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Remark on local degrees of simplicial mappings

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Abstract. Let K^n and M^n be dimensionally homogeneous simplicial n-complexes, M^n being a pseudomanifold and $f : K^n \to M^n$ a simplicial mapping. Both K^n and M^n have orientations, M^n has a coherent one. Let

$$\{\tau_1^n, \tau_1^{n-1}, \tau_2^n, \tau_2^{n-1}, \dots, \tau_{p-1}^{n-1}, \tau_p^n\}$$

be a sequence of alternately n- and $(n-1)$-simplices of M^n such that every $(n-1)$-simplex is the common face of the two neighbouring n-simplices. The simplex τ_i^{n-1} $(i = 1, \dots, p-1)$ has the orientation induced by that of τ_{i+1}^n. Let $d(\tau^n)$ denote the local degree of the mapping f on $\tau^n \in M^n$ and $d(\tau_i^{n-1})$ denote the local degree of the restriction $f \mid \partial K^n$ on τ_i^{n-1}. Then we have the following equality

$$d(\tau_p^n) - d(\tau_1^n) = \sum_{i=1}^{p-1} d(\tau_i^{n-1}),$$

which should be reduced modulo 2 in the non-oriented case. This statement generalizes the main result of the foregoing author's paper (Publ. Math. Debrecen, 1994, **45,** 407–413).

In this short note we wish to show that the main result of the paper [1] (Theorem 1, or "difference formula") is valid under more general assumptions, namely, for simplicial mappings of a dimensionally homogeneous simplicial complex into a simplicial pseudomanifold. We refer the reader to [1] for the necessary definitions.

In what follows let K^n be a finite, n-dimensional simplicial complex that is dimensionally homogeneous, i.e., such that every simplex (of any dimension) of K^n is a face of at least one n-simplex of K^n; let M^n be a

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finite simplicial n-pseudomanifold and $f : K^n \to M^n$ a simplicial mapping. The pseudomanifold M^n is supposed to have a coherent orientation and the complex K^n to have an arbitrary one.

We use some properties of integral n-chains and $(n-1)$-cochains defined on the set of oriented n- and $(n-1)$-simplices of K^n, respectively, as well as of their boundary ∂ and coboundary δ operators (see, for example, [2], p. 297).

Let x^n be the integral n-chain on K^n assuming the value 1 on every oriented n-simplex. Under the *geometrical boundary* (or simply *boundary*) ∂K^n of the complex K^n we understand the set of all its $(n-1)$-simplices σ^{n-1} such that $\partial x^n(\sigma^{n-1}) \neq 0$. Each simplex $\sigma^{n-1} \in \partial K^n$ has the orientation induced by that of K^n.

For a simplicial mapping $f : K^n \to M^n$ we denote by $f \mid \partial K^n$ the restriction of f on the boundary ∂K^n, i.e. the mapping $f \mid \partial K^n : \partial K^n \to skel_{n-1} M^n$. Let τ^{n-1} be a fixed simplex of $skel_{n-1} M^n$. We call the *local degree* of $f \mid \partial K^n$ on τ^{n-1} the difference

$$\sum_i \partial x^n(\sigma_i^{n-1}) - \sum_j \partial x^n(\sigma_j^{n-1})$$

where the sum $\sum\limits_i$ (resp., $\sum\limits_j$) is taken over all the simplices $\sigma^{n-1} \in \partial K^n$ which are mapped by $f \mid \partial K^n$ on τ^{n-1} with preserving (resp., reversing) of the orientation. In the case of a pseudomanifold K^n this definition coincides with that given in the paper [1].

Let us consider a finite sequence

$$\{\tau_1^n, \tau_1^{n-1}, \tau_2^n, \tau_2^{n-1}, \dots, \tau_{p-1}^{n-1}, \tau_p^n\} \tag{1}$$

of alternately n- and $(n-1)$-simplices of M^n such that every $(n-1)$-simplex is the common face of the two neighbouring n-simplices and these two n-simplices are distinct. We assume in what follows that the simplex τ_i^{n-1} $(i = 1, \dots, p-1)$ in the sequence (1) has the orientation induced by that of the simplex τ_{i+1}^n. Let $d(\tau^n)$ denote the local degree of the mapping f on $\tau^n \in M^n$ and let $d(\tau_i^{n-1})$ denote the local degree of the restriction $f \mid \partial K^n$ on τ_i^{n-1} $(i = 1, \dots, p-1)$.

Theorem. *We have the following equality*

$$d(\tau_p^n) - d(\tau_1^n) = \sum_{i=1}^{p-1} d(\tau_i^{n-1}). \tag{2}$$

In the non-oriented case this equality should be reduced modulo 2.

PROOF. It is sufficient to prove equality (2) only in the case when sequence (1) is of the form $\{\tau_1^n, \tau^{n-1}, \tau_2^n\}$. The simplest proof may be received by using the combinatorial form of Stokes' theorem

$$(x^n, \delta y^{n-1}) = (\partial x^n, y^{n-1}) \tag{3}$$

written for any n-chain x^n and any $(n-1)$-cochain y^{n-1} on K^n (see, for example, [2], p. 301). Put the chain x^n being equal to 1 on every oriented n-simplex of K^n and the cochain y^{n-1} being equal to 1 (resp., to -1) on every $(n-1)$-simplex $\sigma^{n-1} \in skel_{n-1}K^n$ that is mapped onto τ^{n-1} with preserving (resp., reversing) of the orientation, and $y^{n-1}(\sigma^{n-1}) = 0$ for any other simplex $\sigma^{n-1} \in skel_{n-1}K^n$. Let us calculate the values of δy^{n-1} on all n-simplices σ^n of K^n. If no $(n-1)$-face of a simpex σ^n is mapped onto τ^{n-1}, then $\delta y^{n-1}(\sigma^n) = 0$. If $f(\sigma^n) = \tau^{n-1}$, then the simplex σ^n has precisely two $(n-1)$-faces, namely σ_1^{n-1} and σ_2^{n-1}, such that $f(\sigma_1^{n-1}) = f(\sigma_2^{n-1}) = \tau^{n-1}$. In this case we have $\delta y^{n-1}(\sigma^n) = 0$, too. Finally, let the simplex σ^n has a unique $(n-1)$-face σ^{n-1} such that $f(\sigma^{n-1}) = \tau^{n-1}$. Then we distinguish the following cases:

1) the simplex σ^n is mapped onto τ_1^n with preserving (resp., reversing) of the orientation, and therefore $\delta y^{n-1}(\sigma^n) = -1$ (resp., $\delta y^{n-1}(\sigma^n) = 1$) (here we take into account that the orientation of τ^{n-1} is induced by that of τ_2^n),
2) the simplex σ^n is mapped onto τ_2^n with preserving (resp., reversing) of the orientation, and therefore $\delta y^{n-1}(\sigma^n) = 1$ (resp., $\delta y^{n-1}(\sigma^n) = -1$).

So the inner product $(x^n, \delta y^{n-1})$ is equal to the difference of local degrees $d(\tau_2^n) - d(\tau_1^n)$. On the other hand, $(\partial x^n, y^{n-1})$ is equal to the local degree $d(\tau^{n-1})$, and Stokes' theorem (3) gives us

$$d(\tau_2^n) - d(\tau_1^n) = d(\tau^{n-1}). \qquad \square$$

Let σ^{n-1} be an $(n-1)$-simplex of a simplicial complex M^n. We call σ^{n-1} a *ramification simplex* if it is the common face of at least three distinct n-simplices of M^n. The following example shows that the difference formula (2), as well as its reduction modulo 2, is false if the complex M^n has a ramification simplex. Let $K^2 = M^2$ be the complex known as "book with three sheets", i.e. the complex with the set of vertices $\{a, b, c, d, e\}$ and with 2-simplices (bac), (abd), and (abe), oriented by these orderings of their vertices. Let $f : K^2 \to M^2$ be the identical mapping. Then $d(bac) = d(abd) = 1$, but $d(ab) = 1$.

Note that our Theorem does not need the simplicial complex M^n to be an n-pseudomanifold. We may instead assume that M^n is dimensionally homogeneous and it has no ramification simplex. The author is indebted to the referee for this remark.

References

[1] Yu. A. Shashkin, Local degrees of simplicial mappings, *Publ. Math. Debrecen* **45** (1994), 407–413.
[2] J. G. Hocking and G. S. Young, Topology, *Addison-Wesley*, *Reading,* Mass., 1961.

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