On complementary inequalities

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1. Introduction

In 1914 SCHWEITZER [15] proved the following so-called complementary inequality for the arithmetic and harmonic mean values:

Let $0 < m \leqq x_k \leqq M$, $k = 1, \ldots, n$. Then

$$\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{x_i}\right) \leqq \frac{(M+m)^2}{4mM}.$$

Using this inequality an inequality complementary to the Cauchy—Bunyakowski—Schwarz inequality can easily be verified (see KANTOROVIC [11]).

Generalizing Scheitzer's result SPECHT [14], and CARGO—SHISHA [5] obtained a complementary inequality for the ratio of power means.

In [3] BECK obtained remarkable results concerning complementary inequalities for quasiarithmetic means (HARDY—LITTLEWOOD—PÓLYA [10]).

In this paper we solve the complementary comparison problem for the *deviation means* defined by DARÓCZY [6, 7]. The class of deviation means contains as a special case the class of quasiaritmetic means and the class of quasiarithmetic means with weightfunction (BAJRAKTAREVIC [2], ACZÉL—DARÓCZY [1]). So our results enrich the theory of these two classes of means too.

2. Deviation means

Let $I \subseteqq \mathbf{R}$ be an interval.

The function $E: I^2 \to \mathbf{R}$ is said to be a *deviation* on I if it has the following properties:

(E1) The function $y \mapsto E(x, y)$ is strictly decreasing and continuous on I for each $x \in I$;

(E2) $E(x, x) = 0$ for every $x \in I$.

Let us denote by $\mathcal{E}(I)$ the set of all deviations on I.

Let $E \in \mathcal{E}(I)$, $\underline{x} = (x_1, \ldots, x_n) \in I^n$, $n \in \mathbf{N}$ and let us consider the equation

$$\sum_{i=1}^{n} E(x_i, y) = 0. \tag{2.1}$$

By (E1) the function $e: y \mapsto \sum_{i=1}^{n} E(x_i, y)$ is strictly decreasing. Using (E2)

$$e(\min\{x_1, \ldots, x_n\}) \geqq 0 \geqq e(\max\{x_1, \ldots, x_n\})$$

thus (2.1) has a unique solution $y = y_0$ satisfying the inequality

$$\min\{x_1, \ldots, x_n\} \leqq y_0 \leqq \max\{x_1, \ldots, x_n\}.$$

Definition 1. Let $E \in \mathscr{E}(I)$, $\underline{x} = (x_1, \ldots, x_n) \in I^n$, $n \in \mathbf{N}$. Then the unique solution y_0 of the equation (2.1) is called the *deviation mean* of $x_1, \ldots, x_n \in I$ and is denoted by $\mathfrak{M}_{n,E}(\underline{x})$ or $\mathfrak{M}_{n,E}(x_1, \ldots, x_n)$.

Let $\Omega(I)$ be the set of all real valued functions which are *continuous* and *strictly monotonously increasing* on I. Furthermore let $\mathscr{P}(I)$ denote the class of *positive* real valued functions on I. If $\varphi \in \Omega(I)$, $f \in \mathscr{P}(I)$ then the function

$$E(x, y) \doteq E_{\varphi, f}(x, y) = f(x)(\varphi(x) - \varphi(y)), \quad (x, y \in I) \tag{2.2}$$

is a deviation on I. If $\underline{x} = (x_1, \ldots, x_n) \in I^n$, $n \in \mathbf{N}$ then we find that the unique solution y_0 of (2.1) for this deviation (2.2) has the form

$$y_0 = \mathfrak{M}_{n, E_{\varphi, f}}(\underline{x}) \doteq M_{n,\varphi}(\underline{x})_f \doteq \varphi^{-1}\left[\sum_{i=1}^{n} f(x_i)\varphi(x_i) \Big/ \sum_{i=1}^{n} f(x_i)\right]. \tag{2.3}$$

The quantity $M_{n,\varphi}(\underline{x})_f$ defined by (2.3) will be called *quasiarithmetic mean with weightfunction* (Bajraktarevich [2], Aczél—Daróczy [1], Daróczy [6, 7]). If $f(x) = p =$ positive constant in (2.3) then we obtain the well-known *quasiarithmetic mean*

$$M_{n,\varphi}(\underline{x}) \doteq \varphi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n} \varphi(x_i)\right]. \tag{2.4}$$

The theory of these means can be found in the book of Hardy—Littlewood—Pólya [10] (see also the books Beckenbach—Bellman [4], Mitrinovic [13]).

In the investigation of the complementary comparison problem of deviation means we need the concept of weighted deviation mean, which has been introduced and examined by Daróczy—Páles [8].

Let $E \in \mathscr{E}(I)$, $x, y \in I$, $\lambda \in [0, 1]$ and let us consider the equation

$$\lambda E(x, t) + (1 - \lambda) E(y, t) = 0. \tag{2.5}$$

It is easy to check that (2.5) has a unique solution $t_0 \in I$ lying between $\min\{x, y\}$ and $\max\{x, y\}$.

Definition 2. Let $E \in \mathscr{E}(I)$, $x, y \in I$, $\lambda \in [0, 1]$. Then the unique solution t_0 of the equation (2.5) is called the *deviation mean* of x and y *weighted* by λ and $1 - \lambda$. This quantity is denoted by $\mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda)$ or $\hat{E}(x, y; \lambda)$.

If $\varphi \in \Omega(I)$, $f \in \mathscr{P}(I)$ and $E_{\varphi, f}$ is the deviation defined by (2.2) then for the solution $t_0 \in I$ of (2.5) we will also use the notation $M_{2,\varphi}(x, y; \lambda, 1-\lambda)_f$.

We need the following results of Daróczy—Páles [8]:

Lemma 2.1. *Let* $E\in\mathcal{E}(I)$, $x, y\in I$, $\lambda\in[0, 1]$. *The inequality*

$$\mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda) \geqq t \tag{2.6}$$

is valid if and only if the inequality

$$\lambda E(x, t)+(1-\lambda)E(y, t) \geqq 0 \tag{2.7}$$

is true.

Lemma 2.2. *Let* $E\in\mathcal{E}(I)$ *and* $x<y$, $(x>y)$ $x, y\in I$. *Then the function*

$$e\colon \lambda \to \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda), \quad \lambda\in[0, 1]$$

is strictly decreasing (increasing) and continuous.

3. Complementary comparison of deviation means

Let $I\subseteq\mathbf{R}$ be a compact interval.

Definition 3. The function $H: I^2\to\mathbf{R}$ is said to be a *comparative function* on I if it has the following properties:
(H1) H is continuous on I^2;
(H2) $H(x, x)=k$ for every $x\in I$, (k=constant);
(H3) (a) For each $y\in I$ the function $x\mapsto H(x, y)$ is strictly increasing;
(b) For each $x\in I$ the function $y\mapsto H(x, y)$ is strictly decreasing.
The set of all comparative functions on I is denoted by $\mathcal{H}(I)$.

Let $E, F\in\mathcal{E}(I)$. The deviation means generated by E and F are said to be *comparable* if for every $\underline{x}\in I^n$, $n\in\mathbf{N}$

$$\mathfrak{M}_{n,F}(\underline{x})\leqq\mathfrak{M}_{n,E}(\underline{x}). \tag{3.1}$$

On the comparison of deviation means Daróczy—Páles [8] proved the following result:

Theorem 3.1. *Let* $E, F\in\mathcal{E}(I)$. *The inequality* (3.1) *holds for all* $\underline{x}\in I^n$, $n\in\mathbf{N}$ *if and only if the inequality*

$$F(x, y)E(z, y) \leqq F(z, y)E(x, y) \tag{3.2}$$

is satisfied for all $x\leqq y\leqq z$ $(x, y, z\in I)$.

For comparable deviation means (3.1) the problem of complementary comparison is the following:

Let $H\in\mathcal{H}(I)$. Find the least upper bound

$$K_H(E, F) \doteq \sup_{\substack{\underline{x}\in I^n\\ n\in\mathbf{N}}} H\big(\mathfrak{M}_{nE}(\underline{x}), \mathfrak{M}_{n,F}(\underline{x})\big). \tag{3.3}$$

If $I=[m, M]$ then by H1—3 we get $K_H(E, F)\leqq H(M, m)$ thus $K_H(E, F)$ is finite.

If in particular $H(x, y)=Q(x, y)\doteq\dfrac{x}{y}$ and $H(x, y)=D(x, y)\doteq x-y$ then by the help of $K_H(E, F)$ the following complementary inequalities can be obtained:

$$\mathfrak{M}_{n,E}(\underline{x}) \leqq K_Q(E, F)\mathfrak{M}_{n,F}(\underline{x}),$$

$$\mathfrak{M}_{n,E}(\underline{x}) \leqq K_D(E, F)+\mathfrak{M}_{n,F}(\underline{x}).$$

For quasiarithmetic means the complementary comparison problem was solved by BECK [3] who proved the next result.

Theorem 3.2. *Let $I=[m, M]$, $\varphi, \psi \in \Omega(I)$, $H\in\mathscr{H}(I)$. Assume that the inequality*

$$M_{n,\varphi}(\underline{x}) \leqq M_{n,\psi}(\underline{x})$$

holds for all $\underline{x}\in I^n$, $n\in\mathbf{N}$ (which means that $\psi\circ\varphi^{-1}$ is a convex function on I). Then

$$\sup_{\substack{\underline{x}\in[m,M]^n\\ n\in\mathbf{N}}} H(M_{n,\psi}(\underline{x}), M_{n,\varphi}(\underline{x})) = \max_{0\leqq\lambda\leqq 1} H(M_{2,\psi}(m, M; \lambda, 1-\lambda), M_{2,\varphi}(m, M; \lambda, 1-\lambda)).$$

We also need the following lemma of CARATHEODORY (see EGGLESTON [9]).

Lemma 3.3. *Let $T\subseteqq\mathbf{R}^n$, $(n\in\mathbf{N})$ be an arbitrary set. Assume that $\underline{0}=(0, \dots, 0)\in\mathbf{R}^n$ is in the convex hull of T. Then there exist $n+1$ points of T such that their convex hull contains $\underline{0}$.*

The following theorem is one of the main results of this paper.

Theorem 3.4. *Let $E, F\in\mathscr{E}(I)$ be arbitrary deviations, and let $H\in\mathscr{H}(I)$ be a comparative function. Then*

$$K_H(E, F) = \sup_{\substack{x,y\in I\\ \lambda\in[0,1]}} H(\mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda), \mathfrak{M}_{2,F}(x, y; \lambda, 1-\lambda)), \tag{3.4}$$

where $K_H(E, F)$ is the quantity defined by (3.3).

PROOF. Let us denote the right hand side of (3.4) by K_0. First we prove the inequality $K_H(E, F)\leqq K_0$.

Let $\underline{x}=(x_1, \dots, x_n)\in I^n$, $n\in\mathbf{N}$ and

$$e \doteq \mathfrak{M}_{n,E}(\underline{x}), \quad f \doteq \mathfrak{M}_{n,F}(\underline{x})$$

Defining the mapping $T: I\to\mathbf{R}^2$ by

$$T(x) \doteq (E(x, e),\ F(x, f)), \quad (x\in I)$$

we get $\sum_{i=1}^{n}\frac{1}{n}T(x_i)=\frac{1}{n}\left(\sum_{i=1}^{n}E(x_i, e), \sum_{i=1}^{n}F(x_i, f)\right)=(0, 0)=\underline{0}$ thus the convex hull of the set $T(I)$ contains $\underline{0}$.

Hence by Lemma 3.3 there exist $z_1, z_2, z_3\in I$ such that $\underline{0}$ is in the convex hull of $T(z_1), T(z_2), T(z_3)$ that is

$$\underline{0}\in\Delta \doteq \{\lambda_1T(z_1)+\lambda_2T(z_2)+\lambda_3T(z_3)|\lambda_1, \lambda_2, \lambda_3\geqq 0, \quad \lambda_1+\lambda_2+\lambda_3 = 1\}.$$

Let $t_0=\sup\{t|(t, 0)\in\Delta\}$. Then $t_0\geqq 0$ since $\underline{0}\in\Delta$, and there exist $x, y\in\{z_1, z_2, z_3\}$, $\lambda\in[0, 1]$ such that

$$(t_0, 0) = \lambda T(x)+(1-\lambda)T(y).$$

Consequently

$$\lambda E(x, e)+(1-\lambda) E(y, e) = t_0 \geqq 0,$$
$$\lambda F(x, f)+(1-\lambda) F(y, f) = 0.$$

By Lemma 2.1. we have

$$\mathfrak{M}_{n,E}(\underline{x}) = e \leqq \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda),$$
$$\mathfrak{M}_{n,F}(\underline{x}) = f = \mathfrak{M}_{2,F}(x, y; \lambda, 1-\lambda).$$

Hence

$$H(\mathfrak{M}_{n,E}(\underline{x}), \mathfrak{M}_{n,F}(\underline{x})) \leqq H(\mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda), \mathfrak{M}_{2,F}(x, y; \lambda, 1-\lambda)) \leqq K_0$$

and thus $K_H(E, F) \leqq K_0$. Now we prove the reversed inequality.

Let $\varepsilon > 0$ be arbitrary. Then there exists $x_0, y_0 \in I, \lambda_0 \in [0, 1]$ such that

$$K_0 \leqq H(\mathfrak{M}_{2,E}(x_0, y_0; \lambda_0, 1-\lambda_0), \mathfrak{M}_{2,F}(x_0, y_0, \lambda_0, 1-\lambda_0)) + \frac{\varepsilon}{2}.$$

Let $\lambda \in [0, 1]$ and

$$h(\lambda) \doteq H(\mathfrak{M}_{2,E}(x_0, y_0; \lambda, 1-\lambda), \mathfrak{M}_{2,F}(x_0, y_0; \lambda, 1-\lambda)).$$

By Lemma 2.2. and property (H1) the function $\lambda \mapsto h(\lambda)$ is continuous on $[0, 1]$. Thus there exists a rational number $\bar{\lambda}_0 \in [0, 1]$ with

$$|h(\lambda_0) - h(\bar{\lambda}_0)| < \frac{\varepsilon}{2}$$

therefore

$$K_0 \leqq h(\bar{\lambda}_0) + \varepsilon.$$

If $\bar{\lambda}_0 = m/n$, $0 \leqq m \leqq n$, $m, n \in \mathbf{N}$ we set

$$\underline{x} = (\underbrace{x_0, \ldots, x_0}_{m}, \underbrace{y_0, \ldots, y_0}_{n-m}) \in I^n.$$

It is clear that

$$\mathfrak{M}_{n,E}(\underline{x}) = \mathfrak{M}_{2,E}(x_0, y_0; \bar{\lambda}_0, 1-\bar{\lambda}_0),$$
$$\mathfrak{M}_{n,F}(\underline{x}) = \mathfrak{M}_{2,F}(x_0, y_0, \bar{\lambda}_0, 1-\bar{\lambda}_0).$$

Hence $H(\mathfrak{M}_{n,E}(\underline{x}), \mathfrak{M}_{n,F}(\underline{x})) = h(\bar{\lambda}_0)$ and $K_0 \leqq h(\bar{\lambda}_0) + \varepsilon \leqq K_H(E, F) + \varepsilon$. Since ε was arbitrary we get $K_0 \leqq K_H(E, F)$ which completes the proof. $\square$

Remarks. (1) In Theorem 3.4. we have not assumed that the means involved are comparable. If we want to calculate $K_H(E, F)$ as a maximum of a function of one variable (as this was the case in Theorem 3.2.) then we shall need a further condition very similar to condition (3.2) (which was the criterion of the comparability of the means).

(2) In the proof of Theorem 3.4. only the properties (H1) and (H2) of H were used. Let $H: I^2 \to \mathbf{R}$ be a continuous function such that for every fixed value $x \in I$ (or $y \in I$) the function $y \to H(x, y)$ (or $x \mapsto H(x, y)$) is monotonously increasing or decreasing. Then (3.4) is valid for this function H, too (provided that $E, F \in \mathcal{E}(I)$ and the definition of $K_H, (E, F)$ is extended to this function H by (3.3)).

Theorem 3.5. *Let* $I=[m, M]\subseteq \mathbf{R}$, $E, F\in\mathscr{E}(I)$, $H\in\mathscr{H}(I)$. *Assume that*

$$E(M, \hat{E}(x, y; \lambda))\cdot F(m, \hat{F}(x, y, \lambda)) \leqq E(m, \hat{E}(x, y; \lambda))F(M, \hat{F}(x, y; \lambda)) \tag{3.5}$$

holds for every $x, y\in I$, $\lambda\in[0, 1]$. *Then*

$$\sup_{\substack{x, y\in I\\ \lambda\in[0,1]}} H(\hat{E}(x, y; \lambda), \hat{F}(x, y; \lambda)) = \max_{0\leqq\lambda\leqq 1} H(\hat{E}(m, M; \lambda), \hat{F}(m, M; \lambda)). \tag{3.6}$$

Proof. Let us denote the right and left hand sides of the inequality (3.6) by K_1 and K_0, respectively. The inequality $K_0\geqq K_1$ is obvious thus it is enough to show that $K_0\leqq K_1$.

For $x, y\in I, \lambda\in[0, 1]$ define p by

$$p=\frac{F(M, \hat{F}(x, y; \lambda))}{F(M, \hat{F}(x, y; \lambda))-F(m, \hat{F}(x, y; \lambda))}. \tag{3.7}$$

By (3.5) and the properties (E1), (E2) we get that $p\in[0, 1]$ and

$$p\leqq\frac{E(M, \hat{E}(x, y; \lambda))}{E(M, \hat{E}(x, y; \lambda))-E(m, \hat{E}(x, y; \lambda))}. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$pF(m, \hat{F}(x, y, \lambda))+(1-p)F(M, \hat{F}(x, y; \lambda)) = 0,$$
$$pE(m, \hat{E}(x, y; \lambda))+(1-p)E(M, \hat{E}(x, y; \lambda)) \geqq 0.$$

Using Lemma 2.1. we get

$$\hat{F}(x, y; \lambda) = \mathfrak{M}_{2,F}(m, M; p, 1-p) = \hat{F}(m, M; p), \tag{3.9}$$

$$\hat{E}(x, y; \lambda) \leqq \mathfrak{M}_{2,E}(m, M; p, 1-p) = \hat{E}(m, M; p). \tag{3.10}$$

By the property (H3) this means that

$$H(\hat{E}(x, y; \lambda), \hat{F}(x, y; \lambda)) \leqq H(\hat{E}(m, M; p),\ \hat{F}(m, M; p)) \leqq K_1$$

therefore $K_0\leqq K_1$.

Remarks. (1) It can be shown that the inequality (3.5) is not only sufficient but also necessary for the existence of such a $p\in[0, 1]$ for which (3.9) and (3.10) simultaneously hold.

(2) From (3.5) we obtain the condition (3.2) if we write y instead of x and substitute $x=m$, $z=M$. This shows that (3.5) is a stronger condition than (3.2).

Theorem 3.6. *Let* $I=[m, M]\subseteq\mathbf{R}$, $E, F\in\mathscr{E}(I)$. *In order that* (3.5) *be valid for* $x, y\in I, \lambda\in[0, 1]$ *it is necessary and in the case* $E=E_{\varphi,g}$ $F=F_{\psi,h}(\varphi, \psi\in\Omega(I), g, h\mathscr{P}(I))$ *also sufficient that*

$$\frac{E(m, M)}{F(m, M)}\leqq\frac{E(x, M)}{F(x, M)}, \tag{3.11}$$

$$\frac{E(x, m)}{F(x, m)}\leqq\frac{E(M, m)}{F(M, m)}. \tag{3.12}$$

hold for every $x\in(m, M)$.

PROOF. *Necessity.*

Let $x\in(m, M)$ and substitute $y=M$ in (3.5). Then for $\lambda\in(0, 1)$

$$\lambda E(x, \hat{E}(x, M; \lambda))+(1-\lambda)E(M, \hat{E}(x, M; \lambda)) = 0,$$

$$\lambda F(x, \hat{F}(x, M; \lambda))+(1-\lambda)F(M, \hat{F}(x, M; \lambda)) = 0.$$

Hence

$$E(M, \hat{E}(x, M; \lambda)) = \frac{\lambda}{\lambda-1} E(x, \hat{E}(x, M; \lambda)), \tag{3.13}$$

$$F(M, \hat{F}(x, M; \lambda)) = \frac{\lambda}{\lambda-1} F(x, \hat{F}(x, M; \lambda)). \tag{3.14}$$

From (3.5), using the properties (E1) and (E2) and applying the relations (3.13) and (3.14), we get

$$\frac{E(m, \hat{E}(x, M; \lambda))}{F(m, \hat{F}(x, M; \lambda))} \leqq \frac{E(M, \hat{E}(x, M; \lambda))}{F(M, \hat{F}(x, M; \lambda))} = \frac{E(x, \hat{E}(x, M; \lambda))}{F(x, \hat{F}(x, M; \lambda))}. \tag{3.15}$$

By Lemma 2.2.

$$\lim_{\lambda\to 0} \hat{E}(x, M; \lambda) = M, \quad \lim_{\lambda\to 0} \hat{F}(x, M; \lambda) = M.$$

Taking the limit $\lambda\to 0$ on both sides of the inequality (3.15) we get

$$\frac{E(m, M)}{F(m, M)} \leqq \frac{E(x, M)}{F(x, M)}$$

which is exactly (3.11). (3.12) can be proved similarly (starting with the substitution $y=m$ in (3.5)).

Sufficiency.

Let $x, y\in I$, $\lambda\in[0, 1]$ be arbitrary. Then

$$\hat{E}(x, y; \lambda) = \varphi^{-1}\left(\frac{\lambda g(x)\varphi(x)+(1-\lambda)g(y)\varphi(y)}{\lambda g(x)+(1-\lambda)g(y)}\right)$$

and

$$\hat{F}(x, y; \lambda) = \psi^{-1}\left(\frac{\lambda h(x)\psi(x)+(1-\lambda)h(y)\psi(y)}{\lambda h(x)+(1-\lambda)h(y)}\right).$$

A simple calculation gives

$$\frac{E(M, \hat{E}(x, y; \lambda))}{E(m, \hat{E}(x, y; \lambda))} = -\frac{E(M, m)}{E(m, M)}\cdot\frac{\lambda E(x, M)+(1-\lambda)E(y, M)}{\lambda E(x, m)+(1-\lambda)E(y, m)}$$

and

$$\frac{F(M, \hat{F}(x, y; \lambda))}{F(m, \hat{F}(x, y; \lambda))} = -\frac{F(M, m)}{F(m, M)}\cdot\frac{\lambda F(x, M)+(1-\lambda)F(y, M)}{\lambda F(x, m)+(1-\lambda)F(y, m)}.$$

Hence instead of (3.5) it is enough to show that

(3.16)
$$\frac{F(M,m)}{F(m,M)}\cdot\frac{\lambda F(x,M)+(1-\lambda)F(y,M)}{\lambda F(x,m)+(1-\lambda)F(y,m)} \leqq \frac{E(M,m)}{E(m,M)}\cdot\frac{\lambda E(x,M)+(1-\lambda)E(y,M)}{\lambda E(x,m)+(1-\lambda)E(y,m)}.$$

Let $x_1 \doteq x$, $x_2 \doteq y$, $\lambda_1 \doteq \lambda$, $\lambda_2 \doteq 1-\lambda$ and $i, j \in \{1, 2\}$ be arbitrary. Then by the conditions (3.11) and (3.12) we have

$$E(m, M)F(x_i, M) \leqq F(m, M)E(x_i, M)$$

and

$$E(x_j, m)F(M, m) \leqq F(x_j, m)E(M, m).$$

Taking the product of these inequalities we obtain

(3.17)
$$E(m, M)F(M, m)E(x_j, m)F(x_i, M) \leqq$$
$$\leqq F(m, M)E(M, m)F(x_j, m)E(x_i, M).$$

Multiplying the inequality (3.17) by $\lambda_i\lambda_j$ and adding the inequalities obtained we have

$$E(m, M)F(M, m)\sum_{j=1}^{2}\lambda_j E(x_j, m)\sum_{i=1}^{2}\lambda_i F(x_i, M) \leqq$$

$$\leqq F(m, M)E(M, m)\sum_{j=1}^{2}\lambda_j F(x_j, m)\sum_{i=1}^{2}\lambda_i E(x_i, M).$$

Rearranging this inequality we get exactly the relation (3.16) which was to be shown. □

Corollary 3.7. *Let $I=[m, M]$, $\varphi, \psi \in \Omega(I)$, $g, h \in \mathscr{P}(I)$. Assume that the deviations $E=E_{\varphi,g}$, $F=E_{\psi,h}$ satisfy the inequalities* (3.11) *and* (3.12) *for each $x \in (m, M)$. Then for any comparative function $H \in \mathscr{H}(I)$*

$$K_H(E_{\varphi,g}, E_{\psi,h}) = \max_{0\leqq\lambda\leqq 1} H(\hat{E}_{\varphi,g}(m, M; \lambda), \hat{E}_{\psi,h}(m, M; \lambda)).$$

Remark. Let in the above corollary be $g(x)\equiv g$, $h(x)\equiv h$ constants and $\psi\circ\varphi^{-1}$ a convex function on I. It is easy to verify that $E=E_{\varphi,g}$ and $F=E_{\psi,h}$ satisfy the conditions (3.11) and (3.12). Thus we obtain Theorem 3.2. as a consequence of Theorem 3.6.

4. Complementary comparison of homogeneous quasiarithmetic means with weightfunction

Let $I=\mathbf{R}_+ \doteq (0, \infty)$, $\varphi \in \Omega(\mathbf{R}_+)$ and let $f \in \mathscr{P}(\mathbf{R}_+)$ be a continuous function. The mean $M_{n,\varphi}(\underline{x})_f$ ($\underline{x} \in \mathbf{R}_+^n$, $n \in \mathbf{N}$) is said to be *homogeneous* if

$$tM_{n,\varphi}(\underline{x})_f = M_{n,\varphi}(t\underline{x})_f$$

holds for every $t \in \mathbf{R}_+$, $\underline{x} \in \mathbf{R}_+^n$, $n \in \mathbf{N}$.

ACZÉL—DARÓCZY [1] proved that the mean $M_{n,\varphi}(\underline{x})_f$ $(\underline{x}\in\mathbf{R}_+^n,\ n\in\mathbf{N})$ is homogeneous if and only if it has one of the following forms:

$$M_{n,a}(\underline{x})_p \doteq \left[\frac{\sum_{i=1}^{n} x_i^{a+p}}{\sum_{i=1}^{n} x_i^p}\right]^{\frac{1}{p}}, \quad a \neq 0,$$

$$M_{n,0}(\underline{x})_p \doteq \exp\left[\frac{\sum_{i=1}^{n} x_i^p \ln x_i}{\sum_{i=1}^{n} x_i^p}\right]$$

where $a, p\in\mathbf{R}$ are arbitrary.

Let $a, p\in\mathbf{R}$, $x, y, z\in\mathbf{R}_+$ and

$$E_{a,p}(x, y) \doteq \begin{cases} x^p(x^a-y^a)/a, & \text{if } a \neq 0, \\ x^p(\ln x-\ln y), & \text{if } a = 0, \end{cases} \tag{4.1}$$

$$j_{a,p}(z) \doteq \begin{cases} (z^{a+p}-z^p)/a, & \text{if } a \neq 0, \\ z^p \ln z, & \text{if } a = 0. \end{cases} \tag{4.2}$$

It is easy to see that $E_{a,p}$ is a deviation on $\mathbf{R}_+$,

$$E_{a,p}(x, y) = y^{a+p}E_{a,p}\left(\frac{x}{y}, 1\right) = y^{a+p}j_{a,p}\left(\frac{x}{y}\right), \quad (x, y\in\mathbf{R}_+) \tag{4.3}$$

and

$$\mathfrak{M}_{n,E_{a,p}}(\underline{x}) = M_{n,a}(\underline{x})_p, \quad (\underline{x}\in\mathbf{R}_+^n, n\in\mathbf{N}).$$

Define g_a by

$$g_a(z) \doteq \begin{cases} -a/(1-z^{-a}), & a \neq 0, \quad z\in\mathbf{R}_+\setminus\{1\}, \\ -1/\ln z, & a = 0, \quad z\in\mathbf{R}_+\setminus\{1\}, \\ -a/2, & z = 1, \quad a\in\mathbf{R}, \end{cases} \tag{4.4}$$

and let

$$h_{a,b}(z) \doteq g_b(z)-g_a(z). \tag{4.5}$$

We need the following results of LOSONCZI [12]:

Lemma 4.1. *The function* $h_{a,b}$ *defined by* (4.5) *is continuous on* $\mathbf{R}_+$ *(even at* $z=1$*!) for all* $a, b\in\mathbf{R}$ *and is*
strictly increasing, if $|a|>|b|$,
constant, if $|a|=|b|$,
strictly decreasing, if $|a|<|b|$.

Theorem 4.2. *Let* $a, b, p, g\in\mathbf{R}$,

$$(a-b)^2+(p-q)^2 > 0, \quad (a+b)^2+(a+p-q)^2 > 0, \tag{4.6}$$

$$m, M\in\mathbf{R}_+, \quad m < M \quad \text{and} \quad A \doteq m/M.$$

Then the following conditions (i), (ii), (iii) *are equivalent:*

(i) *For every* $\underline{x}\in[m, M]^n$, $n\in\mathbf{N}$,

$$M_{n,a}(\underline{x})_p \leqq M_{n,b}(\underline{x})_q; \tag{4.7}$$

(ii) *For every* $z\in[A, 1/A]$,

$$j_{a,p}(z) \leqq j_{b,q}(z); \tag{4.8}$$

(iii) *If* $|a|\geqq|b|$ *then*

$$h_{a,b}(1) < q-p \quad \textit{and} \quad 1 \leqq \frac{j_{b,q}(1/A)}{j_{a,p}(1/A)}, \tag{4.9}$$

If $|a|\leqq|b|$ *then*

$$h_{a,b}(1) < q-p \quad \textit{and} \quad \frac{j_{b,q}(A)}{j_{a,p}(A)} \leqq 1. \tag{4.10}$$

Remark. The inequalities (4.6) exclude the identity

$$M_{n,a}(\underline{x})_p \equiv M_{n,b}(\underline{x})_q, \quad (\underline{x}\in[m, M]^n,\ n\in\mathbf{N}).$$

Theorem 4.3. *Let* $a, p, b, q\in\mathbf{R}$, $m, M\in\mathbf{R}_+$, $m<M$ *and* $A\doteq m/M$. *In order that the deviations* $E=E_{b,q}$, $F=E_{a,p}$ *satisfy* (3.11) *and* (3.12) *for every* $x\in(m, M)$ *it is sufficient and necessary that*

$$q-p \geqq \max\{h_{a,b}(A), h_{a,b}(1/A)\}. \tag{4.11}$$

Proof. Use (4.3) and substitute $y=\frac{x}{M}$ and $y=\frac{m}{x}$ in (3.11) and (3.12), respectively. Then we get the inequalities

$$\frac{j_{b,q}(A)}{j_{a,p}(A)} \leqq \frac{j_{b,q}(y)}{j_{a,p}(y)}, \qquad y\in(A, 1), \tag{4.12}$$

$$\frac{j_{b,q}(y)}{j_{a,p}(y)} \leqq \frac{j_{b,q}(1/A)}{j_{a,p}(1/A)}, \quad y\in(1, 1/A), \tag{4.13}$$

which are equivalent to (3.11) and (3.12) respectively.

Let

$$f(z) \doteq f_{a,b,p,q}(z) \doteq \begin{cases} j_{b,q}(z)/j_{a,p}(z), & z\in\mathbf{R}_+\setminus\{1\}, \\ 1, & z=1. \end{cases}$$

then (4.12) and (4.13) can be written as

$$f(A) \leqq f(y), \quad y\in(A, 1), \tag{4.14}$$

$$f(y) \leqq f(1/A), \quad y\in(1, 1/A). \tag{4.15}$$

A simple calculation shows that f is continuously differentiable on $(0, \infty)$ and

$$f'(z) = (f(z)/z)(q-p-h_{a,b}(z)).$$

If $|a|>|b|$ then by Lemma 4.1. $h_{a,b}$ is strictly increasing. If $q-p<h_{a,b}(1/A)$ then there exists an $y\in(1, 1/A)$ such that $q-p<h(y)$ also holds, but then $f'(z)<0$ for $z\in(y, 1/A)$ hence $f(y)>f(1/A)$. This contradicts to (4.15). So for (4.15) to hold it is necessary that $q-p \geqq h_{a,b}(1/A) > h_{a,b}(A)$. Thus (4.11) is a necessary condition. In the cases $|a|=|b|$, $|a|<|b|$ the necessity of (4.11) can be proved similarly.

If (4.11) holds then $f'(z) \geqq 0$ for $z \in (A, 1/A)$ that is f is monotonous and increasing. Therefore (3.11) and (3.12) are valid for all $x \in (m, M)$. □

Remark. If the inequality (4.11) holds then f is increasing, hence by $f(1)=1$

$$f(z) \leqq 1, \quad z \in [A, 1),$$
$$f(z) \geqq 1, \quad z \in (1, 1/A].$$

Thus the condition (ii) of Theorem 4.2. is satisfied hence (4.7) is valid for every $\underline{x} \in [m, M]^n$, $n \in \mathbf{N}$.

Using Theorem 4.3. and Corollary 3.7. we get

Corollary 4.4. *Let* $m, M \in \mathbf{R}_+$, $m<M$, $A \doteq m/M$ *and* $H \in \mathscr{H}([m, M])$. *Let further* $a, b, p, q \in \mathbf{R}$ *and assume that* (4.11) *holds. Then*

(i) *For every* $\underline{x} \in [m, M]^n$, $n \in \mathbf{N}$ *the inequality* (4.7) *is valid.*

(ii)

$$K_H(E_{b,q}, E_{a,p}) = \max_{0 \leqq \lambda \leqq 1} H\big(M_{2,b}(m, M; \lambda, 1-\lambda)_q, M_{2,a}(m, M; \lambda, 1-\lambda)_p\big).$$

5. Other results

Let $m, M \in \mathbf{R}_+$, $m<M$, $A \doteq m/M$ and $a, b, p, q \in \mathbf{R}$ be real numbers for which the inequality (4.6) is satisfied. *Suppose that* (4.7) *is valid for every* $\underline{x} \in [m, M]^n$, $n \in \mathbf{N}$.

Theorem 5.1. *Let* $H \in \mathscr{H}(\mathbf{R}_+)$ *be a function partially differentiable with respect to both of its variables. Assume that* $|a| \geqq |b|$ ($|a| \leqq |b|$) *and for every* $x \geqq y$, $x, y \in \mathbf{R}_+$ *the function*

$$t \to H(tx, ty), \quad (t \in \mathbf{R}_+) \tag{5.1}$$

is increasing (decreasing). Then

$$K_H(E_{b,q} E_{a,p}) = \max_{0 \leqq \lambda \leqq 1} H\big(M_{2,b}(m, M; \lambda, 1-\lambda)_q, M_{2,a}(m, M; \lambda, 1-\lambda)_p\big).$$

Proof. We deal only with the case $|a| \geqq |b|$, in the case $|a| \leqq |b|$ the proof is similar.

By Theorem 3.4. and the homogeneity of the means

$$K_H(E_{b,q}, E_{a,p}) = \sup_{\substack{m \leqq x \leqq y \leqq M \\ 0 \leqq \lambda \leqq 1}} H\big(M_{2,b}(x, y; \lambda, 1-\lambda)_q, M_{2,a}(x, y; \lambda, 1-\lambda)_p\big) =$$

$$= \sup_{\substack{\frac{m}{y} \leqq \frac{x}{y} \leqq 1 \\ y \in [m, M],\ \lambda \in [0,1]}} H\left(yM_{2,b}\left(\frac{x}{y}, 1; \lambda, 1-\lambda\right)_q, yM_{2,a}\left(\frac{x}{y}, 1; \lambda, 1-\lambda\right)_p\right) \leqq$$

$$\leqq \sup_{\substack{A \leqq u \leqq 1 \\ y \in [m, M],\ \lambda \in [0,1]}} H\big(yM_{2,b}(u, 1; \lambda, 1-\lambda)_q, yM_{2,a}(u, 1; 1-\lambda)_p\big) \doteq K_0.$$

Using (4.7) and the monotonity of the function (5.1) we obtain

$$K_0 \leqq \sup_{\substack{A \leqq u \leqq 1 \\ 0 \leqq \lambda \leqq 1}} H\big(M \cdot M_{2,b}(u, 1; \lambda, 1-\lambda)_q, M \cdot M_{2,a}(u, 1; \lambda, 1-\lambda)_p\big). \tag{5.2}$$

The function of (u, λ) standing behind the sup sign is partially differentiable, hence the supremum is attained on the compact set $[A, 1]\times[0, 1]$. We show that the maximum is attained at $u=A$. Suppose that the maximum is attained at an interior point (u_0, λ_0) of $[A, 1]\times[0, 1]$. Then (u_0, λ_0) satisfies the following equations:

$$\frac{\partial}{\partial u} H\big(M \cdot M_{2,b}(u, 1; \lambda, 1-\lambda)_q, M \cdot M_{2,a}(u, 1; \lambda, 1-\lambda)_p\big)\big|_{(u_0, \lambda_0)} = 0, \tag{5.3}$$

$$\frac{\partial}{\partial \lambda} H\big(M \cdot M_{2,b}(u, 1; \lambda, 1-\lambda)_q, M \cdot M_{2,a}(u, 1; \lambda, 1-\lambda)_p\big)\big|_{(u_0, \lambda_0)} = 0. \tag{5.4}$$

After simple calculations we obtain

$$\frac{\partial \hat{E}_{a,p}(u, 1; \lambda)}{\partial \lambda} = \hat{E}_{a,p}(u, 1; \lambda) \cdot \frac{j_{a,p}(u)}{(\lambda u^p + 1 - \lambda)(\lambda u^{a+p} + 1 - \lambda)},$$

$$\frac{\partial \hat{E}_{a,p}(u, 1; \lambda)}{\partial u} = \hat{E}_{a,p}(u, 1; \lambda) \cdot \frac{\lambda^2 u^{p+2-1} + \lambda(1-\lambda) j'_{a,p}(u)}{(\lambda u^p + 1 - \lambda)(\lambda u^{a+p} + 1 - \lambda)}.$$

Similar formulas hold for $\hat{E}_{b,q}$. Using these relations and equations (5.3) and (5.4) we get

$$\frac{\lambda_0^2 u_0^{b+2q-1} + \lambda_0(1-\lambda_0) j'_{b,q}(u_0)}{\lambda_0^2 u_0^{a+2p-1} + \lambda_0(1-\lambda_0) j'_{a,p}(u_0)} = \frac{j_{b,q}(u_0)}{j_{a,p}(u_0)}.$$

Thus

$$\frac{1-\lambda_0}{\lambda_0} = \frac{j_{b,q}(u_0) u_0^{a+2p-1} - j_{a,p}(u_0) u_0^{b+2q-1}}{j_{a,p}(u_0) j'_{a,p}(u_0) - j'_{b,q}(u_0) j'_{a,p}(u_0)}.$$

It can easily be seen that

$$j_{a,p}(u) = -u^{a+2p} j_{a,p}\left(\frac{1}{u}\right), \quad j_{b,q}(u) = -u^{b+2q} j_{b,q}\left(\frac{1}{u}\right)$$

and

$$j_{a,p}(u) \cdot j'_{b,q}(u) - j_{b,q}(u) j'_{a,p}(u) = \frac{1}{u} j_{b,q}(u) j_{a,p}(u)\big(q - p - h_{a,b}(u)\big).$$

Therefore

$$\frac{1-\lambda_0}{\lambda_0} = \frac{u_0^{a+b+2p+2q}\left(j_{b,q}\left(\frac{1}{u_0}\right) - j_{a,p}\left(\frac{1}{u_0}\right)\right)}{j_{b,q}(u_0) j_{a,p}(u_0)\big(h_{a,b}(u_0) - q + p\big)}. \tag{5.5}$$

Since $\lambda_0 \in (0, 1)$ we have $(1-\lambda_0)/\lambda_0 \in (0, \infty)$ and by

$$A < u_0 < 1, \quad j_{b,q}(u_0)\ j_{a,p}(u_0) > 0.$$

The condition (i) of Theorem (4.2) is satisfied hence $j_{b,q}\left(\frac{1}{u_0}\right)\geqq j_{a,p}\left(\frac{1}{u_0}\right)$. Thus the equation (5.5) implies that

$$h_{a,b}(u_0) > q-p.$$

By the condition (iii) of Theorem 4.2. $q-p>h(1)$ thus

$$h_{a,b}(u_0) > h(1). \tag{5.6}$$

Because of $|a|\geqq|b|$ the function $h_{a,b}$ is increasing so for $u_0\in(A,1)$ the inequality (5.6) is not true. This contradiction shows that in (5.2) the maximum is attained on the boundary of $[A,1]\times[0,1]$. But for the place of the maximum $u_0=1$, $\lambda_0=0$, $\lambda_0=1$ cannot be valid, consequently $u_0=A$. Hence

$$K_H(E_{b,q}, E_{a,p}) \leqq \max_{0\leqq\lambda\leqq 1} H\big(M_{2,b}(m, M; \lambda, 1-\lambda)_q, M_{2,a}(m, M; \lambda, 1-\lambda)_p\big).$$

The reversed inequality is obvious by Theorem 3.4.

Remark. If $\varphi\in\Omega(\mathbf{R}_+)$ is a differentiable function $h\in\Omega(\mathbf{R})$ then it is easy to show that the function $H\colon \mathbf{R}_+^2\to\mathbf{R}$ defined by

$$H(x, y) \doteq h\big(\varphi(x)-\varphi(y)\big) \quad x, y\in\mathbf{R}_+$$

is a comparative function on $\mathbf{R}_+$.

It can also be proved that for every $x, y\in\mathbf{R}_+$, $x\geqq y$ the function defined by (5.1) is increasing if and only if the function $x\mapsto x\varphi'(x)$ is increasing.

Corollary 5.2. *Let* $\lambda_0=\dfrac{1}{z_0+1}$ *and* z_0 *be the root of the equation of second degree*

$$j_{a,p}(A)(A^q+z)(A^{b+q}+z) = j_{b,q}(A)(A^p+z)(A^{a+p}+z) \tag{5.7}$$

lying in $(0,\infty)$. *Then*

$$\sup_{\substack{\underline{x}\in[m,M]^n\\ n\in\mathbf{N}}} \frac{M_{n,b}(\underline{x})_q}{M_{n,a}(\underline{x})_p} = \frac{M_{2,b}(A, 1; \lambda_0, 1-\lambda_0)_q}{M_{2,a}(A, 1; \lambda_0, 1-\lambda_0)_p}.$$

Proof. Let $H(x,y)\geqq Q(x,y)=x/y$, $(x,y\in\mathbf{R}_+)$. Then for $x\geqq y$, $x, y\in\mathbf{R}$ the function (5.1) is constant, hence by Theorem 5.1

$$\sup_{\substack{\underline{x}\in[m,M]^n\\ n\in\mathbf{N}}} \frac{M_{n,b}(\underline{x})_q}{M_{n,a}(\underline{x})_p} = K_Q(E_{b,q}, E_{a,p}) = \max_{0\leqq\lambda\leqq 1} \frac{M_{2,b}(A, 1; \lambda, 1-\lambda)_q}{M_{2,a}(A, 1; \lambda, 1-\lambda)_p}.$$

For $\lambda=0, \lambda=1$ the ratio standing behind the max sign is equal to 1 and for $\lambda\in(0,1)$ it is not less than 1, therefore the maximum is attained at a value $0<\lambda_0<1$ and there

$$\frac{\partial}{\partial\lambda}\frac{M_{2,b}(A, 1; \lambda_0, 1-\lambda_0)_q}{M_{2,a}(A, 1; \lambda_0, 1-\lambda_0)_p} = 0.$$

Thus for $z_0=(1-\lambda_0)/\lambda_0$ we get exactly the equation (4.7).

It can also be proved that (5.7) has in $(0, \infty)$ exactly one root. $\square$

Remark. If $p=q$ then (5.7) is an equation of first degree for z. In the case $p=q=0$ we get the well-known complementary inequality for the ratio of power means of SPECHT [14], CARGO-SHISHA [5].

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References

[1] J. ACZÉL und Z. DARÓCZY, Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind. *Publ. Math.* (Debrecen), **10** (1963), 171—190.
[2] M. BAJRAKTAREVIC, Sur une equation functionelle aux valeurs moyennes. *Glasnik Mat-Fiz. Astr.* **13** (1958), 243—248.
[3] E. BECK, Komplementäre Ungleichungen bei vergleichbaren Mittelwerten. *Monatsh. Math.* **73** (1969), 289—308.
[4] E. BECKENBACH and R. BELLMAN, Inequalities. *Ergebnisse d. Math.* **30** (1961).
[5] G. T. CARGO and O. SHISHA, Bounds on ratios of means. *J. Res. Nat. Bur. Standars Sect. B, 66B*, (1962), 169—170.
[6] Z. DARÓCZY, Über eine Klasse von Mittelwerten. *Publ. Math. (Debrecen)* **19** (1972), 211—217.
[7] Z. DARÓCZY, A general inequality for means. Aequationes. Math., **7** (1972), 16—21.
[8] Z. DARÓCZY and Zs. PÁLES, On comparison of mean values. *Publ. Math.* (Debrecen), **29** (1982), 107—115.
[9] H. G. EGGLESTON, Convexity. *Cambridge Univ. Press, New York and London,* 1967.
[10] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, Inequalities. *Cambridge,* 1934.
[11] L. V. KANTOROVIC, Functional analysis and applied mathematics (Russian). *Uspehi Mat. Nauk.* **3**, *No.* 6 (28) (1948), 89—185.
[12] L. LOSONCZI, Inequalities for Integral Mean Values. *J. Math. Anal. Appl.* **61** (3) (1977), 586—606.
[13] D. S. MITRINOVIC, Analytic inequalities. *Springer-Verlag, Berlin—Heidelberg—New York* 1970.
[14] W. SPECHT, Zur Theorie der elementaren Mittel. Math. Z. **74** (1960), 91—98.
[15] P. SCHWEITZER, An inequality concerning the arithmetic mean *(Hungarian)*. *Math. Phys. Lapok,* **23** (1914), 257—261.

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