

Lattice-like translation ball packings in Nil space

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Abstract. Nil geometry is one of the eight homogeneous Thurston 3-geometries: \mathbf{E}^3 , \mathbf{S}^3 , \mathbf{H}^3 , $\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\widetilde{\mathbf{SL}}_2\mathbf{R}$, \mathbf{Nil} , \mathbf{Sol} . Nil can be derived from W. HEISENBERG's famous real matrix group. The notion of *translation curve* and translation ball can be introduced by initiative of E. MOLNÁR (see [MS], [MSz], [Sz10]). P. SCOTT in [S] defined Nil lattices to which *lattice-like translation ball packings* can be defined. In our work we will use the projective model of Nil geometry introduced by E. MOLNÁR in [M97]. In this paper we have studied the translation balls of Nil space and computed their volume. Moreover, we have proved in Theorems 4.1–4.2 that the density of the optimal lattice-like translation ball packing for every natural lattice parameter $1 \leq k \in \mathbb{N}$ is in interval $(0.7808, 0.7889)$ and if $r \in (0, r_d]$ ($r_d \approx 0.7456$) then the optimal density is $\delta_{\Gamma}^{\text{opt}} \approx 0.7808$. Meanwhile we can apply a nice general estimate of L. FEJES TÓTH [LFT] in our Theorem 4.2. From Corollary 4.2 we shall see that the kissing number of the lattice-like ball packings is less than or equal to 14 and the optimal ball packing is realizable in case of equality. We formulate a conjecture for $\delta_{\Gamma}^{\text{opt}}$, where the density of the conjectural densest packing is $\delta_{\Gamma}^{\text{opt}} \approx 0.7808$ for lattice parameter $k = 1$, larger than the Euclidean one ($\frac{\pi}{\sqrt{18}} \approx 0.74048$), but less than the density of the densest lattice-like *geodesic* ball packing in Nil space known till now [Sz07]. The kissing number of the translation balls in that packing is 14 as well.

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1. Important notions of Nil geometry

Nil geometry can be derived from the famous real matrix group $\mathbf{L}(\mathbf{R})$ discovered by Werner HEISENBERG. The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \quad (1.1)$$

defines “translations” $\mathbf{L}(\mathbf{R}) = \{(x, y, z) : x, y, z \in \mathbf{R}\}$ on the points of $\mathbf{Nil} = \{(a, b, c) : a, b, c \in \mathbf{R}\}$. These translations are not commutative in general. The matrices $\mathbf{K}(z) \triangleleft \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (0, 0, z) \quad (1.2)$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of \mathbf{L} . The elements of \mathbf{K} are called *fibre translations*. **Nil** geometry of the Heisenberg group can be projectively (affinely) interpreted by “right translations” on points as the matrix formula

$$(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \quad (1.3)$$

shows, according to (1.1). Here we consider \mathbf{L} as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0), E_1^\infty(\mathbf{e}_1), E_2^\infty(\mathbf{e}_2), E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$) which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of the projective 3-sphere \mathcal{PS}^3 (or that of the projective space \mathcal{P}^3 where opposite rays (\mathbf{x}) and $(-\mathbf{x})$ are identified). The dual system $\{(\mathbf{e}^i)\}$, $(\{\mathbf{e}^i\} \subset \mathbf{V}_4)$, with $\mathbf{e}_i \mathbf{e}^j = \delta_i^j$ (the Kronecker symbol), describes the simplex planes, especially the plane at infinity $(\mathbf{e}^0) = E_1^\infty E_2^\infty E_3^\infty$, and generally, $\mathbf{v} = \mathbf{u} \frac{1}{c}$ defines a plane $(\mathbf{u}) = (\mathbf{v})$ of \mathcal{PS}^3 (or that of \mathcal{P}^3). Thus $0 = \mathbf{x}\mathbf{u} = \mathbf{y}\mathbf{v}$ defines the incidence of point $(\mathbf{x}) = (\mathbf{y})$ and plane $(\mathbf{u}) = (\mathbf{v})$, as

$(\mathbf{x})\mathbf{I}(\mathbf{u})$ also denotes it. Thus \mathbf{Nil} can be visualized in the affine 3-space \mathbf{A}^3 (so in \mathbf{E}^3) as well [MSz06].

In this context E. MOLNÁR [M97] has derived the well-known infinitesimal arc-length square invariant under translations \mathbf{L} at any point of \mathbf{Nil} as follows

$$\begin{aligned} & (dx)^2 + (dy)^2 + (-xdy + dz)^2 \\ &= (dx)^2 + (1 + x^2)(dy)^2 - 2x(dy)(dz) + (dz)^2 =: (ds)^2 \end{aligned} \tag{1.4}$$

Hence we get the symmetric metric tensor field g on \mathbf{Nil} by components g_{ij} , furthermore its inverse:

$$\begin{aligned} g_{ij} &:= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & -x \\ 0 & -x & 1 \end{pmatrix}, \quad g^{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1 + x^2 \end{pmatrix} \\ &\text{with } \det(g_{ij}) = 1. \end{aligned} \tag{1.5}$$

The translation group \mathbf{L} defined by formula (1.3) can be extended to a larger group \mathbf{G} of collineations, preserving the fibres, that will be equivalent to the (orientation preserving) isometry group of \mathbf{Nil} .

In [M10] E. MOLNÁR has shown that a rotation through angle ω about the z -axis at the origin, as isometry of \mathbf{Nil} , keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z -image \bar{z} as follows:

$$\begin{aligned} \mathcal{M} &= \mathbf{r}(O, \omega) : (1; x, y, z) \rightarrow (1; \bar{x}, \bar{y}, \bar{z}); \\ \bar{x} &= x \cos \omega - y \sin \omega, \quad \bar{y} = x \sin \omega + y \cos \omega, \\ \bar{z} &= z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2) \sin 2\omega + \frac{1}{2}xy \cos 2\omega. \end{aligned} \tag{1.6}$$

This rotation formula \mathcal{M} , however, is conjugate by the quadratic mapping α to the linear rotation Ω in (1.7) as follows

$$\begin{aligned} \alpha^{-1} : (1; x, y, z) &\xrightarrow{\alpha^{-1}} (1; x', y', z') = (1; x, y, z - \frac{1}{2}xy) \quad \text{to} \\ \Omega : (1; x', y', z') &\xrightarrow{\Omega} (1; x'', y'', z'') = (1; x', y', z') \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \omega & \sin \omega & 0 \\ 0 & -\sin \omega & \cos \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \text{with } \alpha : (1; x'', y'', z'') &\xrightarrow{\alpha} (1; \bar{x}, \bar{y}, \bar{z}) = (1; x'', y'', z'' + \frac{1}{2}x''y''). \end{aligned} \tag{1.7}$$

This quadratic conjugacy modifies the **Nil** translations in (1.3), as well. Now a translation with (X, Y, Z) in (1.3) instead of (x, y, z) will be changed by the above conjugacy to the translation

$$(1; x, y, z) \longrightarrow (1; \bar{x}, \bar{y}, \bar{z}) = (1; x, y, z) \begin{pmatrix} 1 & X & Y & Z - \frac{1}{2}XY \\ 0 & 1 & 0 & -\frac{1}{2}Y \\ 0 & 0 & 1 & \frac{1}{2}X \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1.8)$$

that is again an affine collineation. We shall use the following important classification theorem.

Theorem 1.1 (E. MOLNÁR [M10]). **(1)** Any group of **Nil** isometries, containing a 3-dimensional translation lattice, is conjugate by the quadratic mapping in (1.7) to an affine group of the affine (or Euclidean) space $\mathbf{A}^3 = \mathbf{E}^3$ whose projection onto the (x,y) plane is an isometry group of \mathbf{E}^2 . Such an affine group preserves a plane \rightarrow point polarity of signature $(0, 0, \pm 0, +)$.

(2) Of course, the involutive line reflection about the y axis

$$(1; x, y, z) \rightarrow (1; -x, y, -z),$$

preserving the Riemann metric in (1.5), and its conjugates by the above isometries in **(1)** (those of the identity component) are also **Nil**-isometries. There does not exist orientation reversing **Nil**-isometry.

2. Translation curves and balls

We consider a **Nil** curve $(1, x(t), y(t), z(t))$ with a given starting tangent vector at the origin $O(1, 0, 0, 0)$

$$u = \dot{x}(0), \quad v = \dot{y}(0), \quad w = \dot{z}(0). \quad (2.1)$$

For a translation curve let its tangent vector at the point $(1, x(t), y(t), z(t))$ be defined by the matrix (1.3) with the following equation:

$$(0, u, v, w) \begin{pmatrix} 1 & x(t) & y(t) & z(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x(t) \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0, \dot{x}(t), \dot{y}(t), \dot{z}(t)). \quad (2.2)$$

Thus, the *translation curves* in **Nil** geometry (see [MS], [MSz06] [Sz10]) are defined by the above first order differential equation system $\dot{x}(t) = u$, $\dot{y}(t) = v$, $\dot{z}(t) = v \cdot x(t) + w$, whose solution is the following:

$$x(t) = ut, \quad y(t) = vt, \quad z(t) = \frac{1}{2}uvt^2 + wt. \tag{2.3}$$

We assume that the starting point of a translation curve is the origin, because we can transform a curve into an arbitrary starting point by translation (1.3), moreover, unit initial velocity translation can be assumed by “geographic” parameters ϕ and θ :

$$\begin{aligned} x(0) = y(0) = z(0) &= 0; \\ u = \dot{x}(0) = \cos \theta \cos \phi, \quad v = \dot{y}(0) &= \cos \theta \sin \phi, \quad w = \dot{z}(0) = \sin \theta; \\ -\pi \leq \phi \leq \pi, \quad -\frac{\pi}{2} \leq \theta &\leq \frac{\pi}{2}. \end{aligned} \tag{2.4}$$

Definition 2.1. The translation distance $d^t(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the above translation curve from P_1 to P_2 .

Definition 2.2. The sphere of radius $r > 0$ with centre at the origin, (denoted by $S_O^t(r)$), with the usual longitude and altitude parameters ϕ and θ , respectively by (2.4), is specified by the following equations:

$$S_O^t(r) : \begin{cases} x(\phi, \theta) = r \cos \theta \cos \phi, \\ y(\phi, \theta) = r \cos \theta \sin \phi, \\ z(\phi, \theta) = \frac{r^2}{2} \cos^2 \theta \cos \phi \sin \phi + r \sin \theta. \end{cases} \tag{2.5}$$

Definition 2.3. The body of the translation sphere of centre O and of radius r in the **Nil** space is called translation ball, denoted by $B_O^t(r)$, i.e. $Q \in B_O^t(r)$ iff $0 \leq d^t(O, Q) \leq r$.

Remark 2.1. The translation sphere is a simply connected surface without selfintersection in **Nil** space for any radius $0 < r \in \mathbf{R}$.

We apply the quadratic mapping $\alpha^{-1} : \mathbf{Nil} \rightarrow \mathbf{A}^3$ at (1.6) to the translation sphere S^t , its α^{-1} -image is denoted by $S^{t'} =: S' = \alpha^{-1}(S^t)$.

Consider a point $P(x(r, \phi, \theta), y(r, \theta, \alpha), z(r, \phi, \theta))$ lying on a sphere S of radius r with centre at the origin. The coordinates of P are given by parameters $(\phi \in [-\pi, \pi], \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], r > 0)$ (see (1.8), (1.9)), its α^{-1} -image is $P'(x'(r, \phi, \theta), y'(r, \phi, \theta), z'(r, \phi, \theta)) \in S'$ where

$$x'(\phi, \theta) = r \cos \theta \cos \phi,$$

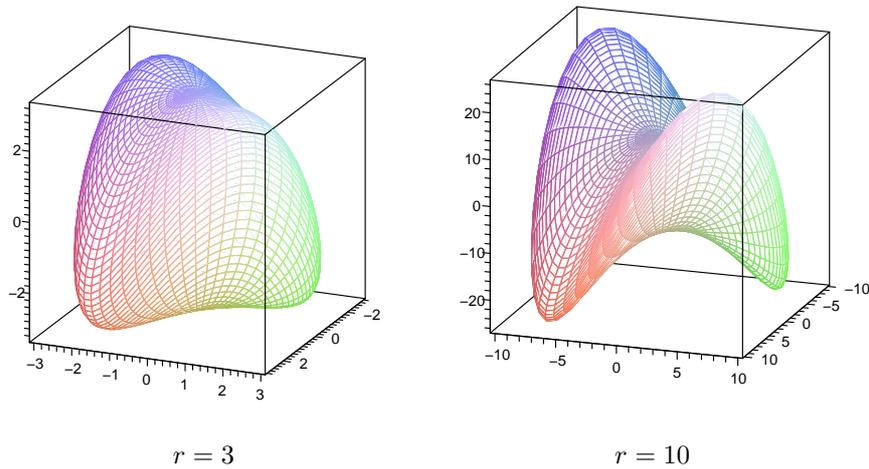


Figure 1. Translation balls of radius r in **Nil** space

$$\begin{aligned} y'(\phi, \theta) &= r \cos \theta \sin \phi, \\ z'(\phi, \theta) &= r \sin \theta. \end{aligned} \tag{2.6}$$

We see that this is a sphere in the Euclidean sense.

Remark 2.2. From the definition of the quadratic mapping α^{-1} at (1.7) it follows that the cross section of the spheres S^t and S' with the plane $[x, z]$, is the same curve, a Euclidean circle of radius r with centre at the origin.

The Jacobi matrix of the quadratic mapping α^{-1} at (1.7) is

$$J(\mathcal{M}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}y & -\frac{1}{2}x & 1 \end{pmatrix}, \text{ i.e. } \det(J(\mathcal{M})) = 1, \tag{2.7}$$

We obtain the volume formula of the translation ball $B_O^t(r)$ of radius r by (2.1) and (2.2):

Theorem 2.1. *The volume of a translation ball of radius r is the same as that of an Euclidean one:*

$$\text{Vol}(B_O^t(r)) = \frac{4}{3}r^3\pi. \tag{2.8}$$

2.1. The convexity of the Nil translation ball in our model. In this subsection we examine the convexity of the translation ball in the Euclidean sense in our affine model. The Nil translation sphere of radius r can be generated by the Nil rotation about the axis z , too (see (1.7) and Remark (2.1)). The parametric equation system (2.5) describes the translation sphere $S^t(r)$ in our model and we have obtained by the derivatives of these parametrically represented functions (by intensive and careful computations with *Maple* through the second fundamental form) the following theorem:

Theorem 2.2. *A translation Nil ball $B^t(S^t(r))$ is convex in the affine-Euclidean sense in our model if and only if $r \in [0, 2]$.*

3. The discrete translation group $L(\mathbb{Z}, k)$

We consider the Nil translations defined in (1.1) and (1.3) and choose first two non-commuting translations

$$\tau_1 = \begin{pmatrix} 1 & t_1^1 & t_1^2 & t_1^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_1^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tau_2 = \begin{pmatrix} 1 & t_2^1 & t_2^2 & t_2^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

now with upper indices for the coordinate variables. Second, we define the translation $(\tau_3)^k$, ($k \in \mathbb{N} = \{1, 2, 3, \dots\}$ k is fixed natural exponent), by the following commutator:

$$(\tau_3)^k = \tau_2^{-1} \tau_1^{-1} \tau_2 \tau_1 = \begin{pmatrix} 1 & 0 & 0 & -t_2^1 t_1^2 + t_1^1 t_2^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and so } \tau_3 \tag{3.2}$$

is also defined. If we take integers as coefficients for τ_1, τ_2, τ_3 , then we generate the discrete group $\langle \tau_1, \tau_2, \tau_3 \rangle$, denoted by $L(\tau_1, \tau_2, k)$ or by $L(\mathbb{Z}, k)$. Here \mathbb{Z} refers to the integers.

We know (see e.g. [S] and [Sz07]) that the orbit space $\text{Nil}/L(\mathbb{Z}, k)$ is a compact manifold, i.e. a Nil space form.

Definition 3.1. The Nil point lattice $\Gamma_P(\tau_1, \tau_2, k)$ is a discrete orbit of point P in the Nil space under group $L(\tau_1, \tau_2, k) = L(\mathbb{Z}, k)$ with an arbitrary starting point P for every fixed $k \in \mathbb{N}$.

Remark 3.1. For simplicity we have chosen the origin as starting point, by the homogeneity of \mathbf{Nil} .

Remark 3.2. We may assume in the following that $t_1^2 = 0$, i.e. the image of the origin by the translation τ_1 lies on the plane $[x, z]$.

We consider by (3.3) a fundamental “parallelepiped complex”

$$\widetilde{\mathcal{F}(k)} = OT_1T_2T_3T_{12}T_{21}T_{23}T_{213}T_{13}, \quad (\text{see Fig. 2 for } k = 2)$$

in the Euclidean sense, which is determined by translations τ_1, τ_2, τ_3 . The images of $\widetilde{\mathcal{F}(k)}$ under $\mathbf{L}(\mathbb{Z}, k)$ fill \mathbf{Nil} without gap. Overlaps occur only on the boundary.

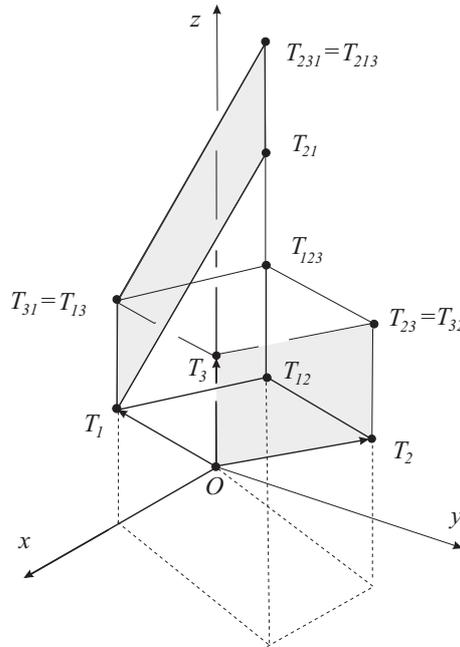


Figure 2. The \mathbf{Nil} parallelepiped $\widetilde{\mathcal{F}(2)}$

Analogously to the Euclidean integer lattice and parallelepiped, $\widetilde{\mathcal{F}(k)}$ ($k \in \mathbb{N}$) can be called a \mathbf{Nil} parallelepiped, endowed by face pairing, as the upper \sim hints to it.

$\widetilde{\mathcal{F}(k)}$ is a *fundamental domain* of $\mathbf{L}(\mathbb{Z}, k)$. It is a tricky task, how to form its solid with great freedom, see e.g. [MP]. We need only its interior for its volume. The homogeneous coordinates of the vertices of $\widetilde{\mathcal{F}(k)}$ can be determined in our

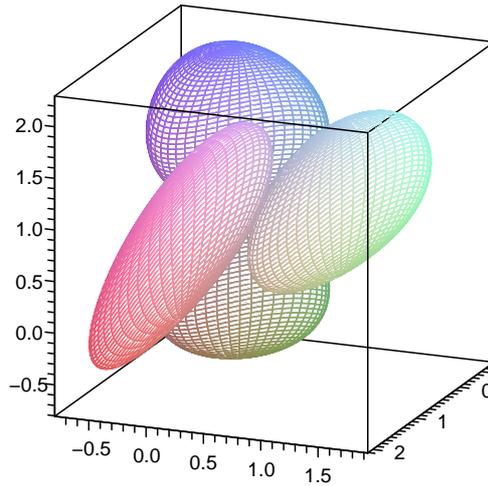


Figure 3. The optimal translation ball arrangement in Nil space

affine model by the translations (3.1) and (3.2) with the parameters t_i^j , $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$ (see (3.3) and Fig. 2).

$$\begin{aligned}
 &T_1(1, t_1^1, 0, t_1^3), \quad T_2(1, t_2^1, t_2^2, t_2^3), \quad T_3\left(1, 0, 0, \frac{t_1^1 t_2^2}{k}\right), \\
 &T_{13}\left(1, t_1^1, 0, \frac{t_1^1 t_2^2}{k} + t_1^3\right), \quad T_{12}(1, t_1^1 + t_2^1, t_2^2, t_2^3 + t_1^3), \\
 &T_{21}(1, t_1^1 + t_2^1, t_2^2, t_1^1 t_2^2 + t_1^3 + t_2^3), \quad T_{23}\left(1, t_2^1, t_2^2, t_2^3 + \frac{t_1^1 t_2^2}{k}\right), \\
 &T_{213} = T_{231}\left(1, t_1^1 + t_2^1, t_2^2, (k+1)\frac{t_1^1 t_2^2}{k} + t_1^3 + t_2^3\right). \quad (3.3)
 \end{aligned}$$

In [Sz07] we have determined the volume of the Nil parallelepiped $\widetilde{\mathcal{F}}(1)$. Analogously to that we get the volume formula of $\widetilde{\mathcal{F}}(k)$ ($k \in \mathbb{N}$) by the usual method:

$$\begin{aligned}
 \iiint_{\widetilde{\mathcal{F}}(k)} \sqrt{\det(g_{ij})} \, dx dy dz &= \text{Vol}(\widetilde{\mathcal{F}}(k)) \\
 &= \frac{1}{k} \int_0^{t_2^2} \int_0^{t_1^1} |t_1^1 \cdot t_2^2| \, dx dy = \frac{(t_1^1 \cdot t_2^2)^2}{k}. \quad (3.4)
 \end{aligned}$$

If the parameter k is given, from this formula it can be seen that the volume of a **Nil** parallelepiped depends on two parameters, i.e. on its projection into the $[x, y]$ plane.

4. Lattice-like translation ball packings

Let $\mathcal{B}_\Gamma^t(r)$ denote a translation ball packing of **Nil** space with balls $B^t(r)$ of radius r where their centres give rise to a **Nil** point lattice $\Gamma(\tau_1, \tau_2, k)$. $\widetilde{\mathcal{F}(k)}_O$ is an arbitrary **Nil** parallelepiped at the origin of this lattice (see (3.1), (3.2), (3.3)). The images of $\widetilde{\mathcal{F}(k)}_O$ by our discrete translation group $\mathbf{L}(\tau_1, \tau_2, k)$ covers **Nil** without open overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $\widetilde{\mathcal{F}(k)}_O$. Analogously to the Euclidean case it can be defined the density $\delta(r, \tau_1, \tau_2, k)$ of the lattice-like translation ball packing $\mathcal{B}_\Gamma^t(r, k)$:

Definition 4.1.

$$\delta(r, \tau_1, \tau_2, k) := \frac{\text{Vol}(\mathcal{B}_\Gamma^t(r) \cap \widetilde{\mathcal{F}(k)}_O)}{\text{Vol}(\widetilde{\mathcal{F}(k)}_O)}, \tag{4.1}$$

if the balls do not overlap each other.

Remark 4.1. By definition of the **Nil** lattice $\mathbf{L}(\tau_1, \tau_2, k)$ (see Definition 3.1) the orbit space $\mathbf{Nil}/\mathbf{L}(\tau_1, \tau_2, k)$ is a compact **Nil** manifold, and (see Section 2),

$$\text{Vol}(\mathcal{B}_\Gamma^t(r) \cap \widetilde{\mathcal{F}(k)}_O) = \text{Vol}(B^t(S^t(r))).$$

4.1. On optimal lattice-like translation ball packing. We consider an arbitrary point lattice $\Gamma_O(\tau_1, \tau_2, k)$ in **Nil** with starting point O .

First we introduce our final optimal arrangement $\mathcal{B}_\Gamma^t(r)$ of translation balls $B^t(r)$, (see Fig. 2–3) where the following equations hold:

- (a) $k = 1, \quad d^t(O, T_3) = 2r,$
 - (b) The ball $B_0^t(r)$ touches the balls $B_{T_1}^t(r)$ and $B_{T_2}^t(r),$
 - (c) The ball $B_{T_3}^t(r)$ touches the balls $B_{T_1}^t(r)$ and $B_{T_2}^t(r),$
 - (d) The ball $B_{T_1}^t(r)$ touches the ball $B_{T_2}^t(r).$
- (4.2)

These assumptions exclude the overlap of any two balls by the lattice structure (its complete **Nil** symmetry group is also described in [MSz06]).

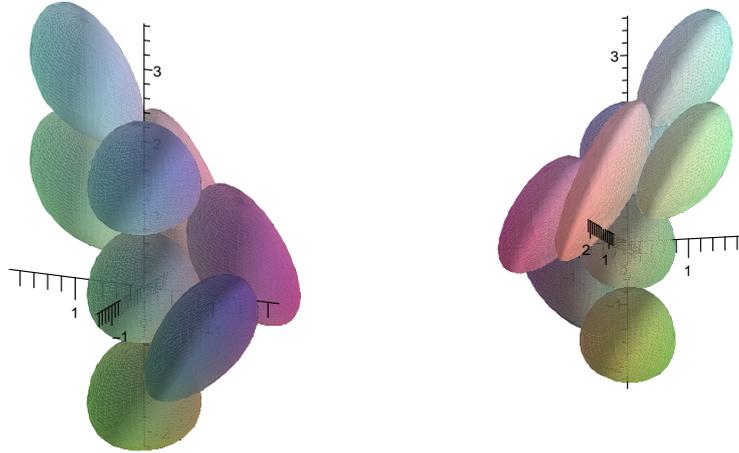


Figure 4. The optimal lattice-like translation ball packing in Nil space

Here d^t is the translation distance function in Nil (see Definition 2.1). By continuity of the distance function, it follows that there is a (unique) solution of the equation system (4.2) for t_i^j and r . We have denoted by $\mathcal{B}_\Gamma^t(r_d)$ this translation ball packing of the balls $B^t(r_d)$. We get the following solution by systematic approximation, where the computations were carried out by *Maple V Release 10* up to 30 decimals:

$$\begin{aligned}
 t_1^{1,d} &\approx 1.31225131; & t_1^{3,d} &= \frac{t_3^3}{2}; & t_2^{1,d} &\approx 0.65612565; \\
 t_2^{2,d} &\approx 1.13644297; & t_2^{3,d} &\approx 1.11847408; \\
 r_d &\approx 0.74564939, & t_3^{3,d} &= 2r_d. & & (4.3)
 \end{aligned}$$

Here index d refers to the densest packing. This translation ball packing can be realized in Nil because a ball of radius r_d is convex in affine sense and this packing can be generated by the translations $\Gamma(\tau_1^d, \tau_2^d, \tau_3^d, 1)$. Thus we obtain the neighbouring balls around an arbitrary ball of the packing \mathcal{B}_Γ^t , the kissing number of the balls is 14. Fig. 3–4 show the typical arrangement of some balls from $\mathcal{B}_\Gamma^t(r_d)$ in our model. We get ball “columns” in z -direction and a “hexagonal” lattice projection in $[x, y]$ -plane.

By formulas (2.3), (3.4) and by Definition 4.1 we can compute the density of this ball packing:

$$\begin{aligned}
 \text{Vol}(\widetilde{\mathcal{F}(1)_0^d}) &\approx 2.22397203, & \text{Vol}(\mathcal{B}_\Gamma^{t,d}) &\approx 1.73657124, \\
 \delta_\Gamma^d &\approx 0.78084221. & & (4.4)
 \end{aligned}$$

Theorem 4.1. *The ball arrangement \mathcal{B}_Γ^d given in formulas (4.2) provides the densest lattice-like translation ball packing in \mathbf{Nil} space if $r \in (0, r_d]$, $k \in \mathbb{N}$.*

PROOF. Let $\mathcal{B}_\Gamma^t(r)$ denote a translation ball packing of \mathbf{Nil} space with balls $B^t(r)$ of radius r where their centres give rise to a \mathbf{Nil} point lattice $\Gamma(\tau_1, \tau_2, k)$ (see (3.1), (3.2) and Definition 3.1). If we give the distance $d^t(O, T_3) = |\frac{t_1 \cdot t_2}{k}|$ then we fix the volume of the \mathbf{Nil} parallelepiped $\widetilde{\mathcal{F}(k)}_O$ generated by translations τ_1, τ_2 . Thus for choosing the radius r of the balls in $\mathcal{B}_\Gamma^t(r)$, we have to minimize the distance $d^t(O, T_3) = |\frac{t_1 \cdot t_2}{k}|$ so that we achieve the densest lattice-like translation ball packing.

We have to minimize the distance $d^t(O, T_3)$ to a given r . This distance is minimal if $d^t(O, T_3) = |\frac{t_1 \cdot t_2}{k}| = 2r$, i.e. *the balls $B_{T_3}^t(r)$ and $B_O^t(r)$ touch each other*. In these cases the volume of the \mathbf{Nil} parallelepiped of the lattice $\Gamma(\tau_1, \tau_2, k)$ is $\text{Vol}(\widetilde{\mathcal{F}(k)}_O) = 4kr^2$. Thus we have to examine the density function (see Definition 4.1 and Remark 4.1)

$$\delta(r, \tau_1, \tau_2, k) := \frac{\text{Vol}(B^t(S^t(r)))}{4kr^2} = \frac{r\pi}{3k}. \tag{4.5}$$

This is an increasing density function for every k , thus the maximum of the density is achieved if $r = r_d$ and $k = 1$. □

We have denoted the density of the densest translation ball arrangement $\mathcal{B}_\Gamma^{\text{opt}}$ by $\delta_\Gamma^{\text{opt}}$.

Theorem 4.2. *If $r \in [r_d, \infty)$ and $k \in \mathbb{N}$, then*

$$\delta_\Gamma^d \leq \delta_\Gamma^{\text{opt}} < \frac{4\pi}{3e \sin \frac{\pi f}{e} \left(\tan^2 \frac{\pi f}{2e} \tan^2 \frac{\pi v}{2e} - 1 \right)}, \tag{4.6}$$

where $f = 14, v = 24, e = 36$.

PROOF. We have seen in Section 2 that the image S' of a translation ball S^t with centre at the origin by quadratic mapping $\alpha^{-1} : \mathbf{Nil} \rightarrow \mathbf{A}^3$ at (1.7) is a sphere in the Euclidean sense. We have introduced the discrete translation group $\mathbf{L}(\mathbb{Z}, k)$ in Section 3 whose image map by the quadratic transformation α^{-1} derives a translation group $\mathbf{L}'(\mathbb{Z}, k)$ in affine (or Euclidean) space $\mathbf{A}^3 = \mathbf{E}^3$. Of course the quadratic conjugacy modifies the \mathbf{Nil} translations in (1.3), as well as formula (1.8) indicates this with corresponding denotations of τ_1, τ_2, τ_3 in (3.3). If we consider in this affine space an Euclidean plane then its image by a “map” translation (generated by (1.3) and (1.6)) is a Euclidean plane, thus

the fundamental domain of $\mathbf{L}'(\mathbb{Z}, k)$ can be realized as an Euclidean polyhedron containing a sphere S' . If its centre is at the origin, then the polyhedron contains a Euclidean sphere.

If the fundamental domain $\mathcal{P}(k)_0$ ($\text{Vol}(\mathcal{P}(k)_0) = \text{Vol}(\widetilde{\mathcal{F}(k)}_O)$) of translation group $\mathbf{L}'(\mathbb{Z}, k)$ is a polyhedron then it is the image of a “Nil-parallelohedron” in \mathbf{A}^3 . These are convex bodies which allow tilings of affine (or Euclidean) space by “map” translations only. In this case, even a face-to-face tiling by lattice vectors can be realized. The translation group $\mathbf{L}'(\mathbb{Z}, k)$ is a non-commutative group, where the translations “parallel to z axis” (see 1.2) form its centre. Thus the maximum number of the faces in the optimal case are 14. It is easy to see, that in this case the maximum number of the vertices is 24. From the nice formula of LÁSZLÓ FEJES TÓTH (see [LFT]) it follows that:

$$F \geq e \sin \frac{\pi f}{e} \left(\tan^2 \frac{\pi f}{2e} \tan^2 \frac{\pi v}{2e} - 1 \right), \tag{4.7}$$

where the surface area of a convex polyhedron containing a unit ball is denoted by F , and equality holds only for regular polyhedra. From (4.7) follows, that the volume of the fundamental domain $\widetilde{\mathcal{F}(k)}_O$ of $\mathbf{L}'(\mathbb{Z}, k)$ is larger than

$$V_{\min} := \frac{1}{3} e \sin \frac{\pi f}{e} \left(\tan^2 \frac{\pi f}{2e} \tan^2 \frac{\pi v}{2e} - 1 \right) \approx 5.30969695$$

where $f = 14, v = 24, e = 36$. □

Remark 4.2. (1) $\delta_{\Gamma}^d \approx 0.78084221 \leq \delta_{\Gamma}^{\text{opt}} < 0.78889440$.

(2) There are Euclidean polyhedra containing unit sphere with parameters $f = 14, v = 24, e = 36$, where

$$5.30969695 < \text{Vol}(\widetilde{\mathcal{F}(k)}_O) < 5.36445154$$

Corollary 4.1. *In cases if the number of the faces of the parallelohedron $\mathcal{P}(k)_0$ is less than 14 then from the above formula (4.7) follows that $F > 16.255054 \Rightarrow \text{Vol}(\mathcal{P}(k)_0) > 5.418351231 \Rightarrow \delta(r, \tau_1, \tau_2, k) < \delta_{\Gamma}^d \approx 0.78084221$.* □

Corollary 4.2. *The kissing number of a ball packing in the Nil space is less than or equal to 14 and the equality is realizable only at ball arrangement \mathcal{B}_{Γ}^d given in formulas (4.2–4.3).* □

Conjecture 4.1. *The ball arrangement \mathcal{B}_{Γ}^d given in formulas (4.2) provides the densest translation ball packing in Nil space.*

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