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Super-paracompactness and continuous sections

By DAVID BUHAGIAR (Msida) and VALENTIN GUTEV (Durban)

Abstract. It is demonstrated that a space X is super-paracompact if and only if for every completely metrizable Y, every l.s.c. mapping from X into the nonempty closed subsets of Y has a compact-valued continuous section.

1. Introduction

For a space Y, we will use 2^Y to denote the *power set* of Y, i.e. the set of all subsets of Y. Also, we will use $\mathscr{F}(Y)$ to denote the set of all nonempty closed subsets of Y, and $\mathscr{C}(Y)$ – that of all compact members of $\mathscr{F}(Y)$. For a set-valued mapping $\varphi : X \to 2^Y$ and $B \subset Y$, let $\varphi^{-1}[B] = \{x \in X : \varphi(x) \cap B \neq \varnothing\}$. The mapping φ is *lower semi-continuous*, or l.s.c., if the set $\varphi^{-1}[U]$ is open in X for every open $U \subset Y$. The mapping φ is *upper semi-continuous*, or u.s.c., if the set

$$\varphi^{\#}[U] = X \setminus \varphi^{-1}[Y \setminus U] = \{x \in X : \varphi(x) \subset U\}$$

is open in X for every open $U \subset Y$. For convenience, we say that φ is *usco* if it is u.s.c. and nonempty-compact-valued, and that φ is *continuous* if it is both l.s.c. and u.s.c.

A mapping $\varphi : X \to 2^Y$ is a *multi-selection* (or, a *set-valued selection*) for $\Phi : X \to 2^Y$ if $\varphi(x) \subset \Phi(x)$ for every $x \in X$; and $\varphi : X \to 2^Y$ is a *section* for $\Phi : X \to 2^Y$ if $\varphi(x) \cap \Phi(x) \neq \varnothing$ for every $x \in X$. If φ is a section for Φ, then both φ and Φ must be nonempty-valued. Of course, every nonempty-valued multi-selection for Φ is also a section for Φ.

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There is a natural relationship between covering properties and multi-selections. Namely, such familiar properties of topological spaces as paracompactness, metacompactness, collectionwise normality, etc., were transformed and thus essentially generalised in terms of multi-selections for l.s.c. mappings in completely metrizable spaces, see, for instance, [4], [5], [6], [8], [10], [12], [14], [17]. The present paper deals with a similar characterisation of another covering property, but now in terms of sections. Let $\mathscr{W}$ be a collection of subsets of a set X. If $U, V \in \mathscr{W}$, then a finite sequence $W_1, W_2, \dots, W_k$ of elements of $\mathscr{W}$ is called a *chain* from U to V if $U = W_1$, $V = W_k$ and $W_i \cap W_{i+1} \neq \varnothing$ for every $i = 1, \dots, k-1$. A subset $\mathscr{P} \subset \mathscr{W}$ is called *connected* if every pair of elements of $\mathscr{P}$ is connected by a chain. The components of $\mathscr{W}$ are defined as the maximal connected subsets of $\mathscr{W}$. A space X is call *super-paracompact* (Pasynkov, see [13]) if every open cover of X has an open finite component (i.e., having finite components) refinement. The purpose of this paper is to prove the following theorem.

Theorem 1.1. *A space X is super-paracompact if and only if for every completely metrizable space Y, every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has a continuous section $\varphi : X \to \mathscr{C}(Y)$.*

Theorem 1.1 can be compared with [7, Proposition 1.1] that a regular space X is paracompact if and only if for every completely metrizable Y, every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has an usco section $\psi : X \to \mathscr{C}(Y)$.

A word should be said also about the paper itself. Theorem 1.1 is proved in Section 3; the preparation for this proof is done in the next section. The technique developed to prove Theorem 1.1 allows to generalise it for a non-metrizable range, see Theorem 3.2.

2. Completeness and special sieves

A partially ordered set $(T, \preceq)$ is a *tree* if $\{s \in T : s \preceq t\}$ is well-ordered for every $t \in T$. For a tree $(T, \preceq)$, we use $T(0)$ to denote the set of the minimal elements of T. Given an ordinal α, if $T(\beta)$ is defined for every $\beta < \alpha$, then $T(\alpha)$ denotes the minimal elements of $T \setminus (T \upharpoonright \alpha)$, where $T \upharpoonright \alpha = \bigcup\{T(\beta) : \beta < \alpha\}$. The set $T(\alpha)$ is called the α^{th}*-level* of T, while the *height* of T is the least ordinal α such that $T \upharpoonright \alpha = T$. We say that $(T, \preceq)$ is an α*-tree* if its height is α. A maximal linearly ordered subset of a tree $(T, \preceq)$ is called a *branch*, and $\mathscr{B}(T)$ is used to denote the set of all branches of T. A tree $(T, \preceq)$ is *pruned* if every element of T has a successor in T, i.e. if for every $s \in T$ there exists $t \in T$, with $s \prec t$. In these terms, an ω-tree $(T,\preceq)$ is pruned if each branch $\beta\in\mathscr{B}(T)$ is infinite. Following Nyikos [15], for every $t\in T$, set

$$\mathscr{O}(t)=\{\beta\in\mathscr{B}(T):t\in\beta\}. \tag{2.1}$$

For a pruned ω-tree $(T,\preceq)$, the family $\{\mathscr{O}(t):t\in T\}$ is a base for a completely metrizable non-Archimedean topology on $\mathscr{B}(T)$. We will refer to this topology on $\mathscr{B}(T)$ as the *branch topology*, and to the resulting topological space as the *branch space*. It is well known that $\mathscr{B}(T)$ is compact if and only if all levels of T are finite.

For a tree $(T,\preceq)$ and $t\in T$, the *node* of t in T is the subset $\mathsf{node}(t)\subset T$ of all immediate successors of t. For convenience, let $\mathsf{node}(\varnothing)=T(0)$. Finally, for a mapping $\Psi:Z\to 2^Y$ and $A\subset Z$, let

$$\Psi[A]=\bigcup\{\Psi(z):z\in A\}.$$

Given a set Y and a pruned ω-tree $(T,\preceq)$, a set-valued mapping $\mathscr{S}:T\to 2^Y$ is a *sieve* on Y if

(i) $Y=\mathscr{S}[\mathsf{node}(\varnothing)]$, and
(ii) $\mathscr{S}(t)=\mathscr{S}[\mathsf{node}(t)]$ for every $t\in T$.

A sieve $\mathscr{S}:T\to 2^Y$ on a space Y is *complete* [3], [11] if for every branch $\beta\in\mathscr{B}(T)$ and every nonempty centred (i.e., with the finite intersection property) family $\mathscr{F}\subset 2^Y$ which refines $\{\mathscr{S}(t):t\in\beta\}$ it follows that $\bigcap\left\{\overline{F}:F\in\mathscr{F}\right\}\neq\varnothing$. In other words, a sieve $\mathscr{S}:T\to 2^Y$ on Y is complete if each family $\{\mathscr{S}(t):t\in\beta\}$, $\beta\in\mathscr{B}(T)$, is a *compact* filter base (i.e., each ultrafilter containing it is convergent) [16].

For a tree $(T,\preceq)$ and $\mathscr{S}:T\to 2^Y$, the *polar mapping* $\Omega_{\mathscr{S}}:\mathscr{B}(T)\to 2^Y$, associated to $\mathscr{S}$, is defined by $\Omega_{\mathscr{S}}(\beta)=\bigcap\{\mathscr{S}(t):t\in\beta\}$, $\beta\in\mathscr{B}(T)$. Also, to the mapping $\mathscr{S}:T\to 2^Y$ we associate the pointwise-closure $\overline{\mathscr{S}}:T\to 2^Y$ of $\mathscr{S}$ by $\overline{\mathscr{S}}(t)=\overline{\mathscr{S}(t)}$, $t\in T$. If $\mathscr{S}:T\to 2^Y$ is a nonempty-valued complete sieve on a space Y, then for every branch $\beta\in\mathscr{B}(T)$, the polar $\Omega_{\overline{\mathscr{S}}}(\beta)$ is a nonempty compact subset of Y, and every open $V\supset\Omega_{\overline{\mathscr{S}}}(\beta)$ contains some $\overline{\mathscr{S}}(t)$ for $t\in\beta$, see, e.g., [3, Proposition 2.10]. In terms of set-valued mappings, this means that the polar mapping $\Omega_{\overline{\mathscr{S}}}:\mathscr{B}(T)\to\mathscr{C}(Y)$ is usco. In this section, we show that every completely metrizable space Y has a special complete sieve $\mathscr{S}:T\to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{S}}}$ is continuous. To this end, recall that a sieve $\mathscr{S}:T\to 2^Y$ is *finitely-additive* if each collection $\{\mathscr{S}(t):t\in T(n)\}$, $n<\omega$, as well as each collection of the form $\{\mathscr{S}(s):s\in\mathsf{node}(t)\}$, $t\in T$, is closed under finite unions.

Lemma 2.1. *Every completely metrizable space Y has a nonempty-open-valued, finitely-additive and complete sieve $\mathscr{S} : T \to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{S}}} : \mathscr{B}(T) \to \mathscr{C}(Y)$ is continuous.*

PROOF. Let d be a complete metric on Y compatible with the topology of Y, and let $\mathscr{R} : D \to 2^Y$ be a nonempty-open-valued sieve on Y with $\mathsf{diam}_d(\mathscr{R}(s)) < 2^{-n}$ for every $s \in D(n)$ and $n < \omega$. According to Cantor's intersection theorem, $\mathscr{R}$ is a complete sieve on Y. In fact, each $\Omega_{\overline{\mathscr{R}}}(\delta)$, $\delta \in \mathscr{B}(D)$, is a singleton, and hence the polar mapping $\Omega_{\overline{\mathscr{R}}} : \mathscr{B}(D) \to \mathscr{C}(Y)$ is singleton-valued and usco (thus, continuous as well). Keeping this in mind, let Σ_D be the set of all nonempty finite subsets of D. By [11, Lemma 2.3], there is a finitely-additive sieve $\mathscr{S} : T \to 2^Y$ on Y generated by the sieve $\mathscr{R}$, where $T \subset \Sigma_D$ and for each $\sigma \in T$, the value of $\mathscr{S}$ in σ is $\mathscr{S}(\sigma) = \mathscr{R}[\sigma] = \bigcup\{\mathscr{R}(s) : s \in \sigma\}$. The order on T is defined in a natural way so that each branch $\beta = \{\sigma_n : n < \omega\} \in \mathscr{B}(T)$ corresponds to a pruned subtree $\bigcup \beta = \bigcup\{\sigma_n : n < \omega\}$ of D such that $\sigma_n \subset D(n)$, for every $n < \omega$, see the proof of [11, Lemma 2.3]. In particular, if $\beta \in \mathscr{B}(T)$, then $\mathscr{B}\left(\bigcup \beta\right) \subset \mathscr{B}(D)$ and, in fact,

$$\Omega_{\overline{\mathscr{S}}}(\beta) = \Omega_{\overline{\mathscr{R}}}\left[\mathscr{B}\left(\bigcup \beta\right)\right]. \tag{2.2}$$

Indeed, the inclusion $\Omega_{\overline{\mathscr{R}}}\left[\mathscr{B}\left(\bigcup \beta\right)\right] \subset \Omega_{\overline{\mathscr{S}}}(\beta)$ is obvious. For the converse, take a point $y \in Omega_{\overline{\mathscr{S}}}(\beta)$ and let $K(y) = \left\{s \in \bigcup \beta : y \in \overline{\mathscr{R}}(s)\right\}$. Because every sieve is order-preserving with respect to the inverse inclusion, $K(y)$ is a subtree of $\bigcup \beta$ such that each $K(y) \cap D(n)$, $n < \omega$, is nonempty and finite. Hence, by Köning's lemma (see Lemma 5.7 in Chapter II of [9]), $K(y)$ contains an infinite branch

$$\delta \in \mathscr{B}(K(y)) \subset \mathscr{B}\left(\bigcup \beta\right) \subset \mathscr{B}(D).$$

Therefore, $y \in \Omega_{\overline{\mathscr{R}}}(\delta) \subset \Omega_{\overline{\mathscr{R}}}\left[\mathscr{B}\left(\bigcup \beta\right)\right]$.

We are now ready to show that the sieve $\mathscr{S}$ is as required. By [11, Lemma 2.3], $\mathscr{S}$ remains nonempty-open-valued and complete, hence the polar mapping $\Omega_{\overline{\mathscr{S}}} : \mathscr{B}(T) \to \mathscr{C}(Y)$ is usco. To show that $\Omega_{\overline{\mathscr{S}}}$ is also l.s.c., take an open set $U \subset Y$ and a branch $\beta \in \mathscr{B}(T)$ such that $\Omega_{\overline{\mathscr{S}}}(\beta) \cap U \neq \varnothing$. According to (2.2), there is a branch $\delta \in \mathscr{B}(\bigcup \beta) \subset \mathscr{B}(D)$ with $\Omega_{\overline{\mathscr{R}}}(\delta) \cap U \neq \varnothing$. Since $\Omega_{\overline{\mathscr{R}}}(\delta)$ is a singleton, we have that $\Omega_{\overline{\mathscr{R}}}(\delta) \subset U$, and, by the completeness of $\mathscr{R}$, we also have that $\overline{\mathscr{R}}(s) \subset U$ for some $s \in \delta$. Then, $s \in \sigma_s$ for some $\sigma_s \in \beta$, and the neighbourhood $\mathscr{O}(\sigma_s)$ of β in $\mathscr{B}(T)$ is such that

$$\varnothing \neq \Omega_{\overline{\mathscr{R}}}(\eta) \subset \Omega_{\overline{\mathscr{S}}}(\gamma) \cap \overline{\mathscr{R}}(s) \subset U$$

for every $\gamma \in \mathscr{O}(\sigma_s)$ and every $\eta \in \mathscr{B}\left(\bigcup \gamma\right)$ with $s \in \eta$, see (2.1). The proof is completed. $\square$

3. Proof of Theorem 1.1

Let X be a super-paracompact space, Y be a completely metrizable space, and let $\Phi : X \to \mathscr{F}(Y)$ be an l.s.c. mapping. According to Lemma 2.1, Y has a nonempty-open-valued, finitely-additive and complete sieve $\mathscr{S} : T \to 2^Y$ such that the polar mapping $\Omega_{\overline{\mathscr{S}}} : \mathscr{B}(T) \to \mathscr{C}(Y)$ is continuous. Consider the composite mapping $\Phi^{-1}[\mathscr{S}(t)]$, $t \in T$, which defines an open-valued finitely-additive sieve $\Phi^{-1} \circ \mathscr{S} : T \to 2^X$ on X because Φ is l.s.c. Since $\{\Phi^{-1}[\mathscr{S}(t)] : t \in T(0)\}$ is a finitely-additive open cover of X and X is super-paracompact, by [1, Proposition 2.3] (see, also, [2, Theorem 2.2]), X has a pairwise disjoint open cover $\{\mathscr{L}(t) : t \in T(0)\}$ such that $\mathscr{L}(t) \subset \Phi^{-1}[\mathscr{S}(t)]$, $t \in T(0)$. Take an element $s \in T(0)$. Then, $\mathscr{L}(s)$ is itself super-paracompact (being a clopen subset of X), while $\{\Phi^{-1}[\mathscr{S}(t)] : t \in \mathsf{node}(s)\}$ is a finitely-additive open cover of $\mathscr{L}(s)$. Hence, just like before, $\mathscr{L}(s)$ has a pairwise disjoint open cover $\{\mathscr{L}(t) : t \in \mathsf{node}(s)\}$ such that $\mathscr{L}(t) \subset \Phi^{-1}[\mathscr{S}(t)]$, $t \in \mathsf{node}(s)$. Proceeding by induction on the levels of the tree T, there exists a clopen-valued sieve $\mathscr{L} : T \to 2^X$ on X such that each family $\{\mathscr{L}(t) : t \in T(n)\}$, $n < \omega$, is discrete and $\mathscr{L}(t) \subset \Phi^{-1}[\mathscr{S}(t)]$, $t \in T$. Consider now the mapping $\mho_{\mathscr{L}} : X \to 2^{\mathscr{B}(T)}$ defined by $\mho_{\mathscr{L}}(x) = \Omega^{-1}_{\mathscr{L}}[\{x\}]$, $x \in X$. By [5, Proposition 5.2 and Lemma 5.3], $\mho_{\mathscr{L}} : X \to \mathscr{C}(\mathscr{B}(T))$ and is continuous. Finally, define $\varphi : X \to \mathscr{C}(Y)$ by $\varphi = \Omega_{\overline{\mathscr{S}}} \circ \mho_{\mathscr{L}}$. By Lemma 2.1, φ is continuous as a composition of continuous set-valued mappings, while, by [5, Lemma 7.1], φ is also a section for Φ.

To show the converse, suppose that X has the section property in Theorem 1.1. Take an open cover $\mathscr{V}$ of X, and let $\mathscr{W}$ be the cover of X consisting of all finite unions of elements of $\mathscr{V}$. Endow $\mathscr{V}$ with the discrete topology, and define a mapping $\Phi : X \to \mathscr{F}(\mathscr{V})$ by $\Phi(x) = \{V \in \mathscr{V} : x \in V\}$, $x \in X$. Since Φ is l.s.c., by assumption, it has a continuous section $\varphi : X \to \mathscr{C}(\mathscr{V})$. Then, $\mathscr{F} = \{\varphi(x) : x \in X\}$ is a family of nonempty finite subsets of $\mathscr{V}$. Set

$$U_F = \{x \in X : \varphi(x) = F\}, \quad F \in \mathscr{F}.$$

Since φ is continuous, U_F is a clopen subset of X, and clearly $\{U_F : F \in \mathscr{F}\}$ is a pairwise disjoint cover of X. Finally, observe that $W_F = \Phi^{-1}[F] \in \mathscr{W}$ for every $F \in \mathscr{F}$ because $\mathscr{W}$ consists of finite union of elements of $\mathscr{V}$. If $x \in U_F$ for some $F \in \mathscr{F}$, then $\varphi(x) = F$ and $\varphi(x) \cap \Phi(x) \neq \varnothing$. Hence, it follows that $\Phi(x) \cap F \neq \varnothing$ and, therefore, $x \in W_F$. Thus, $\{U_F : F \in \mathscr{F}\}$ is a refinement of $\{W_F : F \in \mathscr{F}\}$ and, by [1, Proposition 2.3] (see, also, [2, Theorem 2.2]), X is super-paracompact.

Remark 3.1. The metrizability of Y in Theorem 1.1 was used only in terms of Lemma 2.1, while Lemma 2.1 remains valid as far as Y has a nonempty-open-valued sieve $\mathscr{S} : D \to 2^Y$ for which the polar mapping $\Omega_{\overline{\mathscr{S}}} : \mathscr{B}(D) \to 2^Y$ is singleton-valued and continuous. Spaces with this property were said to have a λ-*base* [3], they are also known as *monotonically developable sieve complete* spaces. Monotonically developable spaces are a natural generalisation of Moore spaces, hence not necessarily metrizable. This gives the following generalisation of Theorem 1.1 to the case of a non-metrizable range.

Theorem 3.2. *If X is super-paracompact and Y is monotonically developable and sieve complete, then every l.s.c. mapping $\Phi : X \to \mathscr{F}(Y)$ has a continuous section $\varphi : X \to \mathscr{C}(Y)$.*

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DAVID BUHAGIAR
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF MALTA
MSIDA MSD.06
MALTA

E-mail: david.buhagiar@um.edu.mt

VALENTIN GUTEV
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF KWAZULU-NATAL
WESTVILLE CAMPUS
PRIVATE BAG X54001
DURBAN 4000
SOUTH AFRICA

E-mail: gutev@ukzn.ac.za

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