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On some class of hypersurfaces of semi-Euclidean spaces

By CENGIZHAN MURATHAN (Bursa), KADRI ARSLAN (Bursa),
RYSZARD DESZCZ (Wrocław),
RIDVAN EZENTAȘ (Bursa) and CIHAN ÖZGÜR (Bursa)

Dedicated to the 60th birthday
of Professor Dr. Hilmi H. Hacisalihoğlu

Abstract. In the paper we prove that under some additional curvature condition the relations $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of semi-Euclidean spaces. We present also examples of hypersurfaces having the curvature tensor expressed by the square, in the sense of the Kulkarni–Nomizu product, of the Ricci tensor.

1. Introduction

Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞ and let ∇ be its Levi–Civita connection. A manifold (M, g) is said to be semisymmetric ([26]) if

$$R \cdot R = 0 \tag{1}$$

on M. Every locally symmetric manifold ($\nabla R = 0$) is semisymmetric. The converse statement is not true. For precise definitions of the symbols used,

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we refer to Section 2. A semi-Riemannian manifold (M, g) is said to be Ricci-semisymmetric if

$$R \cdot S = 0 \tag{2}$$

on M. Every Ricci-symmetric manifold ($\nabla S = 0$) is Ricci-semisymmetric; the converse statement is not true. Ricci-semisymmetric manifolds were investigated by many authors (e.g., see [1], [9], [10], [23] and [24]). It is clear that every semisymmetric manifold is Ricci-semisymmetric. The converse statement is not true (e.g. see [11]). Although (1) and (2) do not coincide for manifolds in general, it is a long standing question whether the conditions $R \cdot R = 0$ and $R \cdot S = 0$ are equivalent for hypersurfaces of Euclidean spaces (see Problem P 808 of [25] by P. J. Ryan). We can also consider the problem whether (1) and (2) are equivalent for hypersurfaces of a semi-Riemannian space form $N^{n+1}(c)$, and more general, the problem of equivalence of (1) and (2) on semi-Riemannian manifolds, i.e. without the assumption that a given manifold can be realized as a hypersurface in a semi-Riemannian space form. A survey of results related to the above subject is given in [12] (see also [1] and [17]).

In [14] it was shown that if M is a hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, with signature $(s, n+1-s)$, $n \geq 4$, satisfying on $U_C \subset M$ the condition $C \cdot R = LQ(g, C)$ then $C \cdot R = 0$ and $R \cdot R = 0$ hold on U_C. Recently, this result was extended to the case when the ambient space is a semi-Riemannian space of constant curvature ([7]). In this paper we consider semi-Riemannian manifolds (M, g), $n \geq 4$, satisfying at every point the condition

$(*)$ the tensors $C \cdot R$ and $Q(g, R)$ are linearly dependent.

We note that $(*)$ is satisfied on (M, g) if and only if

$$C \cdot R = LQ(g, R) \tag{3}$$

on $U_R = \{x \in M \mid R - \frac{\kappa}{n(n-1)} G \neq 0 \text{ at } x\}$, where L is some function on U_R.

In Section 2 we prove that if $(*)$ is satisfied on (M, g), $n \geq 4$, then (M, g) is a manifold with pseudosymmetric Weyl tensor, i.e. a manifold satisfying at every point the condition ([11], Section 12.6)

$(*)_1$ the tensors $C \cdot C$ and $Q(g, C)$ are linearly dependent.

$(*)_1$ is satisfied on (M, g) if and only if $C \cdot C = L_C Q(g, C)$ on $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$, where L_C is some function on U_C. We note that U_C as well as the set $U_S = \{x \in M \mid S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ are subsets of U_R. We denote by U_κ the subset of M consisting of all points of the set $U = U_C \cap U_S \subset M$ at which the scalar curvature κ is nonzero. Semi-Riemannian manifolds fulfilling $(*)$, $(*)_1$ or other conditions of this kind are called manifolds of pseudosymmetry type. Recently, a review of results on pseudosymmetry type manifolds was given in [13].

It is known that every hypersurface of a semi-Euclidean space satisfies the following relation ([18], Corollary 3.1).

$$R \cdot R = Q(S, R). \tag{4}$$

In Section 3 we consider Ricci-semisymmetric manifolds satisfying $(*)$ and (4). In Theorem 3.1 we present curvature properties of such manifolds. In particular, Theorem 3.1(vi) states that if (M, g), $n \geq 4$, is a Ricci-semisymmetric manifold satisfying at a point $x \in U_\kappa \subset M$ the conditions: $L \neq \pm\frac{\kappa}{n-1}$ and $L \neq \frac{\kappa}{(n-2)(n-1)}$, then $R = \left(2(n-2)L + \frac{2(n-2)}{n-1}\kappa\right)^{-1} S \wedge S$ holds on some neighbourhood V of x. We present an example of a semi-Riemannian manifold realizing assumptions of Theorem 3.1(vi). Namely, we check that the warped product $\overline{M} \times_F \widetilde{M}$, $p = \dim \overline{M} \geq 2$, $n - p = \dim \widetilde{M} \geq 2$, with some warping function F, defined in Example 4.1 of [6], is a manifold realizing the assumptions of Theorem 3.1(vi) (see Example 3.1).

In Section 4 we state that the mentioned above warped product can be realized as a hypersurface of $\mathbb{E}^{n+1}_s$ (see Example 4.2). Further, in this section we prove our main result (Theorem 4.1): *For hypersurfaces of* $\mathbb{E}^{n+1}_s$*,* $n \geq 4$*, satisfying* $(*)$ *the conditions of semisymmetry and Ricci-semisymmetry are equivalent*. Furthermore, we prove that (Theorem 4.2): *Every hypersurface* M *of* $\mathbb{E}^{n+1}_s$*,* $n \geq 4$*, having nilpotent shape operator satisfies the following relation*

$$\kappa R = \frac{1}{2} S \wedge S. \tag{5}$$

We note that the last equality implies (3) (Lemma 2.1(ii)). In Example 4.1 we present some semisymmetric hypersurfaces of semi-Euclidean space satisfying (5). We prove also a converse statement (Theorem 4.3): *If* (5) *holds on the subset* U_κ *of a hypersurface* M *of* $\mathbb{E}_s^{n+1}$, $n \geq 4$, *then on* U_κ *the shape operator* $\mathcal{A}$ *of* M *is nilpotent.*

2. Preliminary results

We define on a semi-Riemannian manifold (M, g), $n \geq 3$, the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,$$

$$\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\right),$$

respectively, where the Ricci operator $\mathcal{S}$ is defined by $S(X,Y) = g(X, \mathcal{S}Y)$, S is the Ricci tensor, κ the scalar curvature, A a symmetric $(0,2)$-tensor and $X, Y, Z \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of vector fields of M. The tensor S^2 is defined by $S^2(X,Y) = S(\mathcal{S}X, Y)$. The tensor G, the Riemann–Christoffel curvature tensor R and the Weyl conformal curvature tensor C of (M,g) by $G(X_1,X_2,X_3,X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$, $R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4)$, $C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4)$, respectively. For a $(0,k)$-tensor T, $k \geq 1$, and a symmetric $(0,2)$-tensor A, we define the $(0,k+2)$-tensors $R \cdot T$ and $Q(A,T)$ by

$$\begin{aligned}
&(R \cdot T)(X_1, \ldots, X_k; X, Y) = (\mathcal{R}(X,Y) \cdot T)(X_1, \ldots, X_k) \\
&\quad = -T(\mathcal{R}(X,Y)X_1, X_2, \ldots, X_k) \\
&\qquad - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{R}(X,Y)X_k),
\end{aligned}$$

$$\begin{aligned}
&Q(A,T)(X_1, \ldots, X_k; X, Y) \\
&\quad = ((X \wedge_A Y) \cdot T)(X_1, \ldots, X_k) = -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\
&\qquad - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k),
\end{aligned}$$

respectively. Putting in the above formulas $T = R$, $T = S$, $T = C$ or $T = G$ and $A = g$ or $A = S$, we obtain the tensors: $R \cdot R$, $R \cdot S$, $R \cdot C$,

$Q(g,R)$, $Q(g,C)$, $Q(S,R)$ and $Q(S,C)$. The tensors $C \cdot R$ and $C \cdot C$ we define in the same manner as the tensor $R \cdot R$; the tensor $C \cdot S$ is defined in the same manner as the tensor $R \cdot S$. For $(0,2)$-tensors A and B we define their Kulkarni–Nomizu product $A \wedge B$ by

$$\begin{aligned}(A \wedge B)(X_1, X_2, X_3, X_4) &= A(X_1, X_4)B(X_2, X_3) + A(X_2, X_3)B(X_1, X_4)\\ &\quad - A(X_1, X_3)B(X_2, X_4) - A(X_2, X_4)B(X_1, X_3).\end{aligned}$$

Lemma 2.1. *Let (M,g), $n \geq 3$, be a semi-Riemannian manifold.*

(i) ([20]) *Let B be a symmetric $(0,2)$-tensor on M and let U_B be a set consisting of all points of M at which B is not proportional to g. If at a point $x \in U_B$ the following relation is satisfied: $B \wedge B = 2\alpha g \wedge B + 2\beta G$, $\alpha, \beta \in \mathbb{R}$, then $\alpha^2 = -\beta$ and $\operatorname{rank}(B - \alpha g) = 1$ at x.*

(ii) ([21]) *If at a point $x \in M$ the following condition is satisfied: $R = \frac{\beta}{2} S \wedge S$, $\beta \in \mathbb{R} - \{0\}$, then $R \cdot R = Q(S,R) = 0$ and $C \cdot R = (\frac{1}{(n-2)\beta} - \frac{\kappa}{n-1})Q(g,R)$ at x. In particular, if $\beta = ((n-2)\tau + \frac{n-2}{n-1}\kappa)^{-1}$ and $\tau + \frac{\kappa}{n-1} \neq 0$ then $C \cdot R = \tau Q(g,R)$ at x.*

Examples of manifolds fulfilling $R = \frac{\beta}{2} S \wedge S$ are present in [21] (see also Example 4.1).

Let (M,g), $n \geq 4$, be a semi-Riemannian manifold. We denote by G_{hijk}, C_{hijk} and $(C \cdot R)_{hijklm}$ the local components of the tensors G, C and $C \cdot R$, respectively. Thus, by definition, we have the following relations: $G_{hijk} = g_{hk}g_{ij} - g_{hj}g_{ik}$,

$$\begin{aligned}C_{hijk} &= R_{hijk} - \frac{1}{n-2}(g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} - g_{ik}S_{hj})\\ &\quad + \frac{\kappa}{(n-1)(n-2)}G_{hijk},\end{aligned} \tag{6}$$

$$\begin{aligned}(C \cdot R)_{hijklm} &= g^{rs}(R_{rijk}C_{shlm} + R_{hrjk}C_{silm} + R_{hirk}C_{sjlm}\\ &\quad + R_{hijr}C_{sklm}),\end{aligned} \tag{7}$$

respectively. We have also the following identities:

$$\begin{aligned}g^{rs}Q(g,R)_{hrsklm} &= g_{hl}S_{km} + g_{kl}S_{hm} - g_{hm}S_{kl} - g_{km}S_{hl}\\ &= Q(g,S)_{hklm},\end{aligned} \tag{8}$$

$$g^{rs}Q(g,R)_{rijkls} = g_{kl}S_{ij} - g_{jl}S_{ik} - (n-1)R_{lijk}, \tag{9}$$

$$g^{rs}Q(S,R)_{hrsklm} = A_{lkhm} - A_{lhmk} - A_{mkhl} + A_{mhlk}, \tag{10}$$

$$\begin{aligned} g^{rs}Q(S,R)_{rijkls} = A_{lijk} - A_{iljk} - A_{jilk} - A_{kijl} \\ - \kappa R_{lijk} + S_{kl}S_{ij} - S_{jl}S_{ik}, \end{aligned} \tag{11}$$

$$A_{mijk} = S_{mr}g^{rs}R_{sijk} = S_m^{\ s}R_{sijk}, \quad S_m^{\ p} = g^{rp}S_{mr}. \tag{12}$$

Applying (6) in (7) we get

$$\begin{aligned} (C\cdot R)_{hijklm} &= (R\cdot R)_{hijklm} - \frac{1}{n-2}Q(S,R)_{hijklm} \\ &+ \frac{\kappa}{(n-1)(n-2)}Q(g,R)_{hijklm} \\ &- \frac{1}{n-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}), \end{aligned} \tag{13}$$

where $(R\cdot R)_{hijklm}$, $Q(S,R)_{hijklm}$ and $Q(g,R)_{hijklm}$ are the local components of the tensors $R\cdot R$, $Q(S,R)$ and $Q(g,R)$, respectively.

If (3) is satisfied on the set $U_R \subset M$ then (13) turns into

$$\begin{aligned} (R\cdot R)_{hijklm} &= \frac{1}{n-2}Q(S,R)_{hijklm} \\ &+ \left(L - \frac{\kappa}{(n-1)(n-2)}\right)Q(g,R)_{hijklm} \\ &+ \frac{1}{n-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}). \end{aligned} \tag{14}$$

Contracting this with g^{ij} and using (8) and (10) we obtain

$$R\cdot S = Q(g,D), \tag{15}$$

$$D = \frac{1}{n-2}S^2 + \mu S, \quad \mu = L - \frac{\kappa}{(n-1)(n-2)}. \tag{16}$$

We note that we can present (15) in the form

$$(R\cdot S)_{hijk} = A_{hijk} + A_{ihjk} = g_{hj}D_{ik} + g_{ij}D_{hk} - g_{hk}D_{ij} - g_{ik}D_{hj}. \tag{17}$$

Contracting this with g^{ij} we get

$$B_{km} = -\frac{2}{n-2} S^2_{km} - n\mu S_{km} + \rho g_{km}, \quad \rho = \frac{\mathrm{tr}(S^2)}{n-2} + \kappa\mu, \tag{18}$$

where $B_{km} = S^{rs} R_{rkms}$. Further, summing (17) cyclically in h, j, k, we obtain

$$A_{hijk} + A_{jikh} + A_{kihj} = 0. \tag{19}$$

Now (10) and (11) reduce to

$$g^{rs} Q(S,R)_{hrsklm} = A_{hklm} + A_{khlm}, \tag{20}$$

$$g^{rs} Q(S,R)_{rijkls} = -A_{iljk} - \kappa R_{lijk} + S_{kl} S_{ij} - S_{jl} S_{ik}. \tag{21}$$

Proposition 2.1. *If the condition $C \cdot R = LQ(g,R)$ is satisfied on the subset $U_C \subset M$ of a semi-Riemannian manifold (M,g), $n \geq 4$, then $C \cdot C = LQ(g,C)$ on U_C.*

Proof. Contracting $(C \cdot R)_{hijklm} = LQ(g,R)_{hijklm}$ with g^{ij} and using (6)–(8), we find

$$(C \cdot S)_{hklm} = LQ(g,S)_{hklm}. \tag{22}$$

Further, applying (6) and (22) in the identity

$$(C \cdot C)_{hijklm} = g^{rs}(C_{rijk} C_{shlm} + C_{hrjk} C_{silm} + C_{hirk} C_{sjlm} + C_{hijr} C_{sklm})$$

we obtain

$$(C \cdot C)_{hijklm} = (C \cdot R)_{hijklm} - \frac{1}{n-2}\Big(g_{hk}(C \cdot S)_{ijlm} + g_{ij}(C \cdot S)_{hklm} - g_{hj}(C \cdot S)_{iklm} - g_{ik}(C \cdot S)_{hjlm}\Big),$$

which by making use of (3) and (22), after straightforward calculations, turns into

$$(C \cdot C)_{hijklm} = LQ(g,C)_{hijklm},$$

which completes the proof our proposition. □

From the above result it follows immediately the following

Theorem 2.1. *A semi-Riemannian manifold (M,g), $n \geq 4$, fulfilling $(*)$ is a manifold with pseudosymmetric Weyl tensor.*

3. Ricci-semisymmetric manifolds fulfilling (3) and (4)

Let (3) and (4) be satisfied on the subset U_R of a semi-Riemannian manifold (M, g), $n \geq 4$. Now, by our assumptions, (14) turns on U_R into

$$\begin{aligned}(23)\qquad \frac{n-3}{n-2}Q(S,R)_{hijklm} &= \mu Q(g,R)_{hijklm} \\ &+ \frac{1}{n-2}(g_{hl}A_{mijk} - g_{hm}A_{lijk} - g_{il}A_{mhjk} + g_{im}A_{lhjk} \\ &\quad + g_{jl}A_{mkhi} - g_{jm}A_{lkhi} - g_{kl}A_{mjhi} + g_{km}A_{ljhi}).\end{aligned}$$

Contracting this with g^{hm} and using (9), (17) and (21) we obtain

$$\begin{aligned}(24)\qquad 2A_{lijk} &= -\frac{n-3}{2(n-2)}(S \wedge S)_{lijk} + \left(\frac{n-3}{n-2}\kappa - (n-1)\mu\right) R_{lijk} \\ &+ \mu(g_{kl}S_{ij} - g_{jl}S_{ik}) + \frac{1}{n-2}(g_{kl}B_{ij} - g_{jl}B_{ik}) \\ &+ \frac{n-3}{n-2}(g_{hj}D_{ik} + g_{ij}D_{hk} - g_{hk}D_{ij} - g_{ik}D_{hj}).\end{aligned}$$

We assume now that the hypersurface M is Ricci-semisymmetric. We restrict our considerations to the set $U_S \subset M$. Now (17) yields

$$(25)\qquad (R \cdot S)_{hijk} = A_{hijk} + A_{ihjk} = 0,$$

which, by contraction with g^{hk}, gives $B = S^2$. Applying this in (18) we obtain

$$(26)\qquad \frac{1}{n-2}S^2 + \mu S = \frac{1}{n}\rho g,$$

$$(27)\qquad \frac{1}{n-2}B + \mu S = \frac{1}{n}\rho g.$$

By (26), (16) turns into

$$(28)\qquad D = \frac{1}{n}\rho g.$$

By making use of (27) and (28), (24) reduces to

$$2A_{lijk} = -\frac{n-3}{2(n-2)}(S \wedge S)_{lijk} + \left(\frac{n-3}{n-2}\kappa - (n-1)\mu\right) R_{lijk} + \frac{1}{n}\rho G_{lijk}. \tag{29}$$

Transvecting (29) with $S_m^{\ l}$ and symmetrizing the resulting equality in j, m and applying (25) we find

$$\frac{n-3}{2(n-2)}Q(S,S^2) = \rho Q(g,S). \tag{30}$$

On the other hand, (26) implies

$$-\frac{n}{n-2}Q(S,S^2) = \rho Q(g,S). \tag{31}$$

Further, (30) and (31) yield $\rho Q(g,S) = 0$, whence, in view of Lemma 2.4 (i) of [18], it follows that $\rho = 0$ on U_S. Thus (29) turns into

$$2A_{lijk} = -\frac{n-3}{2(n-2)}(S \wedge S)_{lijk} + (\kappa - (n-1)L)R_{lijk}. \tag{32}$$

In addition, (26) reduces to

$$S^2 = -(n-2)\mu S, \tag{33}$$

Next, transvecting (32) with $S_m^{\ l}$ and using (33) we get

$$(-2(n-2)\mu - \kappa + (n-1)L)A_{lijk} = (n-3)\frac{\mu}{2}(S \wedge S)_{lijk}. \tag{34}$$

We finish this section with the following

Theorem 3.1. *Let the conditions: $C \cdot R = LQ(g,R)$, $R \cdot R = Q(S,R)$ and $R \cdot S = 0$ be satisfied on the subset $U = U_C \cap U_S \subset M$ of a semi-Riemannian manifold (M,g), $n \geq 4$, and let x be a point of U.*

(i) *If μ and κ vanish at x then at x we have: $C \cdot R = 0$ and*

$$Q\left(S,\ R - \frac{1}{2(n-2)} g \wedge S\right) = 0. \tag{35}$$

(ii) *If* $\mu = 0$ *and* $\kappa \neq 0$ *at* x *then at* x *we have:* $\kappa R = \frac{1}{2} S \wedge S$ *and*

$$R \cdot R = Q(S, R) = 0. \tag{36}$$

(iii) *If* $\mu \neq 0$ *and* $\kappa = 0$ *at* x *then at* x *we have:*

$$Q\left(S - \frac{L}{2} g,\ R - \frac{1}{2(n-2)L} S \wedge S\right) = 0. \tag{37}$$

(iv) *If* $\mu \neq 0$, $\kappa \neq 0$ *and* $L - \frac{\kappa}{n-1} = 0$ *at* x *then at* x *we have:*

$$Q\left(S - \frac{\kappa}{n-1} g,\ R - \frac{n-1}{4(n-2)} S \wedge S\right) = 0. \tag{38}$$

(v) *If* $\mu \neq 0$, $\kappa \neq 0$, $L - \frac{\kappa}{n-1} \neq 0$ *and* $L + \frac{\kappa}{n-1} = 0$ *at* x *then at* x *we have:* $\operatorname{rank} S = 1$, $A = \kappa R$ *and* (36).

(vi) *If* $\mu \neq 0$, $\kappa \neq 0$, $L - \frac{\kappa}{n-1} \neq 0$ *and* $L + \frac{\kappa}{n-1} \neq 0$ *at* x *then at* x *we have:* (36) *and*

$$R = \left(2(n-2)L + \frac{2(n-2)}{n-1}\kappa\right)^{-1} S \wedge S. \tag{39}$$

Proof. (i) If μ and κ vanish at x then $L = 0$ and $C \cdot R = 0$ at x. From (32) we get $A = -\frac{n-3}{4(n-2)} S \wedge S$. Now (23) turns into $Q(S, R) = -\frac{1}{4(n-2)} Q(g, S \wedge S)$, which, by making use of the equality (16) of [3], completes the proof in the first case.

(ii) By our assumptions (34) reduces to $(L - \frac{\kappa}{n-1})A = 0$, whence it follows that A vanishes at x. Now (23) and (32) complete the proof of (ii).

(iii) (32), by the assumption $\kappa = 0$, takes the form $A = -\frac{n-3}{4(n-2)} S \wedge S - \frac{(n-1)L}{2} R$. Next, applying the above relations in (23) we get (37).

(iv) We note that (34) reduces to $A = -\frac{n-3}{4(n-2)} S \wedge S$. Applying this in (23) we obtain (38).

(v) (32), by multiplication by μ and making use of (34), yields

$$((n-1)L - \kappa)(A + (n-2)\mu R) = 0, \tag{40}$$

$$A = -(n-2)\mu R. \tag{41}$$

Evidently, from (41) we obtain $A = \kappa R$. Applying this in (32) we get $S \wedge S = 0$, whence $\operatorname{rank} S = 1$. In addition, from (23) it follows that (36) holds at x.

(vi) We note that (32), by making use of (41), turns into (39). Now Lemma 2.1(ii) completes the proof in the last case. Our theorem is thus proved. $\square$

Remark 3.1. From (35) and (38), by making use of results of [16] (Lemma 3.4, Theorem 4.2), we can obtain more information about the curvature tensor R of the semi-Riemannian manifolds considered in the last theorem.

Example 3.1. Let $\overline{M}$ be a nonempty open connected subset of $\mathbb{R}^p$, $p \geq 2$, equipped with the standard metric $\overline{g}$, $\overline{g}_{ab} = \varepsilon_a \delta_{ab}$, $\varepsilon_a = \pm 1$, where $a, b \in \{1, \dots, p\}$. We put $F = F(x^1, \dots, x^p) = k \exp(\xi_a x^a)$, where $k, \xi_1, \dots, \xi_p \in \mathbb{R}$, $\xi_1^2 + \dots + \xi_p^2 > 0$ and $k > 0$. Further, let $\widetilde{N}$ be a nonempty open connected subset of $\mathbb{R}^{n-p}$, $n \geq 4$, equipped with the standard metric $\widetilde{g}$, $\widetilde{g}_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}$, $\varepsilon_\alpha = \pm 1$, where $\alpha, \beta \in \{p+1, \dots, n\}$. We consider the warped product $\overline{M} \times_F \widetilde{N}$ of the manifolds $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ with the warping function F defined above. This warped product satisfies the following relations ([6], Example 4.1):

$$S_{ab} = -\frac{n-p}{4}\xi_a \xi_b, \quad S_{\alpha\beta} = \left(-\frac{\operatorname{tr} T}{2} - \frac{n-p-1}{4F}\Delta_1 F\right)\widetilde{g}_{\alpha\beta},$$

$$T_{ab} = \frac{F}{2}\xi_a \xi_b, \tag{42}$$

$$\Delta_1 F = F^2 \xi^f \xi_f, \quad \operatorname{tr} T = \frac{F}{2}\xi^f \xi_f, \quad \kappa = -\frac{(n-p)(n-p+1)}{4}\xi^f \xi_f,$$

where $\xi^f = \overline{g}^{ef}\xi_e$. We note that if $p = 1$ then the warped product $\overline{M} \times_F \widetilde{N}$ is a conformally flat manifold ([6], Example 4.1). Therefore we consider only the case: $p \geq 2$. In that case the warped product $\overline{M} \times_F \widetilde{N}$ is a non conformally flat semisymmetric manifold satisfying (4) ([6], Example 4.1). Further, using the above relations we can easily check that $\overline{M} \times_F \widetilde{N}$ satisfies

$$\kappa R = \frac{n-p+1}{2(n-p)} S \wedge S. \tag{43}$$

By Lemma 2.1(ii), we have also on $\overline{M} \times_F \widetilde{N}$ the following relation

$$C \cdot R = -\frac{(p-2)\kappa}{(n-2)(n-1)(n-p+1)} Q(g, R). \tag{44}$$

It is clear that there exist the constants ε, ε_a, and ξ_a such that $\xi^f \xi_f$ is nonzero. Now the scalar curvature κ of the manifold $\overline{M} \times_F \widetilde{N}$ is nonzero. Thus we see that the assumptions of Theorem 3.1(vi) are fulfilled.

4. Hypersurfaces satisfying (5)

Let M, $n = \dim M \geq 3$, be a connected hypersurface immersed isometrically in a semi-Riemannian manifold $(N, \widetilde{g})$. We denote by g the metric tensor of M, induced from the metric tensor $\widetilde{g}$. Further, we denote by $\widetilde{\nabla}$ and ∇ the Levi–Civita connections corresponding to the metric tensors $\widetilde{g}$ and g, respectively. Let ξ be a local unit normal vector field on M in N and let $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$. We can present the Gauss formula and the Weingarten formula of M in N by: $\widetilde{\nabla}_X Y = \nabla_X Y + \varepsilon H(X, Y)\xi$ and $\widetilde{\nabla}_X \xi = -\mathcal{A}(X)$, respectively, where X, Y are vector fields tangent to M, H is the second fundamental tensor of M in N, $\mathcal{A}$ is the shape operator of M in N and $H^k(X, Y) = g(\mathcal{A}^k(X), Y)$ and $\mathrm{tr}(H^k) = \mathrm{tr}(\mathcal{A}^k)$, where $k \geq 1$. We assume that the ambient space is a semi-Euclidean space $\mathbb{E}^{n+1}_s$, $n \geq 3$. The Gauss equation of M in $\mathbb{E}^{n+1}_s$ we can present in the form $R = \frac{\varepsilon}{2} H \wedge H$, where R is the curvature tensor of M. Let the equations $x^r = x^r(y^h)$ be the local parametric expression of M in $\mathbb{E}^{n+1}_s$, where y^r and x^r are the local coordinates of M and $\mathbb{E}^{n+1}_s$, respectively, and $h, i, j, k, l \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, n+1\}$. Thus we have

$$R_{hijk} = \varepsilon(H_{hk}H_{ij} - H_{hj}H_{ik}), \quad \varepsilon = \pm 1, \tag{45}$$

where R_{hijk} and H_{hk} are the local components of the tensors R and H, respectively. Contracting (45) with g^{ij} and g^{hk} we obtain

$$S_{hk} = \varepsilon(\mathrm{tr}(H)H_{hk} - H^2_{hk}), \tag{46}$$

$$\kappa = \varepsilon((\mathrm{tr}(H))^2 - \mathrm{tr}(H^2)), \tag{47}$$

respectively, where $H^2_{hk} = g^{ij}H_{ih}H_{jk}$. From (46), by transvection with ${S_l}^h$, we obtain

$$S^2_{lk} = H^4_{lk} - 2\,\mathrm{tr}(H)H^3_{lk} + (\mathrm{tr}(H))^2 H^2_{lk}, \tag{48}$$

where $H^3_{hk} = g^{ij} H^2_{ih} H_{jk}$ and $H^4_{hk} = g^{ij} H^3_{ih} H_{jk}$. It is well known that on every hypersurface M of a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, the following condition is satisfied ([18]): $R \cdot R - Q(S,R) = -\frac{(n-2)\widetilde{\kappa}}{n(n+1)} Q(g,C)$, where $\widetilde{\kappa}$ is the scalar curvature of the ambient space. Evidently, when the ambient space is $\mathbb{E}^{n+1}_s$, $n \geq 4$, the last relation reduces to (4).

Proposition 4.1. *If the conditions: $C \cdot R = LQ(g,R)$ and $R \cdot S = 0$ are satisfied on the subset U_C of a hypersurface M of $\mathbb{E}^{n+1}_s$, $n \geq 4$, then $R \cdot R = 0$ on U_C.*

Proof. From Theorem 2.1 it follows that $C \cdot C = LQ(g,C)$ on U_C. Now, in view of Theorem 4.1 of [8], we obtain (1) on U_C, which completes the proof. □

Theorem 4.1. *For every hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 4$, which satisfy $(*)$ the conditions of semisymmetry and Ricci-semisymmetry are equivalent.*

Proof. (1) and (2) are equivalent on the set $M - U_C$ ([15], Lemma 2). Proposition 4.1 yields the equivalence of both conditions on U_C. Our theorem is thus proved. □

Example 4.1 ([14], Examples 4.1 and 5.1). Let $(\widetilde{N}, \widetilde{g})$, be a 1-dimensional Riemannian manifold. Let $\overline{M}$ be a nonempty open connected subset of $\mathbb{R}^p$, $p = n-1 \geq 3$, equipped with the standard metric $\overline{g}$, $\overline{g}_{ab} = \varepsilon_a \delta_{ab}$, $\varepsilon_a = \pm 1$. We put $F = F(x^1, \ldots, x^p) = k \exp(\xi_a x^a)$, where $\xi_1, \ldots, \xi_p$ and k are constants such that $\xi_1^2 + \cdots + \xi_p^2 > 0$, $\overline{g}^{ab} \xi_a \xi_b = 0$ and $k > 0$. We consider the warped product $\bar{M} \times_F \widetilde{N}$. This warped product satisfies the following relations:

$$
\begin{gathered}
R_{abcd} = 0, \quad R_{nabn} = -\frac{F}{4} \xi_a \xi_b \tilde{g}_{nn}, \quad T_{ab} = \frac{F}{2} \xi_a \xi_b, \quad \operatorname{tr}(T) = 0, \\
\Delta_1 F = 0, \quad S_{ab} = -\frac{1}{4} \xi_a \xi_b, \quad S_{nn} = 0, \quad S^2 = 0 \quad \kappa = 0,
\end{gathered}
\tag{49}
$$

respectively. Using the above formulas we can check that the manifold $\overline{M} \times_F \widetilde{N}$ satisfies the following conditions: $\operatorname{rank} S = 1$, $\kappa = 0$, $S^2 = 0$, $R \cdot R = 0$ and $C(\mathcal{S}X_1, X_2, X_3, X_4) = 0$, for any vector fields $X_1, \ldots, X_4$ on M. These relations imply $R(\mathcal{S}X_1, X_2, X_3, X_4) = 0$, i.e. the tensor A

with the local components A_{hijk} defined by (12), is a zero tensor. It is clear that (5) trivially is fulfilled on M. Let H be the $(0,2)$-tensor on $\overline{M} \times_F \widetilde{N}$, with the local components H_{ij}, defined by $H_{ab} = -\frac{\varepsilon}{4l}\sqrt{F}\xi_a\xi_b$, $H_{an} = 0$, $H_{nn} = l\sqrt{F}\widetilde{g}_{nn}$, where $\varepsilon = \pm 1$ and l is a positive constant. The tensor H fulfils (45) and $H^3 = \mathrm{tr}(H)H^2$. Furthermore, we can check that H is a Codazzi tensor. Thus we see that the manifold $\overline{M} \times_F \widetilde{N}$ can be realized as a hypersurface immersed isometrically in $\mathbb{E}_s^{n+1}$. Therefore on M we have: $R \cdot R = Q(S,R) = 0$. Now (13) reduces on M to $C \cdot R = 0$. In addition, the tensor H^2, with the local components H^2_{hk}, of the hypersurface M is a nonzero tensor. We note also that (M, g) satisfies $S \cdot R = 0$. Hypersurfaces satisfying the last condition were investigated in [2] and [3].

We note that, in virtue of (45)–(47), (5) is equivalent to

$$
\begin{aligned}
(50)\qquad & H^2_{hk}H^2_{ij} - H^2_{hj}H^2_{ik} + \mathrm{tr}(H^2)(H_{hk}H_{ij} - H_{hj}H_{ik}) \\
&= \mathrm{tr}(H)(H_{hk}H^2_{ij} + H_{ij}H^2_{hk} - H_{hj}H^2_{ik} - H_{ik}H^2_{hj}).
\end{aligned}
$$

As an immediate consequence of the above remark we have the following

Theorem 4.2. *Every hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$, having nilpotent shape operator $\mathcal{A}$, satisfies the relation $\kappa R = \frac{1}{2} S \wedge S$.*

Examples of hypersurface of $\mathbb{E}_s^{n+1}$, having nilpotent shape operator are presented in [22].

Proposition 4.2 (cf. [19], Lemma 1)**.** *Let M be a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$. If at a point $x \in U_R \subset M$ the second fundamental tensor H of M satisfies the condition*

$$(51)\qquad H^2 = \alpha H + \beta g, \quad \alpha, \beta \in \mathbb{R},$$

then $R \cdot S = -\varepsilon\beta Q(g,S)$ at x.

Further, as an immediate consequence of Proposition 3.1 of [5], we have

Proposition 4.3. *Let M be a Ricci-semisymmetric hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$. Then at every point $x \in U_S \subset M$ the following relation is satisfied*

$$(52)\qquad H^3 = \mathrm{tr}(H)H^2 + \lambda H, \quad \lambda \in \mathbb{R}.$$

Proposition 4.4. *Let M be a Ricci-semisymmetric hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$. Then at every point $x \in U_S \subset M$ we have:* (52) *and the tensor H^2 is not a linear combination of the tensors H and g, or $H^2 = \alpha H$, $\alpha \neq \operatorname{tr}(H)$, $\alpha \in \mathbb{R}$.*

Proof. We suppose that (51) is satisfied at a point $x \in U_S$. From the fact that the tensor $R \cdot S$ vanishes at x it follows that $\beta = 0$ at x. Thus (51) reduces to $H^2 = \alpha H$. In addition, if $\alpha = \operatorname{tr}(H)$ then $H^2 = \operatorname{tr}(H) H$ and, by (46), $S = 0$, a contradiction. Thus $\alpha \neq \operatorname{tr}(H)$ at x. The last remark completes the proof. □

Let now M be a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$, satisfying (5). First of all we note that (5) holds on $M - U_R$ if and only if its curvature tensor R vanishes on $M - U_R$. Similarly, (5) holds on $M - U_S$ if and only if its Ricci tensor S vanishes on $M - U_S$. Let x be a point of the set $M - U_C$. The identity

$$\kappa C = \kappa R - \frac{\kappa}{n-2} g \wedge S + \frac{\kappa^2}{(n-1)(n-2)} G,$$

by making use of (5) and $C = 0$, turns into

$$S \wedge S = \frac{2\kappa}{n-2} g \wedge S + \frac{2\kappa^2}{(n-1)(n-2)} G.$$

Applying to this Lemma 2.1(i) we get $\frac{4\kappa^2}{(n-2)^2} = \frac{4\kappa^2}{(n-1)(n-2)}$ and $\operatorname{rank}(S - \frac{\kappa}{n-2} g) = 1$, whence it follows that $\kappa = 0$ and $\operatorname{rank} S = 1$ on $M - U_C$. Let now $x \in U_R \subset M$. We note that that from (5) it follows immediately that κ vanishes at x if and only if $\operatorname{rank} S \leq 1$ at x. We recall that in Example 4.1 it was stated that there exist hypersurfaces of a semi-Euclidean space satisfying the conditions: $\kappa = 0$, $\operatorname{rank} S = 1$ and $H^2 \neq 0$. Thus we see, that with respect to the above comments, we can restrict our considerations to the set U_κ consisting of all points of the set $U = U_S \cap U_C \subset M$ at which the scalar curvature κ of M is nonzero. Evidently, from (5) it follows that $R \cdot S = 0$, $S^2 = 0$, and $A = 0$ on U, where the tensor A is defined by (12). The last two relations, by (45)–(48), turn into

$$H^4 = 2 \operatorname{tr}(H) H^3 - (\operatorname{tr}(H))^2 H^2, \tag{53}$$

$$\operatorname{tr}(H)(H^2_{hk} H_{ij} - H^2_{hj} H_{ik}) = H^3_{hk} H_{ij} - H^3_{hj} H_{ik}, \tag{54}$$

respectively. We recall that a hypersurface M is quasiumbilical at a point $x \in M$ if its tensor H has at this point the following decomposition: $H = \alpha g + \beta w \otimes w$, where $w \in T_x^*(M)$ and $\alpha, \beta \in \mathbb{R}$. By the main result of [18], M is quasiumbilical at x if and only if its Weyl conformal curvature tensor C vanishes at this point. Thus we see that U has no quasiumbilical points. Since (U, g) is a Ricci-semisymmetric hypersurface, from Proposition 4.4 it follows that at every point $x \in U$ at least one of the following two equations must be fulfilled: $H^2 = \alpha H$, $\alpha \neq \operatorname{tr}(H)$, or $H^3 = \operatorname{tr}(H) H^2 + \lambda H$, $\lambda \in \mathbb{R}$, and the tensor H^2 is not a linear combination of the tensors H and g. In the first case, (53) turns into $\alpha(\alpha^2 - 2\operatorname{tr}(H)\alpha + (\operatorname{tr}(H)^2)H) = 0$, whence $\alpha(\alpha - \operatorname{tr}(H)) = 0$. Since $\alpha - \operatorname{tr}(H) \neq 0$, the last relation reduces to $\alpha = 0$, i.e. $H^2 = 0$. We consider now the second case. First of all, we note that (54) reduces to $\lambda(H_{ij} H^2_{hk} - H_{ik} H^2_{hj}) = 0$, whence, by symmetrization in h, i we obtain $\lambda Q(H, H^2) = 0$. This, in view of Lemma 2.4(i) of [18], implies $\lambda = 0$. Thus we have at x: $H^3 = \operatorname{tr}(H) H^2$. Transvecting now (50) with H_l^h and using the last relation we obtain $(\operatorname{tr}(H^2) - (\operatorname{tr}(H))^2)(H_{ij} H^2_{lk} - H_{ik} H^2_{lj}) = 0$. Symmetrizing this in l, i we obtain $(\operatorname{tr}(H^2) - (\operatorname{tr}(H))^2) Q(H, H^2) = 0$, which, in view of Lemma 2.4(i) of [18] and (47), implies $\kappa = 0$, a contradiction. Thus we have proved the following

Theorem 4.3. *If M is a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$, such that $\kappa R = \frac{1}{2} S \wedge S$ is fulfilled on the set $U_\kappa \subset M$ then on U_κ the shape operator $\mathcal{A}$ of M is nilpotent.*

Example 4.2 ([6], Example 4.1). Let $\overline{M} \times_F \widetilde{N}$, $p = \dim \overline{M} \geq 2$, $n - p = \dim \widetilde{M}$, $n \geq 4$, be the warped product defined in Example 3.1. Let τ be a function on $\overline{M} \times_F \widetilde{N}$ such that $\tau^2 = -\frac{\varepsilon}{4} \xi^f \xi_f$, $\varepsilon = \pm 1$, on $\overline{M} \times_F \widetilde{N}$. It is clear that there exist constants ε, ε_a, and ξ_a such that the function τ is nonzero at every point x of $\overline{M} \times_F \widetilde{N}$ and the right-hand side of the last relation is positive at every point x. Further, let H be the $(0,2)$-tensor on $\overline{M} \times_F \widetilde{N}$, with local components H_{ij}, defined by $H_{ab} = -\frac{1}{4\tau} \xi_a \xi_b$, $H_{a\alpha} = 0$, $H_{\alpha\beta} = \varepsilon \tau g_{\alpha\beta}$. We can check that the following relations are satisfied on $\overline{M} \times_F \widetilde{N}$: $\nabla_X H(Y, Z) = \nabla_Y H(X, Z)$ and $R(X_1, X_2, X_3, X_4) = \frac{\varepsilon}{2}(H \wedge H)(X_1, X_2, X_3, X_4)$, where $X, Y, Z, X_1, \ldots, X_4$ are vectors fields on $\overline{M} \times_F \widetilde{N}$. Thus we see that the manifold $\overline{M} \times_F \widetilde{N}$ can be realized as a hypersurface of a semi-Euclidean space $\mathbb{E}_s^{n+1}$ ([6] (Example 4.1). Clearly, on such hypersurface (43) is satisfied. Since $p \leq n - 2 < n - 1$, (43) cannot reduce to (5). Thus the constructed above hypersurface do not satisfy the assumptions of Theorem 4.3.

Example 4.3. The product manifold of the p-dimensional standard sphere S^p, $p \geq 2$, and a $(n-p)$-dimensional Euclidean space $\mathbb{E}^{n-p}$, $n \geq 4$, is a semisymmetric manifold satisfying the following condition ([21]): $\kappa R = \frac{p(p-1)}{2} S \wedge S$. It is known that this product manifold can be realized as a hypersurface of a $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$.

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RYSZARD DESZCZ
DEPARTMENT OF MATHEMATICS
AGRICULTURAL UNIVERSITY OF WROCŁAW
UL. GRUNWALDZKA 53
PL – 50–357 WROCŁAW
POLAND

E-mail: rysz@ozi.ar.wroc.pl

CENGIZHAN MURATHAN, KADRI ARSLAN,
RIDVAN EZENTAŞ, CIHAN ÖZGÜR
DEPARTMENT OF MATHEMATICS
ULUDAĞ UNIVERSITY
CAMPUS OF GÖRÜKLE
16059 BURSA
TURKEY

E-mail: arslan@uludag.edu.tr

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