Note on a paper by Ram Singh on star-like functions

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Introduction: The class of star-like functions $f(z)$ such that $\left|\frac{zf'(z)}{f(z)}-1\right| < 1$ in $|z|<1$ has recently been investigated by RAM SINGH [1] who has proved three interesting theorems [1, Theorems 1, 2, 3] on this class of star-like functions. The preliminary object of this note was to generalize these theorems but this generalization led the author to generalize certain fundamental theorems of the complex function theory, some of which are closely connected with these theorems. The fundamental theorems which have been generalized are Schwarz's lemma, Taylor's theorem, Borel—Carathedory theorem [2, § 5. 5, 5. 51], Poisson—Jensen Formula [2, § 3. 62], Hadamard's gap theorem [2, § 7.43] and Fuch's theorem [3, § 10. 11], with special attention to Legendre's and Bessel's differential equations.

1. The transformation $w = z(z-1)$ maps the closed domain bounded by the closed contour $|z(z-1)| = 1$ onto the domain $|w| \leqq 1$. The function z is regular inside and on $|z(z-1)| = 1$ but $z = \frac{1 \pm \sqrt{1+4w}}{2}$ and this function of w has a branch point at $w = -\frac{1}{4}$ inside the circle $|w|=1$, although, it is easy to see that z can be expanded as a power series

$$z = -w + w^2 - 2w^3 + 5w^4 - \cdots$$

inside the closed contour $|z(z-1)| = 1$.

As a generalization of this simple result we have

Theorem 1. *If $f(z)$ is an analytic function, regular inside the closed contour $|\Phi(z)-\Phi(0)| = R$, where $\Phi(z)$ is an analytic function, regular in the region*

$$|\Phi(z)-\Phi(0)| \leqq R,$$

R being a fixed positive number, and if the derivative of $\Phi(z)$ does not vanish at $z=0$, then $f(z)$ can be expanded as a series

$$f(z) = \sum_{n=0}^{\infty} a_n(\Phi(z)-\Phi(0))^n,$$

inside the closed contour $|\Phi(z)-\Phi(0)| = R$ a_n $(n=0, 1, 2, \ldots)$ being constants.

PROOF. Since, by hypothesis, the derivative of $\Phi(z)$ does not vanish at $z=0$, there exists a neighbourhood of $z=0$ in which $\Phi(z)$ takes any value only once [3, p. 153]. Consequently, there exists a fixed positive ε such that $\Phi(z)$ takes any value, inside or on the closed contour $|\Phi(z)-\Phi(0)| = \varepsilon$ only once. Now, by Cauchy's integral formula, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\Phi'(z)\,dz}{\Phi(z)-\Phi(z_0)}, \tag{1}$$

where C denotes the closed contour $|\Phi(z)-\Phi(0)| = \varepsilon$ and z_0 is any point inside C.

Since, by the maximum modulus principle, we have $|\Phi(z_0)-\Phi(0)| < \varepsilon$, it follows easily from (1) that $f(z)$ can be expanded inside C as a series $\sum_{n=0}^{\infty} a_n(\Phi(z)-\Phi(0))^n$, where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\Phi'(z)\,dz}{(\Phi(z)-\Phi(0))^{n+1}}.$$

As a consequence of this result, we have

$$a_n = \frac{1}{2\pi i m}\int_{C_1} \frac{f(z)\Phi'(z)\,dz}{(\Phi(z)-\Phi(0))^{n+1}}, \tag{2}$$

where C_1 denotes the closed contour $|\Phi(z)-\Phi(0)| = R_1 < R$ and m denotes the number of zeros of $\Phi(z)-\Phi(0)$ inside C_1, multiple zeros being counted according to their orders. By (2), we have

$$|a_n| \leqq \frac{M}{R_1^n}, \tag{3}$$

M being independent of n, and the theorem follows from (3).

Corollary 1. If $f(z)$ and $\Phi(z)$ satisfy all the conditions of Theorem 1, and if $f(z)$ vanishes at $z=0$ and is continuous in the domain $|\Phi(z)-\Phi(0)| \leqq R$, then, in this domain we have

$$\left|\frac{f(z)}{\Phi(z)-\Phi(0)}\right| \leqq \frac{M'}{R},$$

where M' denotes the maximum modulus of $f(z)$ on the closed contour

$$|\Phi(z)-\Phi(0)| = R.$$

Corollary 2. If $f(z)$ and $\Phi(z)$ satisfy all the conditions of Corollary 1, then in the domain $|\Phi(z)-\Phi(0)| \leqq R$, we have

$$|f(z)| \leqq \frac{2|\Phi(z)-\Phi(0)|\,A(R)}{R-|\Phi(z)-\Phi(0)|} + \frac{R+|\Phi(z)-\Phi(0)|}{R-|\Phi(z)-\Phi(0)|}\,|f(0)|,$$

where $A(R)$ denotes the maximum of $\operatorname{Re} f(z)$ on the closed contour $|\Phi(z)-\Phi(0)| = R$.

Corollary 3. If $\Phi(z)$ satisfies all the conditions of Theorem 1, and if $f(z)$ is regular and never takes the values 0 or 1 in the region $|\Phi(z)-\Phi(0)| < R$, then

$$R \leqq \frac{2|\Phi'(0)| \operatorname{Im} v(a_0)}{|a_1|\,|v'(a_0)|},$$

where $a_0=f(0)$, $a_1=f'(0)\neq 0$, $f'(0)$ and $\Phi'(0)$ denote the derivatives of $f(z)$ and $\Phi(z)$ at $z=0$, and $v(a_0)$, $v'(a_0)$ have the same meaning as in [3, 15. 52].

Corollary 4. If $\Phi(z)$ satisfies all the conditions of Theorem 1, and if $f(z)$ is regular and never takes the values 0, 1 or a in the region $|\Phi(z)-\Phi(0)| < R$, then

$$R \leqq \frac{2|\Phi'(0)|\,a_0\,|\log a| \operatorname{Im} v(A)}{|a_1|\,|v'(A)|},$$

where a_0, a_1 have the same meaning as before, $\log a$ has its principal value, and A denotes the principal value of $\log_a a_0$.

PROOF. The principal branch of $\log_a f(z)$ is regular and never takes the values 0 and 1 in the region $|\Phi(z)-\Phi(0)| < R$. Replacing $f(z)$ by $\log_a f(z)$ in corollary (3), we get the required result.

Corollary 5. If $\Phi(z)$ satisfies all the conditions of Theorem 1, and if $f(z)$ is regular and never takes the values 0, 1, a or b in the region $|\Phi(z)-\Phi(0)| < R$, then

$$R \leqq \frac{2|\Phi'(0)|\,|a_0 \log a_0| \log(\log_a b)| \operatorname{Im} v(B)}{|a_1|\,|v'(B)|},$$

where a_0, a_1 have the same meaning as before, $\log a_0$, $\log(\log_a b)$ have their principal values and B denotes the principal value of $\log_{(\log_a b)}(\log_a a_0)$.

PROOF. The principal branch of $\log_{(\log_a b)}(\log_a f(z))$ is regular and never takes the values 0 and 1 in the region $|\Phi(z)-\Phi(0)| < R$. The rest of the proof follows the same lines as before.

Theorem 2. *If $f(z)$ is an analytic function, regular inside the circle $|z|=R$, which vanishes at $z=0$, if* $\operatorname{Re} f(z)$ *is bounded on the circle, then*

$$|f(z)| \leqq -a,$$

when $|z|\leqq R$, a being the greatest lower bound of $\operatorname{Re} f(z)$ *on the circle.*

PROOF. Let $\Phi(z) = \dfrac{f(z)}{2lA-f(z)}$, where l is a fixed number greater than 1 and A denotes the least upper bound of $\operatorname{Re} f(z)$ on the circle. Since $f(2)$ vanishes at $z=0$ $\Phi(z)$ also vanishes at $z=0$, by [4, p. 153] it follows that, the transformation $Z=f(z)$ maps a small neigh-bourhood of any interior point z_0 of the circle $|z|=R$ on a small neighbourhood of the point Z_0, where $Z_0=f(z_0)$. Consequently, there exists a continuous curve, with one end-point at $z=0$, such that, at every point of this curve, except $z=0$, $\Phi(z)$ is real and negative. If no zero of $\Phi(z)$, except $z=0$, lies on this curve, the other end-point of the curve will lie on the circle $|z|=R$. If z_1 is a point of the curve, such that z_1 is a zero of $\Phi(z)$ and that no other zero of $\Phi(z)$ lies on the arc of the curve, joining z_1 and $z=0$, then as before, there exists another continuous curve, with one end-point at z_1, such that, at every point of this curve, except z_1, $\Phi(z)$ is real and negative. Proceeding in the same way, it follows that there exists a continuous curve inside the circle $|z|=R$, joining $z=0$ to a point z' of the circle, such that, at every point of the curve, except a zero of $\Phi(z)$ and possibly z', $\Phi(z)$ is real and negative.

Now, if $\Phi(z)$ tends to -1, as z tends to a fixed point of the circle $|z|=R$, through a sequence of interior points of the circle, then given any fixed positive λ, $\lambda<1$, there exists a fixed interior point z_λ of the circle such that

$$f(z_\lambda) = \frac{-2lA\lambda}{1-\lambda}$$

i.e.

$$\operatorname{Re} f(z_\lambda) = \frac{-2lA\lambda}{1-\lambda}.$$

So, it follows that $\operatorname{Re} f(z_\lambda)$ tends to $-\infty$, as λ tends to 1; which is contrary to our hypothesis. We have, thus, proved that $\Phi(z)$ does not tend to -1, as z tends to any fixed point of the circle $|z|=R$, through any sequence of interior points of the circle.

Let

$$(1') \qquad \Phi(z) = -1+u+iv,$$

where u and v are real. We have

$$f(z) = \frac{2lA\Phi(z)}{1+\Phi(z)} = \frac{2lA(-1+u+iv)}{u+iv} = 2lA\left(1-\frac{1}{u+iv}\right) = 2lA\left(1-\frac{u-iv}{u^2+v^2}\right).$$

Therefore

$$(2') \qquad \operatorname{Re} f(z) = 2lA(1-k),$$

where

$$k = \frac{u}{u^2+v^2}.$$

Putting

$$(3') \qquad X = -1+u, \quad Y = v,$$

we have

$$k = \frac{X+2}{(X+1)^2+Y^2}.$$

Since, by hypothesis, $\operatorname{Re} f(z)$ is bounded on the circle $|z|=R$, by (2′), it follows that $\frac{1}{2}<k<\infty$. Under the transformations (1) and (3), the curve $|\Phi(z)|^2=1$ is the circle $X^2+Y^2=1$ and the curve $k=\frac{u}{u^2+v^2}$ i.e. $\frac{u}{k}=u^2+v^2$ is the circle $(X+1)^2+Y^2=$ $=\frac{X+1}{k}$. Since k is greater than $\frac{1}{2}$, the two circles intersect at only one point i.e. $(-1,0)$; and since the centre of the second circle is $\left(\frac{1}{2k}-1, 0\right)$ and its radius is $\frac{1}{2k}$, it lies inside the circle $X^2+Y^2=1$ and touches it at the point $(-1,0)$. Since, we have proved that $\Phi(z)$ i.e. $(X+1+iY)$ does not tend to -1, as z tends to any fixed point of the circle $|z|=R$, it follows that there exists a fixed λ, $0<\lambda<1$, such that $|\Phi(z)|\leqq\lambda$, when $|z|\leqq R$. Also, since we have proved that there exists a continuous curve inside the circle $|z|=R$, joining $z=0$ to a point z' of the circle $|z|=R$, such that, at every point of the curve, except a zero of $\Phi(z)$ and possibly z', $\Phi(z)$ is real and negative, it follows that, there exists a fixed point z'', inside the circle $|z|=R$, such that $\Phi(z'')=-\lambda$ and, consequently, $f(z'')=\dfrac{-2lA\lambda}{1-\lambda}$ i.e. $\dfrac{2lA\lambda}{1-\lambda}=-\operatorname{Re} f(z'')$. So, we have

$$\frac{2lA\lambda}{1-\lambda}\leqq -a. \tag{4}$$

Hence we have

$$|f(z)|\leqq\frac{2lA|\Phi(z)|}{1-|\Phi(z)|}\leqq\frac{2lA\lambda}{1-\lambda}\leqq -a,$$

when $|z|\leqq R$.

Remark: If we put $-f(z)$ for $f(z)$ in the last inequality, we have

$$|f(z)|\leqq A,$$

when $|z|\leqq R$.

Consequently, it follows that

$$|f(z)|\leqq\min(-a, A).$$

Theorem 3. *If $\Phi(z)$ satisfies all the conditions of Theorem 1, if $\Phi(z)$ vanishes at $z=0$, and if $f(z)$ is an analytic function, regular inside and on the closed contour $|\Phi(z)|=R$, having zeros at the points $a_1, a_2, \dots, a_k$, and poles at $b_1, b_2, \dots, b_n$ inside the closed contour, then*

$$\log|f(r'e^{i\theta})|=\frac{1}{2\pi m}\int_L\frac{(R^2-R_1^2)\log|f(re^{i\theta})|\,d\lambda}{R^2-2RR_1\cos(\lambda-\lambda_1)+R_1^2}-$$

$$-\sum_{u=1}^{k}\log\left|\frac{R^2-\overline{\Phi(a_u)}\Phi(r'e^{i\theta'})}{R(\Phi(r'e^{i\theta'}))-\Phi(a_u)}\right|+\sum_{v=1}^{n}\log\left|\frac{R^2-\overline{\Phi(b_v)}\Phi(r'e^{i\theta'})}{R(\Phi(r'e^{i\theta'}))-\Phi(b_v)}\right|,$$

where $r'e^{i\theta'}$ is any point inside the closed contour $|\Phi(z)|=R$, $re^{i\theta}$ is any point on the contour $\Phi(r'e^{i\theta'})=R_1e^{i\lambda_1}$, $\Phi(re^{i\theta})=Re^{i\lambda}$, L denotes the set of values of λ, attained on the closed contour $|\Phi(z)|=R$ and m has the same meaning as in Theorem 1.

PROOF. We consider

$$\frac{1}{2\pi i}\int_C\frac{(R^2-R_1^2)\log F(z)\,d(\Phi(z))}{(\Phi(z)-\Phi(r'e^{i\theta'}))(R^2-\Phi(z)\overline{\Phi(r'e^{i\theta'})})}$$

where

$$F(z)=f(z)\prod_{u=1}^{k}\frac{R^2-\Phi(z)\overline{\Phi(a_u)}}{R(\Phi(z)-\Phi(a_u))}\prod_{v=1}^{n}\frac{R(\Phi(z)-\Phi(a_v))}{R^2-\Phi(z)\overline{\Phi(a_v)}}$$

and C denotes the closed contour $|\Phi(z)|=R$. The rest of the proof follows easily, if we use Theorem 1.

Putting $r'=0$ and supposing that $z=0$ is neither a zero nor a pole of $f(z)$, we have

$$\frac{1}{2\pi m}\int_L \log|f(re^{i\theta})|\,d\lambda-\log|f(0)| = \int_0^R \frac{n(x,0)-n(x,\infty)}{x}\,dx$$

where $n(x, 0)$ and $n(x, \infty)$ denote the number of zeros and the number of poles respectively inside or on the closed contour $|\Phi(z)|=x\leqq R$.

Now, we can define $T(\Phi, R, f)$ by the relation

$$T(\Phi, R, f) = \frac{1}{2\pi m}\int_L \log^+|f(re^{i\theta})|\,d\lambda+\int_0^R \frac{n(x,\infty)}{x}\,dx$$

where $\log^+|a|$ has its usual meaning.

Theorem 4. *If $f(z)$ is an analytic function, regular inside a closed contour $|\Phi(z)|=1$, where $\Phi(z)$ satisfies all the conditions of Theorem 1, with $\Phi(0)=0$ and $R=1$, if in the series*

$$f(z) = \sum_{n=0}^{\infty} a_n\big(\Phi(z)\big)^n,$$

$a_n=0$ except when n belongs to a sequence n_k such that $n_{k+1} > (1+\lambda)n_k$, $\lambda>0$, and if the radius of convergence of the series $\sum\limits_{n=0}^{\infty} a_n z^n$ is 1, then the closed contour $|\Phi(z)|=1$ is a natural boundary of $f(z)$.

Proof. Let $\Phi(z) = aw^p+bw^{p+1}$, where $0<a<1$, $a+b=1$ and p is a positive integer. Clearly $|\Phi(z)|\leqq 1$, if $|w|\leqq 1$ and $|\Phi(z)|<1$ if $|w|\leqq 1$, except that $\Phi(z)=1$ if $w=1$. The rest of the proof, now, follows the same lines as in [2, § 7. 43].

Theorem 5. *If $\Phi(z)$ satisfies all the conditions of Theorem 1 and vanishes at $z=0$, if each of the functions $p(z)$ and $q(z)$ is an analytic function regular at $z=0$, and if A_0 and A_1 are two arbitrary constants, there exists a unique function $w(z)=$ $=\sum\limits_{n=0}^{\infty} a_n\big(\Phi(z)\big)^n$, which is regular and satisfies the differential equation*

$$\frac{d^2w}{dz^2}+\left(p(z)\Phi'(z)-\frac{\Phi''(z)}{\Phi'(z)}\right)\frac{dw}{dz}+q(z)\,(\Phi'(z))^2 w = 0$$

in a region $|\Phi(z)|<r$, and which also satisfies the initial conditions $w(0)=A_0$, $w'(0)=$ $=A_1\cdot\Phi'(0)$.

Proof. We have

$$\frac{dw}{dz} = \Phi'(z)\frac{dw}{d\Phi(z)} \quad\text{and}\quad \frac{d^2w}{dz^2} = (\Phi'(z))^2\frac{d^2w}{d(\Phi(z))^2}+\frac{\Phi''(z)\,dw}{d\Phi(z)}.$$

So the differential equation can be put in the form

$$\frac{d^2w}{d(\Phi(z))^2}+p(z)\frac{dw}{d(\Phi(z))}+q(z)w = 0.$$

Since $p(z)$ and $q(z)$ are regular in a region $|\Phi(z)|<r$, by Theorem 1, they are expansible, in this region, as series of the form

$$p(z)=\sum_0^\infty p_n(\Phi(z))^n,\quad q(z)=\sum_0^\infty q_n(\Phi(z))^n,$$

The rest of the proof, now, follows the same lines as in [3, 10. 11], if we use the inequality (3) of the proof of Theorem 1.

Corollary 6. The generalized Legendre differential equation

$$\frac{d^2w}{dz^2}+\left(\frac{2z\Phi'(z)}{z^2-1}-\frac{\Phi''(z)}{\Phi'(z)}\right)\frac{dw}{dz}+\frac{n(n+1)}{1-z^2}(\Phi'(z))^2w=0$$

has a solution of the form $w(z)=\sum_{n=0}^\infty a_n(\Phi(z))^n$, regular in any region $|\Phi(z)|<r$ which does contain the points ± 1.

Corollary 7. The generalized Bessel differential equation

$$\frac{d^2w}{dz^2}+\left(\frac{\Phi'(z)}{z}-\frac{\Phi''(z)}{\Phi'(z)}\right)\frac{dw}{dz}+\left(1-\frac{v^2}{z^2}\right)(\Phi'(z))^2w=0$$

has a solution of the form $w(z)=\sum_{n=0}^\infty b_n(\Phi(z))^n$, regular in any region $|\Phi(z)|<R$ in which $\Phi(z)$ is regular, provided that $n(n-1)+n\Phi'(0)-v^2(\Phi'(0))^2$ does not vanish for any positive integral or zero value of n.

PROOF. By the method of proof of Theorem 5, the differential equation can be put in the form

$$\frac{d^2w}{d(\Phi(z))^2}+\frac{1}{z}\frac{dw}{d(\Phi(z))}+\left(1-\frac{v^2}{z^2}\right)w=0.$$

Multiplying this differential equation by $(\Phi(z))^2$, we have

$$(\Phi(z))^2\frac{d^2w}{d(\Phi(z))^2}+\frac{(\Phi(z))^2}{z}\frac{dw}{d(\Phi(z))}+\left(1-\frac{v^2}{z^2}\right)(\Phi(z))^2w=0$$

Since the functions $\dfrac{\Phi(z)}{z}$ and $\left(1-\dfrac{v^2}{z^2}\right)(\Phi(z))^2$ are regular in any region $|\Phi(z)|<R$ in which $\Phi(z)$ is regular, by Theorem 1, they can be expanded, in this region, as series of the form $\sum_{n=0}^\infty C_n(\Phi(z))^n$. The rest of the proof, now, follows easily.

2. Theorem 6. *Let each of the functions $\Psi(z)$, $f(z)$ and $\Phi(z)$ be an analytic function of z, and let $\Psi^{-1}(z)$ denote the inverse function of $\Psi(z)$. Let us suppose that the function $z\dfrac{d}{dz}\Psi^{-1}\left(\dfrac{f(z)}{z\Phi(z)}\right)$, where $\dfrac{d}{dz}$ denotes differentian with respect to*

z, is an analytic function of z, regular in the region $|z|<1$, and vanishes at $z=0$. Then a necessary and sufficient condition that the inequality

$$\left| z \frac{d}{dz} \Psi^{-1}\left(\frac{f(z)}{z\Phi(z)}\right)\right| < K,$$

K being a fixed number, may be satisfied at every point of the region $|z|<1$, is that $f(z)$ should be expressible in this region in the form

$$f(z) = z\Phi(z)\Psi\Big(\int_0^z F(t)\,dt\Big), \tag{5}$$

where $F(z)$ is regular and $|F(z)|<K$ in $|z|<1$.

The PROOF follows the same lines as that of [1, Theorem 1], and so it has been omitted.

Theorem 7. *If all the conditions of Theorem 6 are satisfied, we have*

$$\left|\Psi^{-1}\left(\frac{f(z)}{z\Phi(z)}\right)\right| < K|z|, \tag{6}$$

and if, besides these conditions, $\Psi(z)$ is regular in the region $|z|<K$, we have

$$|z\Phi(z)|\,m(K|z|) \leqq |f(z)| \leqq |z\Phi(z)|\,M(K|z|), \tag{7}$$

$$|f'(z)| \geqq m(K|z|)\left(\left|\frac{d(z\Phi(z))}{dz}\right| - |z\Phi(z)|\,N(K|z|)\right) \tag{8}$$

and

$$|f'(z)| \leqq M(K|z|)\left(\left|\frac{d(z\Phi(z))}{dz}\right| + |z\Phi(z)|\,N(K|z|)\right), \tag{9}$$

when $|z|<1$, where $m(r)$ denotes the minimum-modulus of $\Psi(z)$, $M(r)$ and $N(r)$ denote the maximum moduli of $\Psi(z)$ and $\dfrac{\Psi'(z)}{\Psi(z)}$ respectively, on the circle $|z|=r<K$, and $f'(z)$ and $\Psi'(z)$ denote the derivatives of $f(z)$ and $\Psi(z)$ respectively.

PROOF. (7) follows easily from (5), and (8) and (9) follow in the same way as (7), if we differentiate (5).

Theorem 8. *If $f(z)$ does not vanish in the region $|z|<1$, except at $z=0$, and satisfies all the conditions of Theorem 6, if $\Phi(z)$ and $\Psi(z)$ are regular in the regions $|z|\leqq 1$ and $|z|\leqq K$ respectively, and if P and Q denote the maximum-moduli of the functions $\dfrac{\Phi'(z)}{\Phi(z)}$ and $\dfrac{\Psi'(z)}{\Psi(z)}$ on the circles $|z|=1$ and $|z|=K$ respectively, then*

$$|a_{n+1}| \leqq \frac{(P+Q)^n}{n}, \quad \textit{for all} \quad n,\ 1\leqq n\leqq r,$$

and

$$|a_{n+1}| \leqq \frac{(P+Q)^r}{n(r!)}, \quad \text{for all} \quad n,\ n>r,$$

where $\Phi'(z)$ *denotes the derivative of* $\Phi(z)$, r *denotes the greatest integer less than* $(P+Q)$, and $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$.

PROOF. Taking logarithm of each side of (5) and differentiating we easily obtain the relation

$$f_1'(z) = f_1(z)\left(\frac{\Phi'(z)}{\Phi(z)}+\frac{\Psi'\left(\int_0^z F(t)\,dt\right)}{\Psi\left(\int_0^z F(t)\,dt\right)}\right) \tag{10}$$

where

$$f_1(z) = \frac{f(z)}{z} = 1+\sum_{n=2}^{\infty} a_n z^{n-1}.$$

Since, by hypothesis, $f_1(z)$ does not vanish in the region $|z|<1$, the function within brackets on the right-hand side of (10) is regular in this region. Consequently, we can put (10) in the form

(11)

$$\sum_{m=2}^{n}(m-1)a_m z^{m-2}+\sum_{m=n-1}^{\infty} b_m z^m = \left(1+\sum_{m=2}^{n-1} a_m z^{m-1}\right)\left(\frac{\Phi'(z)}{\Phi(z)}+\frac{\Psi'\left(\int_0^z F(t)\,dt\right)}{\Psi\left(\int_0^z F(t)\,dt\right)}\right),$$

Squaring the moduli of both sides of (11) and integrating round the circle $|z|=R<1$, we obtain

$$\sum_{m=2}^{n}(m-1)^2|a_m|^2R^{2m-2}+\sum_{m=n-1}^{\infty}|b_m|^2R^{2m} \leqq (P+Q)^2\left(1+\sum_{m=2}^{n-1}|a_m|^2R^{2m}\right).$$

Making $R\to 1$, we have

$$\sum_{m=2}^{n}(m-1)^2|a_m^2| \leqq (P+Q)^2\left(1+\sum_{m=2}^{n-1}|a_m|^2\right)$$

or

$$(n-1)^2|a_n|^2 \leqq (P+Q)^2+\sum_{m=2}^{n-1}\{(P+Q)^2-(m-1)^2\}|a_m|^2, \tag{12}$$

for all n, $n\leqq r+2$, and

$$(n-1)^2|a_n|^2 \leqq (P+Q)^2+\sum_{m=2}^{r+1}\{P+Q)^2-(m-1)^2\}|a_m|^2, \tag{13}$$

for all n, $n>r+2$.

Now, putting $n=2$ in (12), we have

$$|a_2| \leqq (P+Q) \tag{14}$$

Let us suppose that

$$|a_n| \leqq \frac{(P+Q)^{n-1}}{(n-1)} \tag{15}$$

for all n, $1<n \leqq k<r+1$.

By (12) and (15), it follows easily that

$$|a_{k+1}| \leqq \frac{(P+Q)^k}{k} \tag{16}$$

The rest of the proof follows easily from (13) to (16).

Theorem 9. *If $f(z)$ and $\Phi(z)$ satisfy all the conditions of Theorem 1, with $R=1$, a necessary and sufficient condition that the function*

$$\frac{(\Phi(z)-\Phi(0))f'(z)}{\Phi'(z)f(z)}-1$$

may vanish at $z=0$ and may satisfy the inequality

$$\left|\frac{(\Phi(z)-\Phi(0))f'(z)}{\Phi'(z)f(z)}-1\right| < 1$$

at every point of the region $|\Phi(z)-\Phi(0)| < 1$, is that $f(z)$ should be expressible in this region in the form

$$f(z) = (\Phi(z)-\Phi(0))\, e^{\int_0^z F(t)\Phi(t)\,dt}, \tag{17}$$

where the path of integration is any contour lying inside this region and joining the points 0 and z, $F(z)$ is regular and $|F(z)|<1$ in this region.

The proof depends on Corollary 1 and follows the same lines as that of [1, Theorem 1].

Theorem 10. *If all the conditions of Theorem 9 are satisfied, we have*

$$|\Phi(z)-\Phi(0)|\, e^{-|\Phi(z)-\Phi(0)|} \leqq |f(z)| \leqq |\Phi(z)-\Phi(0)|\, e^{|\Phi(z)-\Phi(0)|},$$

and

$$|\Phi'(z)|\big(1-|\Phi(z)-\Phi(0)|\big)e^{-|\Phi(z)-\Phi(0)|} \leqq |f'(z)| \leqq |\Phi'(z)|\big(1+|\Phi(z)-\Phi(0)|\big)$$

when $|\Phi(z)-\Phi(0)| < 1$.

Proof. Since, under the conditions of Theorem 1, with $R=1$, the transformation $z=\Phi(z)$ maps the region $|\Phi(z)-\Phi(0)| < 1$ on the interior of the circle $|z-\Phi(0)| = 1$ in the z-plane, we can choose the contour of integration in (17), to be that whose image is the segment of the straight line inside the circle, joining the points $\Phi(0)$

and z. Choosing the contour of integration in this way; and putting $\Phi(t)-\Phi(0)=$ $=ue^{iv}$, when t lies on this contour, v being a real constant, we have

$$\int_0^z F(t)\Phi'(t)\,dt = \int_0^u F_1(ue^{iv})e^{iv}\,du, \tag{18}$$

where $\Phi(z)-\Phi(0) = Ue^{iv}$ and $F_1(ue^{iv}) = F(t)$. Since, by Theorem 9, we have $|F(t)|<1$, when t lies on this contour, by (18), it follows that

$$\left|\int_0^z F(t)\Phi'(t)\,dt\right| \leqq \int_0^U du = U = |\Phi(z)-\Phi(0)| \tag{19}$$

Now, the first inequality follows easily from (17) and (19), and the second inequality follows in the same way, if we differentiate each side of (17).

Theorem 11. *If all the conditions of Theorem 9 are satisfied, and if* $f(0)=0$, $f'(0)=\Phi'(0)$, *then*

$$\frac{1}{n!}\left|\left[\left(\frac{d}{d\Phi(z)}\right)^n f(z)\right]_{z=0}\right| \leqq \frac{1}{n-1}$$

for all n, $n \geqq 2$.

Proof. Taking logarithm of each side of (17) and differentiating, we easily obtain the relation

$$f_1'(z) = f_1(z)\Phi'(z)F(z), \tag{20}$$

where

$$f_1(z) = \frac{f(z)}{\Phi(z)-\Phi(0)}.$$

Since, by Theorem 1, $f(z)$ can be expanded, in the region $|\Phi(z)-\Phi(0)|<1$, as a series

$$f(z) = \sum_{n=0}^{\infty} a_n(\Phi(z)-\Phi(0))^n,$$

and since, by hypothesis, $f(0)=0$ and $f'(0)=\Phi'(0)$, we have

$$f_1(z) = 1+\sum_{m=2}^{\infty} a_n(\Phi(z)-\Phi(0))^{n-1},$$

when $|\Phi(z)-\Phi(0)|<1$.
Substituting this series for $f_1(z)$ in (20), we have

$$\sum_{n=2}^{\infty}(n-1)a_n(\Phi(z)-\Phi(0))^{n-2} = F(z)\left(1+\sum_{n=2}^{\infty} a_n(\Phi(z)-\Phi(0))^{n-1}\right) \tag{21}$$

Since, the function $F(z)$ is regular in the region $|\Phi(z)-\Phi(0)|<1$, by Theorem 1, it can be expanded, in this region, as a series

$$F(z) = \sum_{n=0}^{\infty} b_n(\Phi(z)-\Phi(z))^n.$$

Consequently, we can put (21) in the form

$$\sum_{m=2}^{n}(m-1)a_m(\Phi(z)-\Phi(0))^{m-2}+\sum_{m=n-1}^{\infty}c_m(\Phi(z)-\Phi(0))^m = \tag{22}$$

$$=\left(1+\sum_{m=2}^{n-1}a_m(\Phi(z)-\Phi(0))^{m-1}\right)F(z)$$

Since, for every fixed r, $0<r<1$, $|\Phi(z)-\Phi(0)|=r$ is a closed contour which contains the point $z=0$, we have $\Phi(z)-\Phi(0)=re^{i\theta}$, when z lies on the closed contour, where θ attains all values between 0 and 2π.

Now putting $\Phi(z)-\Phi(0)=re^{i\theta}$ in (22), squaring the moduli of both sides of (22), integrating over the interval $0\leqq\theta\leqq 2\pi$, and proceeding in the same way as in the proof of [1, Theorem 3], we can easily obtain the inequality

$$|a_n|\leqq\frac{1}{n-1}$$

for all n, $n\geqq 2$.

Since, we have

$$f(z)=\sum_{n=0}^{n}a_n(\Phi(z)-\Phi(0))^n,$$

when $|\Phi(z)-\Phi(0)|<1$, it follows easily that

$$a_n=\frac{1}{n}\left[\left(\frac{d}{d\Phi(z)}\right)^n f(z)\right]_{z=0}$$

This completes the proof of the theorem.

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