

## Some applications of index form in Finsler geometry

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**Abstract.** In this paper, two results regarding the impact of some curvatures on the topology of a Finsler manifold are established. The first one is a compactness theorem, and the second is an intersection theorem. Their hypotheses involve conditions on an invariant generalizing the Ricci curvature and the proofs are based on the index form along geodesics.

### 1. Introduction

Various differential geometric invariants have strong impact on the topology of differentiable manifolds as illustrated by deep results in the Riemannian geometry such as the theorems of Hopf–Rinow, Myers, Rauch, Synge. The area of these results has been extended along years to the Finslerian setting. The most recent and modern account of them is due to D. BAO, S. S. CHERN and Z. SHEN (see [6, Chapter 6–9]). Their book has been followed by many papers in this field. We cite only few, [4] and [23], as more related to our results.

The main differential geometric invariants involved in the results aiming to establish a topological property are the flag curvature and the Ricci scalar. Among the many others, there exists one, denoted by  $\text{Ric}_k$ , that interpolates between the flag curvature and the Ricci curvature. It is associated to a  $(k + 1)$ -dimensional subspace of the tangent space in a point of a manifold in such a way that for  $k = 1$  it coincides with the flag curvature, and for  $k = \dim M - 1$  it is nothing but the Ricci curvature.

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In this paper, we prove two results related to  $\text{Ric}_k$  which are different in their nature. The first one (see Section 3) provides a sufficient condition on the average of the  $k$ -Ricci curvature in order that the Finsler manifold be compact. The second one (see Section 4) says that if  $\text{Ric}_k$  is positive, then two submanifolds of a Finslerian  $n$ -dimensional manifold, one with asymptotic index  $n - 1$  and one minimal, must intersect. The proofs of both results are based on the index form written in a special frame along geodesics.

## 2. Preliminaries

**2.1. The Morse index form.** We are working in the framework of the standard non-reversible Finsler setting:  $(M, F)$  denotes a Finsler manifold, a pair of non-reversible Finsler metric and a real manifold  $M$  of dimension  $n$ .  $\nabla, \Omega$  denote the Cartan connection, and its curvature tensor, resp., living on the tangent manifold  $TM$ . See for details [6], [21], or an outline in [4].

Now, we recall some facts about the variation of energy and Morse index form, mainly from [18].

Let  $\sigma : [a, b] \rightarrow M$  be a regular curve on  $M$ . Its length with respect to the Finsler metric  $F : TM \rightarrow \mathbb{R}^+$  is given by  $L(\sigma) = \int_a^b F(\dot{\sigma}(t))dt$ , and its energy is given by  $E(\sigma) = \int_a^b F^2(\dot{\sigma}(t))dt$ .

The Finsler metric induces naturally the (Finslerian) distance by  $d(p, q) = \inf_{\sigma \in C(p, q)} L(\sigma)$ , where  $C(p, q)$  is the set of piecewise smooth curves from  $p$  to  $q$ . The properties of a distance, except the symmetry, hold well. The pair  $(M, d)$  is called sometimes a *generalized metric space*. For a non-reversible Finsler metric,  $d$  is not symmetric, because the length of a curve may not coincide with the length of the reverse curve  $\tilde{\sigma}(t) = \sigma(a + b - t) \in M$ . The non-reversibility property is also reflected in the notion of Cauchy sequences.

The classical Hopf–Rinow theorem splits into forward and backward versions (see [6], [10]).

The non-reversibility of the distance implies the existence of two open balls, the forward balls

$$B^+(p, r) = \{x \in M | d(p, x) < r\},$$

where  $p \in M$  and  $r > 0$ , and the backward balls

$$B^-(p, r) = \{x \in M | d(x, p) < r\}.$$

A symmetrized distance can be defined as

$$d_s(p, q) = \frac{1}{2}(d(p, q) + d(q, p)).$$

The closed balls will be denoted by a bar, i.e.  $\overline{B}^+(p, r)$  and  $\overline{B}^-(p, r)$ . The topologies induced by these two kinds of balls agree with the topology of the manifold. We also denote the associated balls of  $d_s$  by  $B_s(x, r)$ . In [10, Proposition 2.2], there is proved a Hopf–Rinow theorem for symmetrized closed balls, i.e. the symmetrized distance  $d_s$  is complete if  $\overline{B}_s(p, r)$  are compact for all  $p \in M$  and  $r > 0$  (or, equivalently,  $\overline{B}^+(p, r) \cap \overline{B}^-(p, r)$  is compact for all  $p \in M$  and  $r > 0$ ). The conditions there are weaker than those in the theorems involving forward or backward completeness. In the same paper [10], an example of Randers type with compact symmetrized balls is constructed, which, however, fails to be forward or backward complete.

The non-reversibility of the metric also induces two types of geodesic completeness, the *forward* type when the domain of the geodesic can always be extended to  $(a, \infty)$  for some  $a \in \mathbb{R}$ , and the *backward* type when it can be extended to  $(-\infty, b)$  for some  $b \in \mathbb{R}$ .

The critical points of the (length) energy functional are the normal geodesics  $\sigma$  in the Finsler manifold  $M$  whenever they are parameterized by arc-length, i.e.  $F(\dot{\sigma}) = 1$ . One proves that the geodesics are characterized also by

**Theorem 1** ([1]). *A regular curve  $\sigma_0$  is geodesic for  $F$  iff*

$$\nabla_{T^H} T^H \equiv 0,$$

where  $T^H(u) = \dot{\sigma}^H = \chi_u(\dot{\sigma}(t)) \in \mathcal{H}_u$ , for all  $u \in \widetilde{M}_{\sigma(t)}$ .

The second variation formula provides the Jacobi fields and suggests the consideration of the index form. It is derived by using a two-parameter geodesic variation. For details, we refer to [1], [16].

Let  $\sigma : [a, b] \rightarrow M$  be a normal geodesic in a Finsler manifold  $M$ . We will denote by  $\mathfrak{X}[a, b]$  the space of piecewise smooth vector fields  $X$  along  $\sigma$  such that

$$\langle X^H, T^H \rangle_T \equiv 0.$$

Furthermore, we shall denote by  $\mathfrak{X}_0[a, b]$  the subspace of all  $X \in \mathfrak{X}[a, b]$  such that  $X(a) = X(b) = 0$ .

*Definition 2* ([1]). The *Morse index form*  $I = I_a^b : \mathfrak{X}[a, b] \times \mathfrak{X}[a, b] \rightarrow \mathbb{R}$  of the normal geodesic  $\sigma : [a, b] \rightarrow M$  is the symmetric bilinear form

$$I(X, Y) = \int_a^b [\langle \nabla_{T^H} X^H, \nabla_{T^H} Y^H \rangle_T - \langle \Omega(T^H, X^H)Y^H, T^H \rangle_T] dt,$$

for all  $X, Y \in \mathfrak{X}[a, b]$ .

After some computations one gets another formula for the Morse index form [1]:

$$I(X, Y) = \langle \nabla_{T^H} X^H, Y^H \rangle_T \Big|_a^b - \int_a^b \langle \nabla_{T^H} \nabla_{T^H} X^H + \Omega(T^H, X^H)T^H, Y^H \rangle_T dt.$$

*Definition 3* ([1]). A *Jacobi field* along a geodesic  $\sigma : [a, b] \rightarrow M$  is a vector field  $J$  which satisfies the Jacobi equation

$$\nabla_{T^H} \nabla_{T^H} J^H + \Omega(T^H, J^H)T^H \equiv 0,$$

where  $J^H(t) = \chi_{\dot{\sigma}(t)}(J(t))$ .

$\dot{\sigma}$  and  $t\dot{\sigma}$  are Jacobi fields; the first one never vanishes, the second one vanishes only at  $t = 0$ .

Two points  $\sigma(t_0)$  and  $\sigma(t_1)$ ,  $t_0, t_1 \in [a, b]$  are said to be conjugate along  $\sigma$  if there exists a nonzero Jacobi field  $J$  along  $\sigma$  with  $J(t_0) = 0$  and  $J(t_1) = 0$ .

We recall from [6, p. 182] the following

**Proposition 4.** *Let  $\sigma(t), 0 \leq t \leq r$  be a geodesic in a Finsler manifold  $(M, F)$ . Suppose that no point  $\sigma(t), 0 < t \leq r$  is conjugate to  $p := \sigma(0)$ . Let  $W$  be any piecewise  $C^\infty$  vector field along  $\sigma$ , and let  $J$  denote the unique Jacobi field along  $\sigma$  that has the same boundary values as  $W$ . That is,  $J(0) = W(0)$  and  $J(r) = W(r)$ . Then*

$$I(W, W) \geq I(J, J).$$

Equality holds if and only if  $W$  is actually a Jacobi field, and in this case  $J = W$ .

**2.2. Minimal submanifolds. Focal points.** Let  $P$  be a submanifold of  $M$  of dimension  $d < n$ . We consider the set

$$A = \{(x, v) | x \in P, v \in T_x M\} = \{\tilde{x} \in \widetilde{M} | \pi(\tilde{x}) \in P\}.$$

Let  $H_{\tilde{x}}T_x M$  and  $H_{\tilde{x}}T_x P$  be the horizontal lifts of  $T_x M$  and  $T_x P$  to  $\tilde{x}$ , and

$$H_P T M = \bigcup_{\tilde{x} \in A} H_{\tilde{x}} T_x M$$

and

$$H_P T P = \bigcup_{\tilde{x} \in A} H_{\tilde{x}} T_x P.$$

For horizontal vector fields  $X, Y \in H_P T P$ , let  $X^*, Y^*$  be some prolongations of them to  $H_P T M$ . The restriction of  $\nabla_{X^*} Y^*$  to  $\tilde{P} = T P \setminus 0$  does not depend on the choice of the prolongations.

Let  $P_{\tilde{x}}^\perp$  be the  $\langle \cdot, \cdot \rangle_{\tilde{x}}$  orthogonal complement of  $H_{\tilde{x}}TP$  in  $H_{\tilde{x}}TM$ . By the orthogonal decomposition

$$H_{\tilde{x}}T_xM = H_{\tilde{x}}T_xP \oplus P_{\tilde{x}}^\perp, \tilde{x} = (x, v) \in A,$$

we obtain that

$$\nabla_{X^*}Y^* = \nabla_X^*Y + \mathbb{I}_v(X, Y).$$

We will call  $\mathbb{I}_v(X, Y)$  the second fundamental form at  $X$  and  $Y$  in the direction of  $v$ . Note that for  $\tilde{x} = (x, v)$  with  $v \in T_xM \setminus T_xP$  we have

$$\langle \nabla_{X^*}Y^*, v^H \rangle_v = \mathbb{I}_v(X, Y). \tag{1}$$

*Definition 5.* Let  $P \subset M$  be an  $r$ -dimensional submanifold of a Finsler manifold  $(M, F)$ . The submanifold  $P$  is called *minimal* if for every tangent vector  $v$  to  $M$  and for any horizontal orthogonal vectors  $V_i^H \in H_PTP, i = \overline{1, r}$  (i.e.  $\langle V_i^H, V_j^H \rangle_v = 0$  for  $i \neq j$ ) we have  $\sum_{i=1}^r \mathbb{I}_v(V_i^H, V_i^H) = 0$ .

This definition is an analytical analogue of the one used for the Riemannian metrics. In the Riemannian case, it comes from the fact that minimal submanifolds are critical points of the volume with respect to a variation of the submanifold in a normal direction. In the Finsler setting, the problem is not so simple. In [2], it is pointed out that totally geodesic submanifolds could not be minimal if one considers the Hausdorff measure. But in our second fundamental form the reference vector is not tangent to submanifold in the problems considered here. In fact, the only condition used is that the second fundamental form is zero on directions orthogonal to the submanifold  $P$ .

The condition of minimality is equivalent with the vanishing of the trace of the linear operator  $A_{v^H}$ , where  $A_{v^H}$  is defined by

$$\langle A_{v^H}X^H, Y^H \rangle_v = \langle \mathbb{I}_v(X^H, Y^H), v^H \rangle_v.$$

For details, we refer to [11], [20].

*Definition 6.* Let  $f : N \rightarrow M$  be an immersion. The *asymptotic index* of the immersion  $f$  in the direction  $v$  is defined by

$$\nu_f = \min_{x \in N} \nu_f(x),$$

where  $\nu_f(x)$  is the maximal dimension of a subspace of  $T_xN$  on which the second fundamental form vanishes in every direction  $v \in T_xM \setminus T_xN$ .

Now, let  $\sigma : [a, b] \rightarrow M$  be a normal geodesic in  $M$  with  $\sigma(a) \in P$  and  $\dot{\sigma}^H(a)$  in the normal bundle of  $P$  (i.e.  $\dot{\sigma}^H(a) \perp (H_{\dot{\sigma}(a)}T_{\sigma(a)}P)$ ).

Let  $\tilde{\mathfrak{X}}^P = \mathfrak{X}^P[a, b]$  be the vector space of all piecewise smooth vector fields  $X$  along  $\sigma$  such that  $X^H(a) \in T_{\dot{\sigma}(a)}\tilde{P}$ , and let  $\mathfrak{X}^P$  be the subspace of  $\tilde{\mathfrak{X}}^P$  consisting of these  $X$  such that  $X^H$  is orthogonal to  $\dot{\sigma}^H$  along the curve  $\sigma$ .

We have that

$$\begin{aligned} \langle \nabla_{T^H} X^H, Y^H \rangle_T &= \langle \nabla_{X^H} T^H + [T^H, X^H] + \theta(T^H, X^h), Y^H \rangle_T \\ &= \langle \nabla_{X^H} T^H, Y^H \rangle_T, \end{aligned} \tag{2}$$

because  $[T^H, X^H]$  and  $\theta(T^H, X^h)$  are vertical vector fields ([1]).

And for  $Y^H$  orthogonal to  $T^H$ , we have that

$$0 = X^H \langle T^H, Y^H \rangle_T = \langle \nabla_{X^H} T^H, Y^H \rangle_T + \langle T^H, \nabla_{X^H} Y^H \rangle_T. \tag{3}$$

By considering the vector fields  $X^H, Y^H$  such that  $X^H(a), Y^H(a) \in T_{\dot{\sigma}(a)}\tilde{P}$  and taking account of formulas (1), (2), (3), the Morse index form  $I^P : \mathfrak{X}^P \times \mathfrak{X}^P \rightarrow \mathbb{R}$  becomes

$$\begin{aligned} I^P(X, Y) &= \langle \nabla_{T^H} X^H, Y^H \rangle_T \Big|_a^b + \langle \mathbb{L}_T(X^H, Y^H), T^H \rangle_T \Big|_a^b \\ &\quad - \int_a^b \langle \nabla_{T^H} \nabla_{T^H} X^H + \Omega(T^H, X^H)T^H, Y^H \rangle_T dt. \end{aligned}$$

From [18] we know that  $I^P$  is symmetric.

*Definition 7* ([18]). Let  $P \subset M$  be a  $d$ -dimensional submanifold of a Finsler manifold  $(M, F)$ . A  $P$ -Jacobi field  $J$  is a Jacobi field which satisfies in addition

$$J(a) \in T_{\sigma(a)}P$$

and

$$\langle \nabla_{T^H} J^H + A_{T^H} J^H, Y^H \rangle_T \Big|_a = 0,$$

for all  $Y \in H_{\dot{\sigma}(a)}T_{\sigma(a)}P$ .

The last condition means in fact that

$$\nabla_{T^H} J^H + A_{T^H} J^H \in P_{\dot{\sigma}(a)}^\perp.$$

The dimension of the vector space of all  $P$ -Jacobi fields along  $\sigma$  is equal to the dimension of  $M$ , and the dimension of the vector space of the  $P$ -Jacobi fields satisfying

$$\langle J^H, T^H \rangle = 0$$

is equal to  $\dim M - 1$ .

If  $P$  is a point, then a  $P$ -Jacobi field is a Jacobi field  $J$  along  $\sigma$  such that  $J(a) = 0$ .

A point  $\sigma(t_0)$ ,  $t_0 \in [a, b]$  is said to be a  $P$ -focal point along  $\sigma$  if there exists a non-null  $P$ -Jacobi field  $J$  along  $\sigma$  with  $J(t_0) = 0$ .

We shall use the following Lemma from [18].

**Lemma 8.** *Let  $(M, F)$  be a Finsler manifold,  $\sigma : [a, b] \rightarrow M$  be a geodesic, and  $P \subset M$  be a submanifold of  $M$ . Suppose that there is no  $P$ -focal point along  $\sigma$ . Let  $X \in \mathfrak{X}^P$  be a vector field orthogonal to  $\sigma$ , and  $J$  a  $P$ -Jacobi field such that  $X(b) = J(b)$ . Then*

$$I^P(X, X) \geq I^P(J, J),$$

with equality if and only if  $X = J$ .

**2.3.  $k$ -Ricci curvature.** We introduce the  $k$ -Ricci curvature  $\text{Ric}_k$  following [21]. For a  $(k + 1)$ -dimensional subspace  $\mathcal{V} \in T_x M$ , the Ricci curvature  $\text{Ric}_y \mathcal{V}$  on  $\mathcal{V}$  is the trace of the Riemann curvature restricted to  $\mathcal{V}$ , with flagpole  $y$ , and is given by:

$$\text{Ric}_y(\mathcal{V}) = \sum_{i=1}^k \langle R_y(b_i), b_i \rangle_y = \sum_{i=1}^k \langle \Omega(y, b_i)y, b_i \rangle_y,$$

where  $R_y(b_i) \equiv \Omega(y, b_i)y$  and  $y, (b_i)_{i=1, \dots, k}$  is an arbitrary orthonormal basis for  $(\mathcal{V}, \langle \cdot, \cdot \rangle_y)$ .  $\text{Ric}_y(\mathcal{V})$  is well-defined and positively homogeneous of degree 2 on  $\mathcal{V}$ ,

$$\text{Ric}_{\lambda y}(\mathcal{V}) = \lambda^2 \text{Ric}_y(\mathcal{V}), \quad \text{for } \lambda > 0, y \in \mathcal{V}.$$

It is clear from the definition that  $\text{Ric}_y(T_x M)$  is nothing but the Ricci curvature  $\text{Ric}(y)$  for  $y \in T_x M$ .

If  $\mathcal{V} = \mathcal{P} \subset T_x M$  is a tangent plane, the flag curvature is given by

$$K(\mathcal{P}, y) = \frac{\langle R_y(u), u \rangle_y}{\langle y, y \rangle_y \langle u, u \rangle_y - \langle u, y \rangle_y^2},$$

where  $u \in \mathcal{P} \setminus \{0\}$ ,  $\text{span}(y, u) = \mathcal{P}$ . This is independent of the choice of  $u \in \mathcal{P} \setminus \{0\}$ . If  $u$  is  $g_y$ -orthogonal to  $y$  and its  $g_y$ -norm is 1, then it becomes

$$K(\mathcal{P}, y) = \frac{\text{Ric}_y \mathcal{P}}{F^2(y)}, \quad y \in \mathcal{P}.$$

Consider the following function on  $M$ :

$$\text{Ric}_k(x) := \inf_{\dim(\mathcal{V})=k+1} \inf_{y \in \mathcal{V}} \frac{\text{Ric}_y(\mathcal{V})}{F^2(y)},$$

the infimum being considered over all  $(k + 1)$ -dimensional subspaces  $\mathcal{V} \subset T_x M$  and  $y \in \mathcal{V} \setminus \{0\}$ . From the above definitions it can be seen that

$$\text{Ric}_1 \leq \dots \leq \frac{\text{Ric}_k}{k} \leq \dots \leq \frac{\text{Ric}_{n-1}}{n-1},$$

and

$$\text{Ric}_1 = \inf_{(P,y)} K(P, y) \quad \text{and} \quad \text{Ric}_{(n-1)} = \inf_{F(y)=1} \text{Ric}(y).$$

We will say that the Finsler manifold  $(M, F)$  has positive  $k$ -Ricci curvature if and only if  $\text{Ric}_k > 0$ .

### 3. A compactness theorem

In this Section, we prove the Finslerian version of a Galloway compactness theorem, see Theorem 2 in [13].

Let  $(M, F)$  be a Finsler manifold such that  $B^+(x, r) \cap B^-(x, r)$  is precompact for all  $x \in M$  and  $r > 0$ . Then, for every  $x \in M$ , the geodesic starting from  $x$  with some initial tangent vector is defined on an interval  $[0, a_x)$ . Consider an increasing function  $f : [0, \infty) \rightarrow [0, A)$  of class  $C^1$  (for example, take  $A = \frac{\pi}{2}$  and  $g(x) = \arctan x$ ). We have the following composition

$$[0, \infty) \xrightarrow{u} [0, A) \xleftrightarrow{v = \frac{ax}{A}t} [0, a_x),$$

and consider the inverse of the composed function  $G := (v \circ u)^{-1} : [0, a_x) \rightarrow [0, \infty)$ .

**Theorem 9.** *Let  $(M, F)$  be a  $n$ -dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \quad \text{is precompact for all } x \in M \text{ and } r > 0. \tag{4}$$

*If there exists a point  $p \in M$  such that along any geodesic  $\sigma : [0, \infty) \rightarrow M$  emanating from  $p$  and parameterized by arc length  $t$  the condition*

$$\int_0^\infty t^\alpha \text{Ric}_k(t) dt = \infty \tag{5}$$

*holds at least for a  $k = 1, 2, \dots, n - 1$  and for some  $\alpha \in [0, 1)$ , then  $M$  is compact. The  $\text{Ric}_k(t)$  means one of the Ricci's curvatures for  $F$  or its reverse metric  $\tilde{F}(u) = F(-u)$ .*

We divide the proof into three lemmas.

**Lemma 10.** *Let  $(M, F)$  be a  $n$ -dimensional Finsler manifold which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \quad \text{is precompact for all } x \in M \text{ and } r > 0. \quad (6)$$

*If there exists a point  $p \in M$  such that every geodesic ray emanating from  $p$  has a point conjugate to  $p$  along the ray with respect to  $F$  or  $\tilde{F}$ , then  $M$  is compact.*

PROOF. Let  $S_p$  be the indicatrix in the point  $p \in M$ . For each  $y \in S_p$  issue the unit speed geodesic from  $p$  with the initial unity velocity  $y$ . Let  $c_y$  be the value of  $t$  in the first conjugate point of  $p$ , and  $i_y$  the value of  $t$  in the cut point of  $p$ . By the hypothesis, the set of  $c_y$  is forwardly bounded from above, and since one has  $i_y \leq c_y$ , it follows that  $\sup_{y \in S_p} i_y \leq \sup_{y \in S_p} c_y$ , and because the diameter of  $M$  is less or equal to  $\sup_{y \in S_p} i_y$ , it comes out that  $M$  is forwardly bounded from the above. As  $M$  is closed in its own topology, by the Hopf–Rinow theorem it is compact.  $\square$

Before going on, we recall that a differential equation

$$x'' + h(t)x = 0, \quad (7)$$

where  $h$  is a continuous function on an interval  $I$ , is called of *Jacobi type* and it is said to be *conjugate* if there exists a nontrivial solution  $\phi$  which vanishes for at least two values  $t_1$  and  $t_2$  in  $I$ .

The equation (7) is called oscillatory on  $[0, \infty)$  if each solution of it on  $[0, \infty)$  has arbitrary large, and hence infinitely many zeros. If (7) is oscillatory, then it is conjugate, too.

**Lemma 11.** *Suppose that there exists a point  $p \in M$  such that along a geodesic  $\sigma : [0, \infty) \rightarrow M$  emanating from  $p$  and parameterized by the arc length  $t$ , the Jacobi type equation*

$$x'' + \frac{\text{Ric}_k(t)}{k}x = 0 \quad (8)$$

*is conjugate on  $[0, \infty)$ . Then  $p$  has a conjugate point on  $\sigma$ .*

PROOF. Suppose, by contrary, that  $p$  has no conjugate points on  $\sigma$ . Since the equation (8) is conjugate, there exists a nontrivial solution  $\phi : [0, \infty) \rightarrow \mathbb{R}$  of it such that  $\phi(t_1) = \phi(t_2) = 0$  for  $0 \leq t_1 < t_2$ , and  $\phi(t) \neq 0$  for  $t \in (t_1, t_2)$ . Define a function  $f$  that is null on  $[0, t_1)$  and coincides with  $\phi$  for  $t \in [t_1, t_2]$ .

We consider a  $g_T$ -orthonormal frame  $(e_i(t))$  along  $\sigma$  with each  $(e_i(t))$  parallel along  $\sigma$  and  $e_n = T$ . We set  $W_\alpha(t) = f(t)e_\alpha(t)$ ,  $\alpha = 1, 2, \dots, n - 1$ . These  $W_\alpha$  are

$C^\infty$ -vector fields on  $[0, t_1]$  and  $[t_1, t_2]$ . We compute the index form  $I(W_\alpha, W_\alpha)$  on the interval  $[0, t_2]$ . Using  $D_T W_\alpha = f'e_\alpha$ ,  $D_T D_T W_\alpha = f''e_\alpha$  and the definition of  $f$ , we get

$$I(W_\alpha, W_\alpha) = - \int_0^{t_2} (f'' + K(e_\alpha \wedge T)f)f dt. \tag{9}$$

Summing up from 1 to a fixed  $k = 1, 2, \dots, n - 1$ , one yields

$$\sum_{\alpha=1}^k I(W_\alpha, W_\alpha) = -k \int_0^{t_2} f f'' dt - \int_0^{t_2} f^2 \sum_{\alpha=1}^k K(e_\alpha \wedge T) dt. \tag{10}$$

By the definition of  $\text{Ric}_k$ , we get

$$-f^2 \sum_{\alpha=1}^k K(e_\alpha \wedge T) \leq -f^2 \text{Ric}_k(t), \tag{11}$$

and so we obtain

$$\begin{aligned} \sum_{\alpha=1}^k I(W_\alpha, W_\alpha) &\leq -k \int_0^{t_2} f f'' dt - \int_0^{t_2} f^2 \text{Ric}_k(t) dt \\ &= -k \int_{t_1}^{t_2} (\phi''(t) + \frac{\text{Ric}_k(t)}{k} \phi) \phi(t) dt = 0, \end{aligned} \tag{12}$$

because  $\phi$  is a solution of equation (8). Thus, there exists at least an  $\alpha$  such that  $I(W_\alpha, W_\alpha) \leq 0$ . We denote that  $W_\alpha$  by  $W$  and then proceed by contradiction using Proposition 4. The vector field  $W$  satisfies  $W(t) = 0$  for  $t \in [t_1, t_2]$  and  $W(t_2) = 0$ , and it cannot be a Jacobi field since is zero on the interval  $(0, t_1)$  (by unicity of solution of a second order differential equation). By Proposition 4, we have  $0 = I(J, J) < I(W, W) < 0$ , which is a contradiction. Thus on the geodesic  $\sigma$  there exists some point conjugate to  $p$ .  $\square$

Theorem 2 from the paper [17] by R. A. MOORE, in some particular conditions gives the following

**Lemma 12.** *Consider equation (7) with  $t \in [0, \infty)$ . If for some  $\alpha$ ,  $0 \leq \alpha < 1$ , we have*

$$\int_0^\infty t^\alpha h(t) dt = +\infty, \tag{13}$$

*then equation (7) is oscillatory.*

Now, let us combine the above three Lemmas. By Lemma 11, taking  $h(t) = \frac{\text{Ric}_k(t)}{k}$ , the assumptions of Theorem 14 imply that the equation (8) is oscillatory, hence conjugate. Thus by Lemma 10, there exists a point  $p \in M$  such that each geodesic starting from  $p$  has a conjugate point of  $p$ . By Lemma 9, the Finsler manifold  $(M, F)$  is compact. Thus Theorem 14 is proved.  $\square$

The observations in the beginning of Subsection 2.1 and the previous theorem lead to the following

**Theorem 13.** *Let  $(M, F)$  be a forward (resp. backward) complete Finsler manifold. If there exists a point  $p \in M$  such that along any geodesic  $\sigma : [0, \infty) \rightarrow M$  emanating from  $p$  and parameterized by arc length  $t$  the condition*

$$\int_0^\infty t^\alpha \text{Ric}_k(t) dt = \infty \tag{14}$$

*holds at least for a  $k = 1, 2, \dots, n - 1$  and some  $\alpha \in [0, 1)$ , then  $M$  is compact.*

*Remark.* If the condition  $\int_0^\infty t^\alpha \text{Ric}_k(t) dt = \infty$  holds for  $k = 1$ , that is for  $\text{Ric}_1 = \inf_{(P,y)} K(P, y)$ , then the inequalities on  $\text{Ric}_k$  in Subsection 2.3 give the same condition for the same  $k = 2, 3, \dots, n - 1$ .

#### 4. An intersection theorem

In this section, we prove a generalization of an intersection theorem in Riemannian setting (see [14, Theorem 1]).

**Theorem 14.** *Let  $(M, F)$  be a  $n$ -dimensional Finsler manifold of nonnegative  $k$ -Ricci curvature which satisfies the condition*

$$B^+(x, r) \cap B^-(x, r) \text{ is precompact for all } x \in M \text{ and } r > 0. \tag{15}$$

*Let  $Q$  be a complete immersed submanifold of  $\dim Q = n - 1$  and with asymptotic index  $n - 1$ , and  $P$  be a  $r$ -dimensional closed, minimal submanifold of  $M$ , with  $r \geq k$ . Suppose that both  $P$  and  $Q$  are closed, and one of them is compact. If  $M$  has positive  $k$ -Ricci curvature either in all points of  $P$  or in all points of  $Q$ , then  $P$  and  $Q$  must intersect.*

**PROOF.** Suppose, by contrary, that  $P$  and  $Q$  do not intersect. The discussions of Theorem 5.2 in [8], the proof of Theorem 3.1 in [9], and the assumption 15

imply that both  $P$  and  $Q$  should be compact, or one of them is forward bounded and the other is backward bounded.

From [15, Theorem 2], see also [8, Theorem 5.2] and [9, Proposition 3.1], there exists a normal geodesic  $\gamma : [0, l] \rightarrow M$  which minimizes the distance between  $P$  and  $Q$  (or between  $Q$  and  $P$ ). This geodesic strikes  $P$  and  $Q$  orthogonally with respect to the inner product  $\langle \cdot, \cdot \rangle_{\gamma'}$ , and is free of focal points except the initial or final point of the geodesic. Consider now a vector  $v$  tangent to  $P$ . By the parallel transport along  $\gamma$  induced by the Cartan connection, it gives rise to a vector field along  $\gamma$ , and at the endpoint  $q = \gamma(l)$  will be a vector tangent to  $Q$ , because  $Q$  has codimension 1.

Consider now a basis of  $T_pP$ ,  $v_1, \dots, v_r$  orthonormal with respect to the inner product  $\langle \cdot, \cdot \rangle_{\gamma'(0)}$ . The parallel transport induced by the Cartan connection generates the vector fields  $V_1, \dots, V_r$ , which are orthogonal along  $\gamma$  with respect to the inner product  $\langle \cdot, \cdot \rangle_{\gamma'}$ .

The Morse index form, along each vector field  $V_i$ , is

$$I^{(P,Q)}(V_i, V_i) = \langle \mathbb{I}_T^P(V_i, V_i), T^H \rangle_T \Big|_l - \langle \mathbb{I}_T^Q(V_i, V_i), T^H \rangle_T \Big|_0 - \int_0^l \langle \nabla_{T^H} \nabla_{T^H} V_i^H + \Omega(T^H, V_i^H)T^H, V_i^H \rangle_T dt.$$

The minimality of  $P$  implies that, in all the points of  $P$ , we have

$$\sum_{i=1}^r \langle \mathbb{I}_T^P(V_i, V_i), T^H \rangle_T = 0.$$

From  $\text{Ric}_k \geq 0$  in all the points of  $P$  and  $r \geq k$  it follows that, for any subset  $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ , we have

$$\sum_{j=1}^k K(V_{i_j}, T) \geq 0,$$

and

$$\sum_{i=1}^r K(V_i, T) = \frac{r}{kC_r^k} \sum_{1 \leq i_1 \leq \dots \leq i_k} \sum_{j=1}^k K(V_{i_j}, T) \geq 0 \quad \forall p \in P.$$

Further, the fact that  $M$  has positive  $k$ -Ricci curvature either in all points of  $P$  or at all points of  $Q$  implies that either

$$\sum_{j=1}^k K(V_{i_j}, T) \Big|_0 > 0 \quad \text{or} \quad \sum_{j=1}^k K(V_{i_j}, T) \Big|_l > 0,$$

for any distinct indices. Hence, we have either

$$\sum_{i=1}^r K(V_i, T) \Big|_0 > 0 \quad \text{or} \quad \sum_{i=1}^r K(V_i, T) \Big|_l > 0.$$

Hence, being  $\sum_{i=1}^r K(V_i, T) \Big|_l > 0$ , we get

$$\sum_{i=1}^r I^{(P,Q)}(V_i, V_i) = - \sum_{i=1}^r \int_0^l K(V_i, T) dt < 0.$$

Thus,  $I^{(P,Q)}(V_i, V_i) < 0$  for some index  $i$ , which contradicts the minimality of the geodesic. Hence,  $P$  and  $Q$  must intersect.  $\square$

One of the reviewers observed that the above theorem can be improved by relaxing conditions about  $P$  and  $Q$ . The conclusion of the theorem remains valid if  $Q$  is a closed, backward bounded hypersurface (without boundary) which is the boundary of a connected open subset  $D$ , and  $\langle \mathbb{I}_v(X, X), v^H \rangle_v \geq 0$  for any  $X \in H_P TP$  and  $v \in T_x M$  is a normal vector to  $Q$  (i.e.  $\langle w, v \rangle_v = 0$  for all  $w \in T_x Q$ ) pointing inside  $D$ , and  $P$  is a forward bounded closed immersed minimal submanifold of dimension  $r \geq k$  contained in  $M \setminus D$ . Moreover, if  $Q$  satisfies  $\langle \mathbb{I}_v(X, X), v^H \rangle_v > 0$ , the assumption that  $\text{Ric}_k$  is positive on all points of  $P$  or  $Q$  can be removed. This condition leads to analyzing the relation between the sign of  $\langle \mathbb{I}_v(X, X), v^H \rangle_v$  and the convexity of  $Q$  in the sense that its normal curvatures with respect to a normal vector pointing outside  $D$  is positive semidefinite (see [5]).

If the Finsler metric is forward (backward) complete, it follows

**Theorem 15.** *Let  $(M, F)$  be a forward (resp. backward)  $n$ -dimensional Finsler manifold of nonnegative  $k$ -Ricci curvature. Let  $Q$  be a complete immersed submanifold of  $\dim Q = n - 1$  and with asymptotic index  $n - 1$ , and  $P$  be an  $r$ -dimensional complete, minimal submanifold of  $M$ , with  $r \geq k$ . Suppose that both  $P$  and  $Q$  are closed, and one of them is compact. If  $M$  has positive  $k$ -Ricci curvature either in all points of  $P$  or in all points of  $Q$ , then  $P$  and  $Q$  must intersect.*

*Remark.* A Finsler manifold is called of *Berwald type* if the Cartan connection does not depend on the reference vector. In this case, the Cartan connection is a linear connection on the manifold  $M$ . The connection coefficients do not depend on the tangent vectors. In this situation, the fact that the second fundamental

form of a submanifold vanishes is equivalent to the fact that the submanifold is totally geodesic. In the Berwald category, one has  $\nu_f = \dim P$  iff  $P$  is a totally geodesic submanifold.

Consequently, for Berwald manifolds, in Theorems 14 and 15 we can replace the condition that the asymptotic index of  $Q$  is  $n - 1$  with that the submanifold  $Q$  is a totally geodesic submanifold of dimension  $n - 1$ .

The same situation holds for Riemannian metrics, and so our results in the above-mentioned theorems also generalize the results of KENMOTSU and XIA [14].

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