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normally solvable linear relations satisfying some conditions by small perturbations.

Linear relations made their appearance in Functional Analysis in J. VON NEUMANN [19] motivated by the need to consider adjoints of non-densely defined operators used in applications to the theory of generalized equations [6] and also by the need to consider the inverses of certain operators, used, for example, in the study of some Cauchy problems associated to parabolic type equations in Banach spaces [8]. Interesting works on multivalued linear operators include the treatment of degenerate boundary value problems (see, for instance, [9] and [13]), the development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory (see, for instance, [1] and [12]), the application of multivalued methods to invariant subspace problem (see, [14] and [20]), the application of the spectral theory of linear relations to the study of many problems of operators, as, for example, the spectral theory of ordered pair of operators and of linear bundles (see, for instance, [5] and the references therein) and several papers on linear relations type semiFredholm and other classes related to them (see, for instance, [3], [4] and [7] among others).

Recall that a normed space $(X, \|\cdot\|)$ is called paracomplete or operator range if there exists a stronger norm $\|\cdot\|_s$ on X such that the space $X_s := (X, \|\cdot\|_s)$ is complete.

There are many motivations for the investigation of paracomplete spaces. We cited some of them.

1. The notion of paracomplete subspace of a Banach space is a good generalization of closed subspace. Indeed, the sum of two closed subspaces need not be closed but the sum of two paracomplete subspaces is again a paracomplete subspace. Many subspaces of a Banach space are paracomplete; for example, the domain and the range of a closed linear relation.

2. Many incomplete normed spaces appearing in applications are paracomplete. For example, the space $C[0, 1]$ with the norm of $L_2[0, 1]$ or some Sobolev spaces with suitable L_2 -norms.

3. Compactness of the spectrum of a bounded operator on a complex paracomplete space. It is very known that if T is a bounded operator on a complex Banach space X , then its spectrum is a compact set, but this property is not true if X is incomplete. In [2] it is proved that bounded operators on complex paracomplete spaces have a compact spectrum.

4. Applications related to reductivity, reflexivity and invariant subspaces of operators algebras. For example, the famous Burnside theorem on invariant

subspaces of algebras of operators in finite dimensional spaces admits an adequate generalization to strongly closed algebras of operators on Hilbert spaces in terms of invariant paracomplete spaces (see, for instance, [10]).

It is important to remark a recent work of LABROUSSE, SANDOVICI, DE SNOO and WINKLER [17]. In this interesting paper the authors proves that many of the results of LABROUSSE [16] for quasi-Fredholm operators remain valid in the context of multivalued linear operators in Hilbert spaces. We note that some of the results of our paper are closely related to results of [17].

2. Preliminary and auxiliary results

In this Section we collect some results of the theory of linear relations needed in the sequel, in the attempt of making our paper as selfcontained as possible. Before beginning let us recall some basic definitions following the notation and terminology of the book [7]. Let X, Y, \dots denote infinite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . A linear relation or multivalued linear operator $T : X \rightarrow Y$ is a mapping from a subspace $D(T) \subset X$, called the domain of T , into the collection of nonempty subsets of Y such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero scalars α, β and $x_1, x_2 \in D(T)$. The class of such linear relations T is denoted by $LR(X, Y)$. If T maps the points of its domain to singletons, then T is said to be a single valued or simply an operator. We note that T is single valued if and only if $T(0) = \{0\}$. A linear relation $T \in LR(X, Y)$ is uniquely determined by its graph, $G(T)$, which is defined by $G(T) := \{(x, y) \in X \times Y : x \in D(T), y \in Tx\}$.

Let $T \in LR(X, Y)$. The inverse of T is the linear relation T^{-1} given by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. The subspace $T^{-1}(0)$, denoted by $N(T)$, is called the null space of T and we say that T is injective if $N(T) = \{0\}$. The range of T is the subspace $R(T) := T(D(T))$ and T is said to be surjective if its range coincides with Y . The quantities $\alpha(T) := \dim N(T)$ and $\beta(T) := \dim Y/R(T)$ are called the nullity and the deficiency of T , respectively. We also write $\bar{\beta}(T) := \dim Y/\overline{R(T)}$ and the index of T is defined by $k(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite. If $\alpha(T)$ and $\beta(T)$ are both infinite, then T is said to have no index.

Let M be a subspace of X such that $M \cap D(T) \neq \emptyset$. Then the restriction $T|_M$ is the linear relation given by $G(T|_M) := \{(m, y) : m \in M, y \in Tm\}$. We note that $T|_M \in LR(X, Y)$ but $TJ_M \in LR(M, Y)$ where J_M denotes the natural injection map of M into X . Let $S, T \in LR(X, Y)$. The sum $T + S$ is the linear relation given by $G(T + S) := \{(x, y + z) : (x, y) \in G(T), (x, z) \in G(S)\}$. Let

$T \in L(X, Y)$, $S \in LR(Y, Z)$ such that $R(T) \cap D(S) \neq \emptyset$, then the composition ST is the linear relation given by $G(ST) := \{(x, z) : (x, y) \in G(T), (y, z) \in G(S) \text{ for some } y \in Y\}$.

In the sequel X and Y will denote infinite dimensional normed spaces. For a given closed subspace M of X let Q_M denote the natural quotient map from X onto X/M . If $T \in LR(X, Y)$ then we shall denote $Q_{\overline{T(0)}}$ by Q_T . Clearly $Q_T T$ is single valued. For $x \in D(T)$, $\|Tx\| := \|Q_T Tx\|$ and the norm of T is defined by $\|T\| := \|Q_T T\|$. We note that $\|\cdot\|$ is not a true norm since $\|T\| = 0$ does not imply $T = 0$. Let M and N be subspaces of X and X' (the dual space of X) respectively. Then $M^\perp := \{x' \in X' : x'(M) = 0\}$ and $N^\top := \{x \in X : N(x) = 0\}$. The adjoint T' of T is defined by $G(T') := G(-T^{-1})^\perp \subset Y' \times X'$. This means that $(y', x') \in G(T')$ if and only if $y'(y) - x'(x) = 0$ for all $(x, y) \in G(T)$.

Let $T \in LR(X, Y)$. We say that T is closed if its graph is a closed subspace, normally solvable if it is closed with closed range, continuous if for each neighbourhood V in $R(T)$, the inverse image $T^{-1}(V)$ is a neighbourhood in $D(T)$, open if its inverse is continuous and T is called bounded if $D(T) = X$ and T is continuous.

The above classes of linear relations can be characterized as follows:

Lemma 1 ([7, II.3.2 and II.53]). *Let $T \in LR(X, Y)$. Then*

- (i) *T is continuous if and only if $\|T\| < \infty$.*
- (ii) *T is open if and only if $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\}$ is a positive number.*
- (iii) *T is closed if and only if $Q_T T$ is a closed operator and $T(0)$ is a closed subspace.*

There exist closed (respectively, continuous) linear relations S and T such that ST is not closed (respectively, continuous). We shall use the following result which gives sufficient conditions for the composition of two closed (respectively, continuous) linear relations to be closed (respectively, continuous).

Lemma 2 ([7, II.3.13 and II.5.18]). *Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. We have:*

- (i) *If $T(0) \subset D(S)$, then $\|ST\| \leq \|S\|\|T\|$.*
- (ii) *If S is closed and T is a bounded single valued, then ST is closed.*

We list the following useful properties of the adjoint of a linear relation.

Proposition 3 ([7, III.1.4, III.1.5 and III.4.6]). *Let $T \in LR(X, Y)$. We have:*

- (i) T' is a closed linear relation such that $T'(0) = D(T)^\perp$, $N(T') = R(T)^\perp$ and if T is closed then $T(0) = D(T')^\top$.
- (ii) If $S \in LR(X, Y)$ is continuous and $D(T) \subset D(S)$, then $(T + S)' = T' + S'$.
- (iii) T is continuous if and only if $D(T') = T(0)^\perp$. In such case T' is continuous and $\|T\| = \|T'\|$.
- (iv) T is open if and only if $R(T') = N(T)^\perp$. In such case $\gamma(T) = \gamma(T')$.

We shall make frequent use of the following result which is the multivalued version of the corresponding result for operators.

Proposition 4 ([7, III.5.3 and III.5.4]). *Let $T \in LR(X, Y)$ be closed. We have:*

- (i) If X is complete, then $\gamma(T) = \gamma(T')$. Moreover, $R(T)$ is closed if T is open.
- (ii) *Closed Graph Theorem:* If X and Y are complete and $D(T)$ is closed, then T is continuous.
- (iii) *Closed Range Theorem:* If X and Y are complete, then T is open if and only if T' is open if and only if $R(T)$ is closed if and only if $R(T')$ is closed.

The next proposition investigates the stability of certain Fredholm type properties of a linear relation under small perturbation.

Proposition 5 ([7, III.7.4, III.7.5 and III.7.6]). *Let $S, T \in LR(X, Y)$ such that $\gamma(T) > 0$, $S(0) \subset \overline{T(0)}$, $D(T) \subset D(S)$ and $\|S\| < \gamma(T)$. Then*

- (i) $\alpha(T + S) \leq \alpha(T)$ and $\overline{\beta}(T + S) \leq \overline{\beta}(T)$.
- (ii) $R(T)$ dense implies $R(T + S)$ dense.
- (iii) If T is injective, then $T + S$ is open and $\overline{\beta}(T + S) = \overline{\beta}(T)$.

Finally, we conclude this Section with a result for future use.

Proposition 6 ([7, I.6.1 and V.15.5]). *We have:*

- (i) *Invariance of finite codimensionality:* Let $T \in LR(X, Y)$ and let M be a subspace of X . Then $\dim R(T)/TM \leq \dim D(T)/D(T) \cap M \leq \dim X/M$.
- (ii) *Finite dimensional extensions:* Let $S, T \in LR(X, Y)$ and let S be an extension of T (that is, $S|_{D(T)} = T$) such that $\dim D(S)/D(T) := n < \infty$ and T has an index. Then $k(S) = k(T) + n$.

3. Small perturbation of a normally solvable linear relation

In this Section we investigate the stability of the index, nullity and the deficiency of a normally solvable linear relation in paracomplete spaces under small perturbation. For this end, we first prove some auxiliary results.

Let X be a paracomplete space and let $\|\cdot\|_s$ be a stronger norm on X such that $X_s := (X, \|\cdot\|_s)$ is complete. We will denote the inclusion map from X_s onto X by α_X and the inverse of α_X by β_X . It is clear that α_X and β_X are bijective everywhere defined and closed with α_X bounded and β_X open.

Lemma 7. *Let $T \in LR(X, Y)$. We have:*

- (i) *If X is paracomplete, then $R(T) = R(T\alpha_X)$, $\alpha(T) = \alpha(T\alpha_X)$ and $T\alpha_X$ is closed if so is T .*
- (ii) *If Y is paracomplete, then $N(T) = N(\beta_Y T)$, $\beta(T) = \beta(\beta_Y T)$ and $\beta_Y T$ is closed if so is T .*

PROOF. (i) $R(T\alpha_X) = T\alpha_X D(T\alpha_X) = T\alpha_X \alpha_X^{-1} D(T) = TD(T) = R(T)$. $N(T\alpha_X) = (T\alpha_X)^{-1}(0) = \alpha_X^{-1} T^{-1}(0) = \beta_X N(T)$ and since β_X is bijective and open we infer that $\dim N(T) = \dim N(T\alpha_X)$.

Assume now that T is closed. Then, since α_X is a bounded operator it follows from Lemma 2 that $T\alpha_X$ is closed.

(ii) $N(\beta_Y T) = (\beta_Y T)^{-1}(0) = T^{-1}N(\beta_Y) = N(T)$. This last property combined with the equality $\beta(\beta_Y T) + \alpha(T) + \alpha(\beta_Y) = \alpha(\beta_Y T) + \beta(T) + \beta(\beta_Y) + \dim\{T(0) \cap N(\beta_Y)\}$ ([7, I.6.11]) leads to $\beta(T) = \beta(\beta_Y T)$.

Suppose now that T is closed equivalently T^{-1} is closed and thus we infer from (i) that $T^{-1}\alpha_Y$ is closed, that is, $T^{-1}\beta_Y^{-1} = (\beta_Y T)^{-1}$ is closed equivalently $\beta_Y T$ is closed. \square

Proposition 8 (Generalized Closed Graph and Open Mapping Theorem). *Let $T \in LR(X, Y)$ be closed. We have:*

- (i) *If X is complete, Y is paracomplete and $D(T)$ is closed, then T is continuous.*
- (ii) *If X is paracomplete, Y is complete and $R(T)$ is closed, then T is open.*

PROOF. (i) By Lemma 7, $\beta_Y T \in LR(Y, Z)$ is closed, $D(T) = D(\beta_Y T)$ and Y_s obviously complete. Then by Proposition 4, $\beta_Y T$ is continuous and since $T = \alpha_Y \beta_Y T$ and α_Y is bounded it follows from Lemma 2 that T is continuous.

(ii) Follows from the part (i) upon substituting T^{-1} for T . \square

As an immediate consequence of Proposition 4 we have the following result of duality

Corollary 9. *Let X be complete and let $T \in LR(X, Y)$ be closed such that T' is normally solvable. Then T is normally solvable.*

Example 10 below shows that Corollary 9 fails if X is not complete.

Example 10. Let $X = c_{oo}$ be the space of all scalar sequences which at most finitely many nonzero coordinates normed by the norm $\|(\alpha_n)\| = \sup\{|\alpha_n| : n \in \mathbb{N}\}$ and we define S by $S : (\alpha_1, \alpha_2, \dots, \alpha_n, \dots) \in c_{oo} \rightarrow (0, \alpha_1, \alpha_2/2, \dots, \alpha_n/n, \dots) \in c_{oo}$.

Then, it is easy to see (see, for instance, [18]) that S is a precompact operator such that $R(I - S)$ is a proper dense subspace of X . Therefore $T := I - S$ is not normally solvable and clearly $T' = I - S'$ is normally solvable since S' is a bounded compact operator.

Proposition 11. *Let $T \in LR(X, Y)$. Then*

- (i) $\alpha(T') = \bar{\beta}(T)$.
- (ii) *If X is paracompact, Y is complete and T is normally solvable then T' is normally solvable and if T has an index then $k(T) = -k(T')$.*

PROOF. (i) $\alpha(T') = \dim N(T') = \dim R(T)^\perp$ (Proposition 3) = $\dim(Y/\overline{R(T)})' = \dim Y/\overline{R(T)} = \bar{\beta}(T)$.

(ii) By Proposition 8, T is open equivalently $R(T') = N(T)^\perp$ (Proposition 3) so that $R(T')$ is closed and $\beta(T') = \dim X'/R(T') = \dim X'/N(T)^\perp = \dim N(T)' = \dim N(T) = \alpha(T)$. This last property together with the part (i) implies that if $k(T)$ exists then $k(T) = -k(T')$. \square

The following example illustrates that the completeness of Y is essential in the part (ii) of Proposition 11.

Example 12. Let $T \in LR(X, X)$ be an injective everywhere defined precompact operator where X is complete, $Y := R(S)$ and let T be the operator S considered as an element of $LR(X, Y)$.

It is clear that T is closed, injective and surjective and hence T is normally solvable with $k(T) = 0$. However, T' is not normally solvable since clearly T' is compact.

The following lemma is elementary but it is essential to prove Theorem 16.

Lemma 13. *Let M and N be subspaces of X such that $M \subset N$ and M is closed. Then*

- (i) *N is closed if and only if N/M is closed.*
- (ii) *If N is closed then $X/N \cong (X/M)/(N/M)$ and $Q_N = Q_{N/M}Q_M$ where \cong is a canonical isometry.*

Lemma 14. *Let $T \in LR(X, Y)$ be closed and let Y be complete. If $S \in LR(X, Y)$ is continuous with $S(0) \subset T(0)$ and $D(T) \subset D(S)$, then $T+S$ is closed.*

PROOF. Suppose S and T single valued and let (x_n) be a sequence in $D(T+S) = D(T) \cap D(S) = D(T)$ (as $D(T) \subset D(S)$) such that $x_n \rightarrow x$ and $(T+S)x_n \rightarrow y$ for some $x \in X$ and $y \in Y$. Then for $m, n \in \mathbb{N}$ we have that

$$\|Tx_n - Tx_m\| \leq \|(T+S)(x_n - x_m)\| + \|S\| \|x_n - x_m\|$$

which implies that (Tx_n) is a Cauchy sequence in the Banach space Y , and hence there exists $z \in Y$ such that $Tx_n \rightarrow z$. Since T is closed, it follows that $x \in D(T)$ and $Tx = z$. Since S is continuous with $D(T) \subset D(S)$, we have that $Sx_n \rightarrow Sx$. Therefore $(T+S)x_n \rightarrow (T+S)x = y$, that is, $T+S$ is closed.

Passing to the general case, it follows from Lemma 1 that $Q_T T$ is a closed operator and $T(0)$ is a closed subspace. Furthermore, as $S(0) \subset \overline{T(0)} = T(0)$ is $(T+S)(0) = T(0)$ (so that $Q_{T+S} = Q_T$) and we deduce from Lemma 13 that $Q_T = Q_{\overline{S(0)}/T(0)} Q_S$ and thus we deduce from Lemma 1 that $Q_T S$ is a continuous single valued. By what has already been shown, $Q_{T+S}(T+S) = Q_T T + Q_T S$ is closed. Applying again Lemma 1, $T+S$ is closed, as desired. \square

Now, we are in the position to give the main theorems of this paper.

Theorem 15. *Let X be paracomplete, Y complete and let $T \in LR(X, Y)$ be normally solvable with an index. Then for any $S \in LR(X, Y)$ satisfying $D(T) \subset D(S)$, $S(0) \subset T(0)$ and $\|S\| < (1/\|\alpha_X\|)\gamma(T\alpha_X)$. Then $T+S$ is normally solvable and $k(T) = k(T+S)$.*

PROOF. By Proposition 4 and Lemma 7, $T\alpha_X$ is closed and open with $\alpha(T) = \alpha(T\alpha_X)$ and $\beta(T) = \beta(T\alpha_X)$. Furthermore, $T+S$ is closed by virtue of Lemma 14 and we observe that by Lemma 2, $\|S\alpha_X\| \leq \|S\|\|\alpha_X\|$, so that $S\alpha_X$ is continuous and $\|S\alpha_X\| < \gamma(T\alpha_X)$. Hence the proof can be reduced to the case when X and Y are complete.

(i) Let us consider two cases for T :

Case 1: $\dim N(T) < \infty$. Then there exists a closed finite codimensional subspace M of $D(T)$ such that $T|_M$ is injective and open. Since M is closed, $T|_M$ is closed and it follows from Proposition 4 that $R(T|_M)$ is closed. We first deduce the conclusion for the case $\|S\| < \gamma(T|_M)$. Applying Proposition 5 we obtain that $(T+S)|_M$ is injective, open and $\overline{\beta}((T+S)|_M) = \overline{\beta}(T|_M)$ and thus since $(T+S)|_M$ is closed applying Proposition 4 again, $R((T+S)|_M)$ is closed.

In consequence

$$0 = \alpha(T|_M) = \alpha((T + S)|_M) \quad \text{and} \\ \bar{\beta}(T|_M) = \beta(T|_M) = \beta((T + S)|_M) = \bar{\beta}((T + S)|_M).$$

A combination of these equalities and Proposition 6 leads to

$$k(T + S) = k((T + S)|_M) + \alpha(T) = k(T|_M) + \alpha(T) = k(T)$$

provided $\|S\| < \gamma(T|_M)$.

Passing to the case $\|S\| < \gamma(T)$, let I denote the closed interval $[0, 1]$ with the usual topology and let $\mathcal{Z} := \mathbb{Z} \cup \{-\infty, +\infty\}$ with the discrete topology. Define

$$\phi : I \rightarrow \mathcal{Z} \text{ by } \phi(\lambda) := k(T + \lambda S).$$

It follows from the above that

$$\phi(\lambda) = k(T + \lambda_o S + (\lambda - \lambda_o)S) = k(T + \lambda_o S) = \phi(\lambda_o)$$

provided λ_o is sufficiently close to λ .

Therefore, ϕ is a continuous map. Consequently $\phi(I)$ is a connected set and hence consists of just one point. Hence

$$k(T) = \phi(0) = \phi(1) = k(T + S).$$

Case 2: $\dim Y/R(T) < \infty$. By Propositions 3 and 4 we have that

$$T' \text{ is normally solvable, } \alpha(T') < \infty, (T + S)' = T' + S' \\ \text{and } \|S'\| < \gamma(T'). \quad (1)$$

Furthermore, since $S'(0) = D(S)^\perp$ and $T'(0) = D(T)^\perp$ (again Proposition 3) and $D(T) \subset D(S)$ it follows trivially that

$$S'(0) \subset T'(0). \quad (2)$$

We note that as $S(0) \subset T(0)$ we have that $D(T') \subset (D(T'))^\top{}^\perp = T(0)^\perp \subset S(0)^\perp = D(S')$ (again Proposition 3). Hence

$$D(T') \subset D(S'). \quad (3)$$

From (1), (2) and (3) we can apply the Case 1 to T' and S' and then it follows that $R(T' + S')$ is closed and $k(T' + S') = k(T')$. Therefore

$$k(T) = -k(T') = -k(T' + S') = -k((T + S)') = k(T + S). \quad \square$$

Theorem 16. *Let $X, Y, T,$ and S satisfy the hypothesis in Theorem 15 and suppose that $S(0)$ is closed. Then there exists a number $\eta > 0$ such that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \eta$.*

PROOF. Let us consider two possibilities for T :

Case 1: $\dim N(T) < \infty$. Let $\lambda \neq 0$ and let $x \in N(T + \lambda S)$. Then

$$-\lambda Sx \subset Tx$$

whence

$$Sx \subset R(T) := R_1 \quad \text{and} \quad x \in S^{-1}R_1 := D_1.$$

Thus

$$-\lambda Sx \subset Tx \subset TD_1 := R_2 \quad \text{and} \quad x \in S^{-1}R_2 := D_2.$$

Continuing in this way, we obtain

$$R_{k+1} := TD_k \quad \text{where} \quad D_k := S^{-1}R_k.$$

It follows from the construction that

$$(R_n) \text{ and } (D_n) \text{ are decreasing sequences and } N(T + \lambda S) \subset \bigcap_{k=1}^{\infty} D_k. \quad (4)$$

Now we shall see that

$$\text{For } n \in \mathbb{N}, R_n \text{ and } D_n \text{ are closed subspaces of } X \text{ and } Y \text{ respectively.} \quad (5)$$

Before beginning we recall the following entirely algebraic property due to Cross [7, I.3.1]

(*) Let $T \in LR(X, Y)$ where X and Y are vector spaces and let M be a subspace of X . Then $T^{-1}TM = \{M \cap D(T)\} + N(T)$.

The proof of (5) is by induction. Assume that $n = 1$. Then $R_1 := R(T)$ is closed (Proposition 4) and we have that $Q_S R_1 = (R(T) + S(0))/S(0) = R(T)/S(0)$ (as $S(0) = \overline{S(0)} \subset T(0) \subset R(T)$) and this subspace is closed by virtue of Lemma 13. This last property combined with the fact that $Q_S S$ is a continuous operator (Lemma 1) leads to $(Q_S S)^{-1} Q_S R_1$ is closed and since

$$\begin{aligned} (Q_S S)^{-1} Q_S R_1 &= S^{-1} Q_S^{-1} Q_S R_1 = S^{-1} \{ \{R_1 \cap D(Q_S)\} + N(Q_S) \} \quad (**) \\ &= S^{-1} \{ R(T) + \overline{S(0)} \} = S^{-1} R(T) \quad (\text{as } \overline{S(0)} = S(0) \subset T(0) \subset R(T)) = S^{-1} R_1 \end{aligned}$$

we conclude that D_1 is closed.

Suppose now that R_n and D_n are closed. We first prove that R_{n+1} is closed. For this, we show that

$$T|_{(N(T)+D_n)} \text{ is closed.} \tag{a}$$

Indeed, T is closed and $N(T) + D_n$ is closed (since $N(T)$ is finite dimensional and D_n is closed by the induction hypothesis).

$$T|_{(N(T)+D_n)} \text{ is open.} \tag{b}$$

It is sufficient to observe that as T and $T|_{(N(T)+D_n)}$ have the same null space, $\gamma(T) \leq \gamma(T|_{(N(T)+D_n)})$.

The properties (a) and (b) combined with Proposition 4 lead to that $R(T|_{(N(T)+D_n)})$ is closed. Now, that R_{n+1} is closed follows immediately upon observing that

$$\begin{aligned} R(T|_{(N(T)+D_n)}) &= T(N(T) + D_n) = TT^{-1}(0) + TD_n = T(0) + TD_n \\ &= TD_n := R_{n+1}. \end{aligned}$$

To show that D_{n+1} is closed we can proceed exactly as in the case $n = 1$ using that R_n is closed.

Define

$$D_o := \bigcap_{n=1}^{\infty} D_n \quad \text{and} \quad R_o := \bigcap_{n=1}^{\infty} R_n$$

and

$$T_o := T|_{(D(T) \cap D_o)} \quad \text{and} \quad S_o := S|_{(D(T) \cap D_o)}.$$

By the definitions it follows that $R(T_o) \subset R_o$ and $R(S_o) \subset R_o$, and since T is closed and D_o is closed, T_o is a closed linear relation. To see that T_o is surjective, let $y \in R_o = R(T) \cap (\bigcap_{n=1}^{\infty} TD_n)$. Then for each $n \geq 1$, there exists $x_n \in D_n$ such that $y \in TD_n$. Since $\alpha(T) < \infty$ and $D_{n+1} \subset D_n$, there exists m such that for $n \geq m$,

$$N(T) \cap D_m = N(T) \cap D_n$$

and for $x_m \in D_m$, and $x_n \in D_n$,

$$x_n - x_m \in N(T) \cap D_m = N(T) \cap D_n.$$

From this it follows that

$$x_m \in \bigcap_{n \geq m} D_n = D_o, \quad \text{and} \quad y \in Tx_m.$$

Thus T_o is surjective and so by Open Mapping Theorem (Proposition 4), T_o is open.

By Theorem 15, there exists a number $\eta > 0$ such that for $0 < |\lambda| < \eta$ we have

$$k(T + \lambda S) = k(T). \quad (6)$$

Since

$$\beta(T_o + \lambda S_o) \leq \beta(T_o) = \bar{\beta}(T_o) = 0, \text{ it follows that } \beta(T_o + \lambda S_o) = 0, \quad (7)$$

and hence

$$\alpha(T_o + \lambda S_o) = k(T_o + \lambda S_o) = k(T_o) = \alpha(T_o). \quad (8)$$

By (4), it follows that for $\lambda \neq 0$,

$$N(T + \lambda S) = N(T_o + \lambda S_o). \quad (9)$$

By (6), (7), (8) and (9) we conclude that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \eta$ if $\dim N(T) < \infty$.

Case 2: $\dim Y/R(T) < \infty$. The result is obtained by passing to the adjoints, arguing as in the proof of Case 2 of the above Theorem. \square

Corollary 17. *Let X be paracomplete, Y complete and let $S, T \in LR(X, Y)$ such that $D(T) \subset D(S)$, $S(0) = \overline{S(0)} \subset T(0)$ and S is continuous. Define U to be the set of $\lambda \in \mathbb{K}$ for which $T + \lambda S$ is normally solvable and has an index. Then*

- (i) U is an open set.
- (ii) If C is a component of U , then on C , with the possible exception of isolated points, $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ have constant values n_1 and n_2 , respectively. At the isolated points,

$$n_1 < \alpha(T + \lambda S) < \infty \quad \text{and} \quad n_2 < \beta(T + \lambda S) < \infty.$$

PROOF. (i) For $\lambda \in U$, apply Theorem 15 to $T + \lambda S$ in place of T .

(ii) Since any component of an open set in \mathbb{C} is open, we have that C is open. Define $\alpha(\lambda) := \alpha(\lambda - T)$, and choose λ_o such that $\alpha(\lambda_o) := n_1$ is the smallest nonnegative integer attained by $\alpha(\lambda)$ on C . Suppose $\alpha(\lambda') \neq n_1$. Owing to the connectivity of C , there exists an arc Γ lying in C with endpoints λ_o and λ' . It follows from Theorem 16 and the fact that C is open, that about each $\eta \in \Gamma$ there exists an open ball $B_C(\eta; r)$ contained in C such that $\alpha(\lambda)$ is constant on $B_C(\eta; r) \setminus \{\eta\}$.

Since Γ is compact and connected, there exists points $\lambda_1, \dots, \lambda_n = \lambda'$ on Γ such that

$$B_C(\lambda_o; r_o), B_C(\lambda_1; r_1), \dots, B_C(\lambda_n; r_n) \text{ cover } \Gamma \text{ and}$$

$$B_C(\lambda_i; r_i) \cap B_C(\lambda_{i+1}; r_{i+1}) \neq \emptyset, \quad 0 \leq i \leq n-1. \quad (10)$$

We assert that $\alpha(\lambda) = \alpha(\lambda_o)$ on all of $B_C(\lambda_o; r_o)$. It follows from Theorem 15 that $\alpha(\lambda) \leq \alpha(\lambda_o)$ for λ sufficiently closed to λ_o . Therefore, since $\alpha(\lambda_o)$ is the minimum of $\alpha(\lambda)$ on C , $\alpha(\lambda) = \alpha(\lambda_o)$ for λ sufficiently closed to λ_o . Since $\alpha(\lambda)$ is constant for all $\lambda \neq \lambda_o$ in $B_C(\lambda_o; r_o)$, this constant must be $\alpha(\lambda_o)$. Now $\alpha(\lambda)$ is constant on $B_C(\lambda_i; r_i)$ with the point λ_i deleted, $1 \leq i \leq n$. Hence, it follows from (10) and the observation $\alpha(\lambda) = \alpha(\lambda_o)$ for all $\lambda \in B_C(\lambda_o; r_o)$, that $\alpha(\lambda) = \alpha(\lambda_o)$ for all $\lambda \neq \lambda'$ in $B_C(\lambda'; r_n)$ and $\alpha(\lambda') > n_1$.

To see that the result holds for $\beta(\lambda - T)$, we pass to the adjoint of T and apply the above and the equality $\alpha(\lambda - T') = \beta(\lambda - T)$. \square

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