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On the diophantine equation $D_1x^2 + D_2 = k^n$

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Abstract. Let D, D_1, D_2, k be positive integers such that $D = D_1D_2$, $D_1 > 1$, $D_2 > 1$, $k > 1$ and $\gcd(D_1, D_2) = \gcd(D, k) = 1$. Let $\omega(k)$ be the number of distinct prime factors of k. Further, let $N(D_1, D_2, k)$ be the number of positive integer solutions (x, n) of the equation $D_1x^2 + D_2 = k^n$. In this paper, we prove that if $2 \nmid k$ and $\max(D_1, D_2) > \exp\exp\exp 105$, then $N(D_1, D_2, k) \leq 2^{\omega(k)-1} + 1$ or $2^{\omega(k)-1}$ according as the triple (D_1, D_2, k) is exceptional or not. The above upper bound is the best possible if k is a prime.

1. Introduction

Let $\mathbb{Z}, \mathbb{N}$ be the sets of integers and positive integers, respectively. Let $D, D_1, D_2, k \in \mathbb{N}$ be such that $D = D_1D_2$, $D_1 > 1$, $D_2 > 1$, $k > 1$ and $\gcd(D_1, D_2) = \gcd(D, k) = 1$. Let $\omega(k)$ be the number of distinct prime factors of k. Further let $N(D_1, D_2, k)$ be the number of solutions (x, n) of the equation

$$D_1x^2 + D_2 = k^n, \quad x, n \in \mathbb{N}. \tag{1}$$

In [2], Bender and Herzberg proved that if $2 \nmid k$ and $k > \Gamma(D)$, where $\Gamma(D)$ is an effectively computable constant depending on D, then $N(D_1, D_2, k) \leq 2^{\omega(k)-1}$. In [6] and [7], the author proved that if k is a prime and $\max(D_1, D_2) > C$, where C is an effectively computable absolute constant, then $N(D_1, D_2, k) \leq 1$ except in some explicit cases.

For any $m \in \mathbb{Z}$ with $m \geq 0$, let F_m be the m-th Fibonacci number. A triple (D_1, D_2, k) will be called *exceptional* if D_1, D_2 and k satisfy either

$$3D_1s_1^2 - D_2 = \delta,\ 4D_1s_1^2 - \delta = k^{r_1},\ 4D_2 + \delta = 3k^{r_1}, \qquad \delta \in \{-1, 1\}, \quad r_1, s_1 \in \mathbb{N}, \tag{2}$$

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or

$$D_1 s_2^2 = \frac{F_{6m}}{4}, \quad D_2 = \begin{cases} \frac{3}{4}F_{6m} - F_{6m-1}, \\ \frac{3}{4}F_{6m} + F_{6m+1}, \end{cases} \tag{3}$$
$$k^{r_2} = \begin{cases} F_{6m-2}, \\ F_{6m+2}, \end{cases} \quad m, r_2, s_2 \in \mathbb{N}.$$

In this paper, we prove the following general result:

Theorem. *If $2 \nmid k$ and $\max(D_1, D_2) > \exp\exp\exp 105$, then we have*

$$N(D_1, D_2, k) \le \begin{cases} 2^{\omega(k)-1} + 1, & \text{if } (D_1, D_2, k) \text{ is exceptional}, \\ 2^{\omega(k)-1}, & \text{otherwise}. \end{cases} \tag{4}$$

Moreover, all solutions (x, n) of (1) satisfy $n < 10\sqrt{D}\log 2e\sqrt{D}/\pi$.

If (D_1, D_2, k) is exceptional, then (1) has at least two solutions, namely

$$(x, n) = \begin{cases} (s_1, r_1),\ (s_1|D_1 s_1^2 - 3D_2|, 3r_1), \\ \qquad \text{if (2) holds}, \\ (s_2, r_2),\ (s_2|D_1^2 s_2^4 - 10D_1D_2s_2^2 + 5D_2^2|, 5r_2), \\ \qquad \text{if (3) holds}. \end{cases} \tag{5}$$

The upper bound (4) is the best possible if k is a prime.

2. Preliminaries

Let $h(-4D)$ be the class number of the primitive binary quadratic forms with discriminant $-4D$.

Lemma 1. $h(-4D) < 4\sqrt{D}\log 2e\sqrt{D}/\pi$.

Proof. By [4, Theorem 12.10.1], we have $h(-4D) = 2\sqrt{D}K(-4D)/\pi$, and by [4, Theorem 12.14.2], $K(-4D) < \log 4D + 2$. This implies the lemma.

Lemma 2 ([8, Theorems 1 and 3]). *If $2 \nmid k$ and the equation*

$$D_1 X^2 + D_2 Y^2 = k^Z,\ X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \tag{6}$$

has solutions (X, Y, Z), then all solutions of (6) belong to at most $2^{\omega(k)-1}$ classes. Further, for any fixed class S, there exists a unique solution

(X_1, Y_1, Z_1) in S such that $X_1 > 0$, $Y_1 > 0$, $Z_1 \le Z$ and $h(-4D) \equiv 0 \pmod{2Z_1}$, where Z runs through all solutions in S. Further, every solution (X, Y, Z) in S can be expressed as

$$Z = Z_1 t, X\sqrt{D_1} + Y\sqrt{-D_2} = \lambda_1 \left(X_1\sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2}\right)^t \qquad t \in \mathbb{N},\ 2 \nmid t,\ \lambda_1, \lambda_2 \in \{-1, 1\}.$$

The solution (X_1, Y_1, Z_1) is called the least solution of S.

Lemma 3 ([5, the proof of the Theorem]). *Let $\varepsilon = X_1\sqrt{D_1} + \sqrt{-D_2}$ and $\bar\varepsilon = X_1\sqrt{D_1} - \sqrt{-D_2}$, where $X_1 \in \mathbb{N}$. If*

$$|\varepsilon^t - \bar\varepsilon^t| \le |\varepsilon - \bar\varepsilon|, \tag{7}$$

for some $t \in \mathbb{N}$, then $t < 8 \cdot 10^6$. Moreover, if $t \ge 7$ and $2 \nmid t$, then $\max(D_1, D_2) < \exp\exp\exp 105$.

3. Proof of the Theorem

Let (x, n) be a solution of (1). Then $(X, Y, Z) = (x, 1, n)$ is a solution of (6). By Lemma 2, we may assume that $(x, 1, n)$ belongs to a certain class S. Let (X_1, Y_1, Z_1) be the least solution of S. Then we have

$$n = Z_1 t, \quad t \in \mathbb{N},\ 2 \nmid t, \tag{8}$$

$$x\sqrt{D_1} + \sqrt{-D_2} = \lambda_1 \left(X_1\sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2}\right)^t, \quad \lambda_1, \lambda_2 \in \{-1, 1\}. \tag{9}$$

From (9), we get

$$1 = \lambda_1\lambda_2 Y_1 \left(\binom{t}{1} (D_1X_1^2)^{(t-1)/2} + \binom{t}{3} (D_1X_1^2)^{(t-3)/2}(-D_2Y_1^2) + \dots + \binom{t}{t} (-D_2Y_1^2)^{(t-1)/2} \right),$$

whence we obtain $Y_1 = 1$. This implies that $(x, n) = (X_1, Z_1)$ is a solution of (1). Moreover, if (1) has another solution (x, n) such that $(x, n) \ne (X_1, Z_1)$ and $(x, 1, n)$ also belongs to S, then x and n satisfy (8) and (9) for $Y_1 = 1$ and $t > 1$.

Let $\varepsilon = X_1\sqrt{D_1} + \sqrt{-D_2}$ and $\bar\varepsilon = X_1\sqrt{D_1} - \sqrt{-D_2}$. Since $Y_1 = 1$, if the other solution exists, (9) implies that (7) holds for $t > 1$. Therefore, by Lemma 3, if $\max(D_1, D_2) > \exp\exp\exp 105$, then we have $t \le 7$. Since $2 \nmid t$, we get $t = 3$ or 5.

For any nonnegative integer m, let L_m and F_m be the m-th Lucas number and Fibonacci number, respectively. For $t=3$ and $t=5$, by (9) we get (2) and (3), respectively. Since D_1, D_2 and k are fixed, the integers $r_1, s_1, \delta, r_2, s_2, m$ in (2) and (3) are given. Now we proceed to prove that (2) and (3) cannot hold at the same time. Notice that $F_{6m-2} = L_{3m-1}F_{3m-1}$ and

$$3F_{6m} - 4F_{6m-1} + \delta = 3F_{6m-2} - F_{6m-1} + \delta$$
$$= \begin{cases} (L_{3m-1} + L_{3m-3})F_{3m-1}, & \text{if } \delta{=}1 \text{ and } 2 \mid m \text{ or} \\ & \delta = -1 \text{ and } 2 \nmid m, \\ (F_{3m-1} + F_{3m-3})L_{3m-1}, & \text{if } \delta{=}1 \text{ and } 2 \nmid m \text{ or} \\ & \delta = -1 \text{ and } 2 \mid m. \end{cases}$$

If (2) and (3) were hold at the same time together with $k^{r_2} = F_{6m-2}$, then we would have

$$\begin{aligned} &3^{r_2}(L_{3m-1}F_{3m-1})^{r_1} = \big((L_{3m-1} + L_{3m-3})F_{3m-1}\big)^{r_2} \quad \text{or} \\ &\big((F_{3m-1} + F_{3m-3})L_{3m-1}\big)^{r_2}, \quad r_1, r_2 \in \mathbb{N}. \end{aligned} \tag{10}$$

Since $\gcd(L_{3m-1}, L_{3m-2}L_{3m-3}) = \gcd(F_{3m-1}, F_{3m-2}F_{3m-3}) = 1$, (10) is impossible. Using the same method, we can prove a contradiction in the case $k^{r_2} = F_{6m+2}$. Therefore, if (D_1, D_2, k) is not exceptional, then (1) has at most one solution (x, n) such that $(x, 1, n)$ belongs to a fixed class. Moreover, if (D_1, D_2, k) is exceptional, then there exists exactly one class, say S, such that (1) has exactly two solutions (5) with $(x, 1, n)$ belonging to S, and the other classes have most one. Thus, by Lemma 2, (4) is proved.

On the other hand, by (8), we have $n \le 5Z$, and from Lemma 2, $2Z_1 \le h(-4D)$. Thus, by Lemma 1, we obtain $n < 10\sqrt{D}\log 2e\sqrt{D}/\pi$. This completes the proof.

Remark 1. The proof of the condition "$\max(D_1, D_2) > \exp\exp\exp 105$" in Lemma 3 involves an upper bound (of Baker [1]) for the solutions of Thue's equations. Using the sharper bounds Győry and Papp [3], the condition could be improved.

Remark 2. Using a similar argument as in the proof of our Theorem, we can obtain an analogous result for the equation

$$D_1x^2 + D_2 = 4k^n, \quad x, n \in \mathbb{N}.$$

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