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Weakly-symmetry of the Sasakian lifts on tangent bundles

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This paper is dedicated to Professor Lajos Tamássy on the occasion of 90th birthday

Abstract. The weakly symmetry of the Sasakian lift G of a Riemannian metric g is characterized in terms of flatness for g and G. The cases of recurrent or pseudo-symmetric G studied by Binh and Tamássy are obtained in particular.

1. Introduction

The notion of *weakly symmetric Riemannian manifold* was introduced by Lajos Tamássy and Tran Quoc Binh in [9]. Since then, this type of Riemannian geometry was the subject of several papers: [3]–[8], [12]. For example, the authors of this concept study the case of Einstein and Sasaki manifold in [10], respectively the situation of Kähler manifolds in [11]; the case of decomposable (i.e. product) space appears in [1].

Two weaker variants of weakly symmetries, namely recurrence and pseudo-symmetry, are considered, again by Tamássy and Binh, in [2] having as prescribed metric the Sasakian lift G to TM of a Riemannian metric g on the base manifold M. Their result is as follows: *If (TM, G) is recurrent or pseudo-symmetric then (M, g) must be flat and thus (TM, G) must be flat too. The converse is trivially true.* The aim of this short note is to extend this reduction result to the general case of weakly symmetry for G.

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2. The Sasaki lift of a Riemannian metric

Fix a pair (M, g) with M_n a smooth $n(\geq 3)$-dimensional manifold and g a Riemannian metric on M. Let $\pi : TM \to M$ its tangent bundle. Let $q = (q^i) = (q^1, \ldots, q^n)$ be the coordinates on the base manifold M and the corresponding bundle coordinates $(q, v) = (q^i, v^i) = (q^1, \ldots, q^n, v^1, \ldots, v^n)$ on TM; then the metric g has the local coefficients $g_{ij} = g\big(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\big)$. For $t \in TM$ let $V_tTM = \ker \pi_{*,t}$ be *the vertical subspace* of T_tTM. The basis of the vertical distribution $V(TM)$ is given by $\{\frac{\partial}{\partial v^i}; 1 \leq i \leq n\}$. The Levi–Civita connection ∇ of g has the Christoffel symbols (Γ^k_{ij}) and yields the decomposition:

$$T_tTM = V_tTM \oplus H_tTM \tag{2.1}$$

with H_tTM *the horizontal subspace* spanned by $\left\{\frac{\delta}{\delta q^i}; 1 \leq i \leq n\right\}$ where:

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} - \Gamma^k_{ij} v^j \frac{\partial}{\partial v^k}. \tag{2.2}$$

Then every vector field $\tilde{X}$ on TM has the decomposition: $\tilde{X} = X^v + X^h$ with respect to (2.1). Also, a vector field $X = X^i(q)\frac{\partial}{\partial q^i}$ on M has the following lifts: a vertical one, $X^V = X^i \frac{\partial}{\partial v^i}$, respectively a horizontal one, $X^H = X^i \frac{\delta}{\delta q^i}$. Let us denote by R the tensor field of curvature of g.

The Sasaki lift of g to TM is the Riemannian metric G of diagonal form:

$$G = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{2.3}$$

with respect to the decomposition (2.1). Let $\tilde{\nabla}$ and $\tilde{R}$ be the Levi–Civita connection and respectively the curvature of G; we have in the point $t \in TM$:

$$\begin{cases} (\tilde{\nabla}_{X^H} Y^H)|_t = (\nabla_X Y)^H|_t - \dfrac{1}{2}(R(X,Y)t)^V|_t, \\ (\tilde{\nabla}_{X^H} Y^V)|_t = (\nabla_X Y)^V|_t + \dfrac{1}{2}(R(t,Y)X)^H|_t, \\ (\tilde{\nabla}_{X^V} Y^H)|_t = \dfrac{1}{2}(R(t,X)Y)^H|_t, (\tilde{\nabla}_{X^V} Y^V)|_t = 0. \end{cases} \tag{2.4}$$

The expression of $\tilde{R}$ is, [2, p. 556–557]:

$$\begin{cases} \tilde{R}(X^V, Y^V)Z^V|_t = 0, \\ \tilde{R}(X^V, Y^V)Z^H_t = \Big[R(X,Y)Z + \frac{1}{4}R(t,X)R(t,Y)Z \\ \quad -R(t,Y)R(t,X)Z\Big]^H_t, \\ \tilde{R}(X^H, Y^V)Z^V|_t = -\left[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(t,Y)R(t,Z)X\right]^H |_t, \\ 2\tilde{R}(X^H, Y^V)Z^H_t = [(\nabla_X R)(t,Y,Z)]^H_t \\ \quad + \left[R(X,Z)Y + \frac{1}{2}R(R(t,Y)Z,X)t\right]^V_t, \\ \tilde{R}(X^H, Y^H)Z^V|_t = \frac{1}{2}\left[(\nabla_X R)(t,Z,Y) - (\nabla_Y R)(t,Z,X)\right]^H|_t \\ \quad + \left[R(X,Y)Z + \frac{1}{4}R(R(t,Z)Y,X)t - \frac{1}{4}R(R(t,Z)X,Y)t\right]^V |_t, \\ \tilde{R}(X^H, Y^H)Z^H|_t = \frac{1}{2}\left[(\nabla_Z R)(X,Y,t)\right]^V|_t \\ \quad + \Big[R(X,Y)Z + \frac{1}{4}R(t,R(Z,Y)t)X + \frac{1}{4}R(t,R(X,Z)t)Y \\ \quad + \frac{1}{2}R(t,R(X,Y)t)Z\Big]^H_t \end{cases} \tag{2.5}$$

where t from the above expressions $R(t,\cdot)\cdot$, $R(\cdot,\cdot)t$ is thought as a vector field on M, namely $t = v^i\frac{\partial}{\partial q^i}$. Then $t^V = v^i\frac{\partial}{\partial v^i}$ is *the Liouville vector field* while $t^H = v^i\frac{\delta}{\delta q^i}$ is exactly *the geodesic spray* of the metric g.

3. Weakly symmetric Sasakian lifts

Definition ([9]). The Riemannian manifold (M,g) is called *weakly symmetric* if there exist four 1-forms $\alpha_1,\dots,\alpha_4$ and a vector field A, all on M, such that:

$$\begin{aligned}(\nabla_W R)(X,Y,Z) = {}& \alpha_1(W)R(X,Y)Z + \alpha_2(X)R(W,Y)Z + \alpha_3(Y)R(X,W)Z \\ & + \alpha_4(Z)R(X,Y)W + g(R(X,Y)Z,W)A. \end{aligned}\tag{3.1}$$

De and Bandyopadhyay proved in [3] that the following relations are necessary:

$$\begin{cases} \alpha_2 = \alpha_3 = \alpha_4 \\ A = (\alpha_2)^\sharp \end{cases} \tag{3.2}$$

i.e. A is the g-dual vector field of the 1-form α_2. Therefore a weakly symmetric Riemannian manifold is characterized by:

$$\begin{aligned}(\nabla_W R)(X,Y,Z) = \alpha_1(W)R(X,Y)Z + \alpha_2(X)R(W,Y)Z + \alpha_2(Y)R(X,W)Z \\ + \alpha_2(Z)R(X,Y)W + g(R(X,Y)Z,W)(\alpha_2)^\sharp. \end{aligned}\tag{3.3}$$

The aim of this note is to study the weakly symmetry of the Sasakian lift (2.3). More precisely, we have:

Theorem. *The Riemannian manifold (TM,G) is weakly symmetric if and only if the base manifold (M,g) is flat. Hence, (TM,G) is flat.*

Proof. If $R=0$ it results that $\tilde{R}=0$ and we have (3.3) as null equality. For a proof of the first part we use several times the techniques of [2] using the formulae (2.5). Firstly, we consider the condition (3.3) for W^H, X^H, Y^V and Z^V and we get:

$$\begin{aligned}&\alpha_1(W^H)\tilde{R}(X^H,Y^V)Z^V + \alpha_2(X^H)\tilde{R}(W^H,Y^V)Z^V + \alpha_2(Y^V)\tilde{R}(X^H,W^H)Z^V \\ &\quad + \alpha_2(Z^V)\tilde{R}(X^H,Y^V)W^H + G(\tilde{R}(X^H,Y^V)Z^V,W^H)(\alpha_2)^\sharp \\ &= -\tilde{\nabla}_{W^H}\left[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(t,Y)R(t,Z)X\right]^H_t \\ &\quad - \tilde{R}\left[(\nabla_W X)^H_t - \frac{1}{2}(R(W,X)t)^V_t)Y^V\right]Z^V \\ &\quad - \tilde{R}(X^H, \frac{1}{2}(R(t,Y)W)^H_t + (\nabla_W Y)^V_t)Z^V \\ &\quad - \tilde{R}(X^H,Y^V)\left[\frac{1}{2}(R(t,Z)W)^H_t + (\nabla_W Z)^V_t\right]. \end{aligned}\tag{3.4}$$

Secondly, in the above equation we consider only the four times the vertical part of both sides and then:

$$\begin{aligned}&\alpha_2(Y^V)\left[4R(X,W)Z + R(R(t,Z)W,X)t - R(R(t,Z)X,W)t\right] \\ &\quad + \alpha_2(Z^V)(2R(X,W)Y + R(R(t,Y)W,X)t) - g(2R(Y,Z)X \\ &\quad + R(t,Y)R(t,Z)X,W)\alpha_2^\sharp = \frac{1}{2}R\left[W, 2R(Y,Z)X + R(t,Y)R(t,Z)X\right]t \\ &\quad - 2R(X,R(t,Y)W)Z + \frac{1}{2}R[R(t,Z)R(t,Y)W,X]t - \frac{1}{2}R[R(t,Z)X,R(t,Y)W] \\ &\quad - \frac{1}{2}R[R(t,Y)R(t,Z)W,X]t + R(X,R(t,Z)W)Y \end{aligned}\tag{3.5}$$

which is exactly four times the first relation on the page 559 of [2]. Thus in the following, the arguments are as in [2, p. 559-560]. We choose in (3.5) consequently:

I) $Y = t$ and then:

$$\alpha_2(t^V)\left[4R(X,W)Z + R(R(t,Z)W,X)t - R(R(t,Z)X,W)t\right] + 2\alpha_2(Z^V)R(X,W)t \\ -2g(R(t,Z)X,W)(\alpha_2)^\sharp = R(W,R(t,Z)X)t + R(X,R(t,Z)W)t \tag{3.6}$$

II) $Z = t$ and then:

$$4\alpha_2(Y^V)R(X,W)t + \alpha_2(t^V)[2R(X,W)Y + R(R(t,Y)W,X)t] - 2g(R(Y,t)X,W)\alpha_2^\sharp \\ = R(W,R(Y,t)X)t - 2R(X,R(t,Y)W)t. \tag{3.7}$$

In the last relation we replace Y with Z:

$$4\alpha_2(Z^V)R(X,W)t + \alpha_2(t^V)\left[2R(X,W)Z + R(R(t,Z)W,X)t\right] - 2g(R(Z,t)X,W)\alpha_2^\sharp \\ = R(W,R(Z,t)X)t - 2R(X,R(t,Z)W)t \tag{3.8}$$

and by adding (3.6) and (3.8) we derive:

$$6\alpha_2(Z^V)R(X,W)t + \alpha_2(t^V)\left[6R(X,W)Z + 2R(R(t,Z)W,X)t - R(R(t,Z)X,W)t\right] \\ = -R(X,R(t,Z)W)t. \tag{3.9}$$

With $Z = t$ we get:

$$\alpha_2(t^V)R(X,W)t = 0 \tag{3.10}$$

and if $\alpha_2(t^V) \neq 0$ we have the conclusion. Suppose now that $\alpha_2(t^V) = 0$ then $(\alpha_2)^{\sharp V} = 0$; returning to (3.6) it results:

$$4\alpha_2(Z^V)R(X,W)t = 2R(X,R(t,Z)W)t + 2R(W,R(t,Z)X)t. \tag{3.11}$$

With $W = X$ we obtain:

$$R(X,R(t,Z)X)t = 0 \tag{3.12}$$

and we take the g-product with Z: $g(R(t,Z)X,R(t,Z)X) = 0$ which means:

$$R(t,Z)X = 0. \tag{3.13}$$

Again the g-product with an arbitrary Y gives:

$$g(R(X,Y)t,Z) = 0. \tag{3.14}$$

The vector field Z being arbitrary we get: $R(X,Y)t = 0$, for every X, Y and t. Thus, we have the Conclusion. $\square$

For $\alpha_2 = 0$ respectively $\alpha_1 = 2\alpha_2$ in (3.3) we get the Tamássy–Binh result of Introduction:

Corollary. *The Riemannian manifold (TM, G) is recurrent or pseudo-symmetric or locally symmetric ($\tilde{\nabla}\tilde{R} = 0$) if and only if the base manifold (M, g) is flat. Hence, (TM, G) is flat.*

The following *open problem* is natural: to extend the present Theorem to other classes of metrics on tangent bundles. The possible (first) candidates are from the natural metrics of [4] or [5].

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References

[1] T. Q. Binh, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **42** (1993), 103–107.

[2] T. Q. Binh and L. Tamássy, On recurrence or pseudo-symmetry of the Sasakian metric on the tangent bundle of a Riemannian manifold, *Indian J. Pure Appl. Math.* **35** (2004), 555–560.

[3] U. C. De and S. Bandyopadhyay, On weakly symmetric Riemannian spaces, *Publ. Math. Debrecen* **54** (1999), 377–381.

[4] V. Oproiu, Some classes of natural almost Hermitian structures on the tangent bundles, *Publ. Math. Debrecen* **62** (2003), 561–576.

[5] N. Papaghiuc, A locally symmetric pseudo-Riemannian structure on the tangent bundle, *Publ. Math. Debrecen* **59** (2001), 303–315.

[6] M. Prvanović, On weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen* **46** (1995), 19–25.

[7] A. A. Shaikh and S. K. Jana, On weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen* **71** (2007), 27–41.

[8] H. Singh and Q. Khan, On special weakly symmetric Riemannian manifolds, *Publ. Math. Debrecen* **58** (2001), 523–536.

[9] L. Tamássy and T. Q Binh,, On weakly symmetric and weakly projective symmetric Riemannian manifolds, Differential geometry and its applications (Eger, 1989), North-Holland, Amsterdam, *Colloq. Math. Soc. János Bolyai* **56** (1992), 663–670.

[10] L. Tamássy and T. Q. Binh, On weak symmetries of Einstein and Sasakian manifolds, *Tensor* **53** (1993), 140–148.

[11] L. Tamássy, U. C. De and T. Q. Binh, On weak symmetries of Kaehler manifolds, *Balkan J. Geom. Appl.* **5** (2000), 149–155.

[12] S. A. Uysal and R. Ö. Laleoglu, On weakly symmetric spaces with semi-symmetric metric connection, *Publ. Math. Debrecen* **67** (2005), 145–154.

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