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A note on the diophantine equation $D_1x^2 + D_2 = 2y^n$

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Abstract. Let D_1, D_2 be positive integers such that $2 \nmid D_1D_2$, $\gcd(D_1, D_2) = 1$, $D_1D_2 \not\equiv 7 \pmod 8$ and D_1, D_2 are square free. Let h denote the class number of $\mathbb{Q}(\sqrt{-D_1D_2})$. In this note we prove that the equation $D_1x^2 + D_2 = 2y^n$, $x, y, n \in \mathbb{N}$, $\gcd(x, y) = 1$, $n > 1$, $\gcd(n, 2h) = 1$, has only finitely many non-trivial solutions (x, y, n). Moreover, if (x, y, n) is a non-trivial solution, then n is an odd prime with $7 \le n < 212603$ and $y <$expexpexp 24.17.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let D_1, D_2 be positive integers such that $2 \nmid D_1D_2$, $\gcd(D_1, D_2) = 1$, $D_1D_2 \not\equiv 7 \pmod 8$ and D_1, D_2 are square free. Let h denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D_1D_2})$. In [4], Ljunggren discussed the solvability of the equation

$$\begin{aligned} D_1x^2 + D_2 = 2y^2,\ x, y, n \in \mathbb{N},\ y > 1,\ \gcd(x, y) = 1,\\ n > 1,\ \gcd(n, 2h) = 1. \end{aligned} \tag{1}$$

In this note, using Baker's method, we prove a general result concerning (1).

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For any non-negative integer m, let F_m, L_m denote the mth Fibonacci number and the mth Lucas number respectively. Before proceeding we note that if

$$D_1 r^2 = \frac{1}{2}\left(s + (-1)^{(s-1)/2}\right),\ D_2 = \frac{1}{2}\left(3s - (-1)^{(s-1)/2}\right), \tag{2}$$
$$3 \nmid h,\ r, s \in \mathbb{N},\ s > 1,\ 2 \nmid s,$$

or

$$D_1 r^2 = \begin{cases} \frac{1}{2}F_{6k-3}, \\ \frac{1}{2}F_{6k+3}, \end{cases} \quad D_2 = \frac{1}{2}L_{6k},\ 5 \nmid h,\ k, r \in \mathbb{N}, \tag{3}$$

then (1) has solutions

$$(x, y, n) = \left(\frac{r}{2}(3D_2 - D_1 r^2), s, 3\right), \tag{4}$$

or

$$(x, y, n) = \tag{5}$$
$$= \begin{cases} \left(\frac{r}{4}(D_1^2 r^4 - 10 D_1 D_2 r^2 + 5 D_2^2), F_{6k-1}, 5\right), & \text{if } D_1 r^2 = \frac{1}{2}F_{6k-3}, \\ \left(\frac{r}{4}(-D_1^2 r^4 + 10 D_1 D_2 r^2 - 5 D_2^2), F_{6k+1}, 5\right), & \text{if } D_1 r^2 = \frac{1}{2}F_{6k+3}. \end{cases}$$

The solutions (4) and (5) are called the trivial solutions of (1). This implies that (1) has possible infinitely many trivial solutions. For the non-trivial solutions of (1), we prove the following result.

Theorem. *The equation* (1) *has only finitely many non-trivial solutions* (x, y, n). *Moreover, if* (x, y, n) *is a non-trivial solution of* (1), *then* n *is an odd prime with* $7 \le n < 212603$ *and* $y < \exp\exp\exp\ 24.17$.

When $D_1 = 7$ and $D_2 = 11$, (1) has a non-trivial solution $(x, y, n) = (1169, 9, 7)$. This is the only example of non-trivial solutions that we know. It is natural to conjecture that $(x, y, n) = (1169, 9, 7)$ is the only non-trivial solution of (1).

2. Preliminaries

Lemma 1 ([3, Formula 3.76]). *For any positive integer n and any complex numbers α and β, we have*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} (\alpha+\beta)^{n-2i}(\alpha\beta)^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!\,n}{(n-2i)!\,i\,!}, \quad i = 0, \dots, [n/2]$$

are positive integers.

Let α be an algebraic number with the minimal polynomial

$$a_0z^d + a_1z^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (z - \sigma_i\alpha), \quad a_0 > 0,$$

where $\sigma_1\alpha, \dots, \sigma_d\alpha$ are conjugates of α. Then

$$h(\alpha) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\max(1, |\sigma_i\alpha|)\right)$$

is called Weil's height of α.

Lemma 2 ([2, Theorem 3]). *Let $\varepsilon = (X_1\sqrt{D_1} + \sqrt{-D_2})/\sqrt{2}$ and $\overline{\varepsilon} = (X_1\sqrt{D_1} - \sqrt{-D_2})/\sqrt{2}$ for some positive integers X_1. Let $\alpha = \varepsilon/\overline{\varepsilon}$ and $\Lambda = n\log\alpha - k\pi\sqrt{-1}$ for some integers n, k with $0 \le |k| < n$. If $\Lambda \neq 0$, then we have*

$$\log|\Lambda| \ge -9AB^2,$$

where $\quad A = \max(20, 12.85|\log\alpha| + h(\alpha)),\ d = [\mathbb{Q}(\alpha) : \mathbb{Q}]/2,$

$\quad B = \max(17, d\log n(1/2A + 1/25.7\pi) + 4.6d + 3.25).$

Lemma 3 ([1, Theorem 3]). *Let $k \in \mathbb{Z}$ with $k \neq 0$, and let $F(X,Y) \in \mathbb{Z}[X,Y]$ be an irreducible binary form of degree r with $r \geq 3$. Then all solutions (X,Y) of the equation*

$$f(X,Y) = k, \quad X, Y \in \mathbb{Z}, \quad \gcd(X,Y) = 1,$$

satisfy

$$\max(|X|,|Y|) < \exp\left(3^{3(r+9)} r^{18(r+1)} H^{2r-2} (\log H)^{2r-1} \log|k|\right),$$

where H is the maximum absolute value of the coefficients of f.

3. Proof of Theorem

Let (x,y,n) be a solution of (1). Since $D_1 D_2 \not\equiv 7 \pmod 8$ and $\gcd(n, 2h) = 1$, it follows from the analysis in [3] that $2 \nmid y$ and that there exist suitable positive integers X_1 and Y_1 such that

$$D_1 X_1^2 + D_2 Y_1^2 = 2y, \quad \gcd(X_1, Y_1) = 1, \quad 2 \nmid X_1 Y_1, \tag{6}$$

$$x\sqrt{D_1} + \sqrt{-D_2} = \frac{\lambda_1}{2^{(n-1)/2}} \left(X_1\sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2}\right)^n, \tag{7}$$
$$\lambda_1, \lambda_2 \in \{-1, 1\}.$$

Let $\alpha = \lambda_1(X_1\sqrt{D_1} + \lambda_2 Y_1\sqrt{-D_2})/\sqrt{2}$ and $\beta = \lambda_1(X_1\sqrt{D_1} - \lambda_2 Y_1\sqrt{-D_2})/\sqrt{2}$. Since $\alpha - \beta = \lambda_1\lambda_2 Y_1\sqrt{-2D_2}$ and $\alpha\beta = y$, by (7), we get

$$\sqrt{-2D_2} = \alpha^n - \beta^n = \lambda_1\lambda_2 Y_1 \sqrt{-2D_2}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right). \tag{8}$$

Applying Lemma 1, we obtain from (8) that

$$\begin{aligned} 1 &= \lambda_1\lambda_2 Y_1 \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (\alpha - \beta)^{n-2i-1} (\alpha\beta)^i \\ &= \lambda_1\lambda_2 Y_1 \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-2D_2 Y_1^2)^{(n-1)/2-i} y^i. \end{aligned}$$

It follows that $Y_1 = 1$ and

$$\sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-2D_2)^{(n-1)/2-i} y^i = \pm 1. \tag{9}$$

Let $\varepsilon = (X_1\sqrt{D_1} + \sqrt{-D_2})/\sqrt{2}$ and $\overline{\varepsilon} = (X_1\sqrt{D_1} - \sqrt{-D_2})/\sqrt{2}$. For any $m \in \mathbb{N}$ with $2 \nmid m$, let $Y_m = (\varepsilon^m - \overline{\varepsilon}^m)/(\varepsilon - \overline{\varepsilon})$. From (9), we get

$$\frac{\varepsilon^n - \overline{\varepsilon}^n}{\varepsilon - \overline{\varepsilon}} = \pm 1. \tag{10}$$

If n is not a prime, then n has an odd prime factor p and $n = pq$, where $q \in \mathbb{N}$ with $q > 1$ and $2 \nmid q$. By Lemma 1, we get from (10) that

$$Y_p \left(\frac{(\varepsilon^p)^q - (\overline{\varepsilon}^p)^q}{\varepsilon^p - \overline{\varepsilon}^p} \right) = Y_p \sum_{j=0}^{(q-1)/2} \begin{bmatrix} q \\ j \end{bmatrix} (-2D_2Y_p^2)^{(q-1)/2-j} y^{pj} = \pm 1.$$

This implies that $Y_p = \pm 1$ and

$$\frac{\varepsilon^p - \overline{\varepsilon}^p}{\varepsilon - \overline{\varepsilon}} = \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2)^{(p-1)/2-i} y^i = \pm 1. \tag{11}$$

Similarly, by (10), we have

$$Y_q \left(\frac{(\varepsilon^q)^p - (\overline{\varepsilon}^q)^p}{\varepsilon^q - \overline{\varepsilon}^q} \right) = Y_q \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2Y_q^2)^{(p-1)/2-i} y^{qi} = \pm 1,$$

whence we get $Y_q = \pm 1$ and

$$\sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2)^{(p-1)/2-i} y^{qi} = \pm 1. \tag{12}$$

By (12), there exists a suitable $\lambda \in \{-1, 1\}$ such that

$$(-2D_2)^{(p-1)/2} - \lambda \equiv 0 \pmod{y^q}. \tag{13}$$

Since $\gcd\big((-2D_2)^{(p-1)/2}-\lambda,(-2D_2)^{(p-1)/2}+\lambda\big)=1$, we see from (11) that

$$\begin{aligned}&\Big((-2D_2)^{(p-1)/2}-\lambda\Big)+p(-2D_2)^{(p-3)/2}y\\ &+p\binom{p-3}{2}(-2D_2)^{(p-5)/2}y^2\equiv 0 \pmod{y^3}.\end{aligned} \tag{14}$$

Since $q\geq 3$ and $\gcd(y,2D_2)=1$, we find from (13) and (14) that $p(-2D_2+y(p-3)/2)\equiv 0 \pmod{y^2}$, and hence, $y\mid 2D_2$, a contradiction. Thus n must be an odd prime.

If $n=3$, then from (9) we get $-2D_2+3y=\pm 1$. Since $2\nmid D_1D_2y$ and $y\equiv(-1)^{(y-1)/2} \pmod 4$, we see from (6) that D_1,D_2 satisfy (2) and (1) has solutions (4).

If $n=5$, then from (9) we get $4D_2^2-10D_2y+5y^2=\pm 1$. Then we have $(4D_2-5y)^2-5y^2=\pm 4$, and hence,

$$|4D_2-5y|=L_m,\ y=F_m,\ m\in\mathbb{N},\ m>1. \tag{15}$$

Notice that $2\nmid L_mF_m$, $2\parallel F_m$ and $2\parallel L_m$ if and only if $3\nmid m$, $m\equiv 3 \pmod 6$ and $6\mid m$, respectively. We find from (6) and (15) that D_1,D_2 satisfy (3) and (1) has solutions (5). Therefore, if (x,y,n) is a non-trivial solution of (1), then $n\geq 7$.

For any complex number z, we have either $|e^z-1|>1/2$ or $|e^z-1|\geq |z-k\pi\sqrt{-1}\,|/2$ for some $k\in\mathbb{Z}$. Hence, by (10), we get

$$\begin{aligned}\log|\varepsilon-\overline{\varepsilon}|&=\log|\varepsilon^n-\overline{\varepsilon}^n|=n\log|\overline{\varepsilon}|+\log|(\varepsilon/\overline{\varepsilon})^n-1|\\ &\geq n\log|\overline{\varepsilon}|+\log\big|n\log\alpha-k\pi\sqrt{-1}\,\big|-\log 2,\end{aligned} \tag{16}$$

where $k\in\mathbb{Z}$ with $|k|<n$. Let $\Lambda=n\log\alpha-k\pi\sqrt{-1}$. Since $y\geq 3$ and α satisfies

$$y\alpha^2-(D_1X_1^2-D_2)\alpha+y=0, \tag{17}$$

α is not a root of unity, and hence, $\Lambda\neq 0$. Notice that $[\mathbb{Q}(\alpha):\mathbb{Q}]=2$, $0<|\log\alpha|<\pi$ and $h(\alpha)=\log\sqrt{y}$ by (17). Applying Lemma 2, we get

$$\begin{aligned}&\log|\Lambda|\geq -9(12.85\pi+\log\sqrt{y})\\ &\times(\max(17,7.85+\log n(1/(25.7\pi+\log\sqrt{y})+1/25.7\pi)))^2.\end{aligned} \tag{18}$$

If $7.85 + \log 2n/25.7\pi \geq 17$, then $n > 380038$. Further, by (16) and (18), we get

$$\log \sqrt{2y} + 9(12.85\pi + \log \sqrt{y})(7.85 + \log 2n/25.7\pi)^2 > n \log \sqrt{y}.$$

It follows that

$$\begin{aligned} &\log \sqrt{2} + 735.65(7.85 + \log 0.0247712n)^2 \\ &> \log \sqrt{2} + 9\left(\frac{12.85\pi}{\log \sqrt{y}} + 1\right)\left(7.85 + \log \frac{2n}{25.7\pi}\right)^2 > n, \end{aligned}$$

whence we conclude $n < 380000$, a contradiction. So we have $7.85 + \log 2n/25.7\pi < 17$ and

$$\log |\Lambda| \geq -2601(12.85\pi + \log \sqrt{y}). \tag{19}$$

The combination of (16) and (19) yields

$$n \leq 212603. \tag{20}$$

Let

$$f(X, Y) = \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} X^{(n-1)/2-i} Y^i.$$

Notice that n is an odd prime and

$$\begin{bmatrix} n \\ (n-1)/2 \end{bmatrix} = n, \quad n \;\Big|\; \begin{bmatrix} n \\ j \end{bmatrix}, \quad j = 1, \ldots, (n-1)/2.$$

Then $f(X, Y) \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $(n-1)/2$ with

$$H = \max_{i=0,\ldots,(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} = \max_{i=0,\ldots,(n-1)/2} \frac{n}{n-2i} \binom{n-i-1}{i} < 2^{n-1}.$$

We see from (9) that $(X, Y) = (-2D_2, Y)$ is a solution of the equation

$$f(X, Y) = \pm 1, \quad X, Y \in \mathbb{Z}.$$

Therefore, by Lemma 3, if $n \geq 7$, then we have

$$\begin{aligned} (21) \quad y &\leq \max(2D_2, Y) \\ &< \exp\left(3^{3(n+17)/2}\left(\frac{n-1}{2}\right)^{9(n+1)} 2^{(n-1)(n-3)}\big((n-1)\log 2\big)^{n-3}\right). \end{aligned}$$

Substituting (20) into (21), we deduce that $y < \exp\exp\exp 24.17$. The proof is complete.

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