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# Strict complete regularity in the categories of bitopological spaces and ordered topological spaces

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**Abstract.** We introduce the notion of a strictly completely regular bitopological space and show that the category of strictly completely regular bitopological spaces is isomorphic to the category of strictly completely regular ordered spaces. It is shown by means of examples that the category of strictly completely regular bitopological spaces is more flexible than its counterpart in ordered spaces. We also determine quasi-uniformities that induce strictly completely regular bitopologies.

## 1. Introduction

The notion of a strictly completely regular ordered space was introduced in [7]. It was investigated in detail in [4], where an example of a completely regular ordered space which is not a strictly completely regular ordered space was given. In this paper we introduce the notion of strict complete regularity in bitopological spaces and investigate properties of this notion.

For ordered spaces we refer the reader to [1] and [11], for bitopological spaces to [3], [6] and [15], for quasi-uniformities we refer the reader to Fletcher and Lindgren [1].

We shall also present examples to show that complete regularity and strict complete regularity in bitopological spaces do not coincide. This distinction is also emphasized in the last section where we study quasi-uniformities which induce strictly completely regular bitopologies.

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## 2. Strictly completely regular bitopological spaces

Let 2Top be the category of bitopological spaces and bicontinuous functions, CR2Top be the category of completely regular bitopological spaces and bicontinuous functions, CRTopOrd be the category of completely regular ordered spaces and continuous order-preserving functions. For an ordered space $(X, \tau, \leq)$, there is a naturally associated bitopological space $(X, \tau^\sharp, \tau^\flat)$ where $\tau^\sharp$ consists of the open upper sets and $\tau^\flat$ consists of the open lower sets.

We shall use $\mathbb{R}_0$ to denote the set of real numbers with the usual topology and the usual order. For a subset $A$ of the set of real numbers, we use $\mathbb{A}_0$ to denote $\mathbb{A}$ as an ordered subspace of $\mathbb{R}_0$ e.g. $\mathbb{I}_0$ denotes the closed interval with the usual topology and the usual order. A mapping $f : (X, \tau, \leq) \to (Y, \tau', \leq')$ between two ordered topological spaces $(X, \tau, \leq)$ and $(Y, \tau', \leq')$ is said to be *order-preserving* (*order-reversing*) if $f(x) \leq' f(y)$ $(f(y) \leq' f(x))$ whenever $x, y \in X$ and $x \leq y$, and is continuous if it is continuous with respect to the given topologies. Let $A = \{x \in X : f(x) \leq 0\}$ where $f : (X, \tau, \leq) \to \mathbb{R}_0$ is a continuous order-preserving function. In [10] sets of this form are called the decreasing zero sets, and sets of the form $B = \{x \in X : f(x) \geq 0\}$ where $f : (X, \tau, \leq) \to \mathbb{R}_0$ is continuous order-preserving function are called the increasing zero sets. Let $\mathcal{A}_0$ denote the collection of the decreasing zero sets and $\mathcal{B}_0$ denote the collection of the increasing zero sets. In [16] the functor $I$ : CRTopOrd $\to$ CR2Top given by $I(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ where $\tau_{\mathcal{A}_0}$ is the topology having $\mathcal{A}_0$ as a base for closed sets, and $\tau_{\mathcal{B}_0}$ is the topology having $\mathcal{B}_0$ as a base for closed sets was discussed.

We now consider the functors $M$ : CR2Top $\to$ CRTopOrd, and $S$ : CRTopOrd $\to$ 2Top, given by $M(X, \tau_1, \tau_2) = (X, \tau_1 \vee \tau_2, \leq_{\tau_1})$, where $x \leq_{\tau_1} y \Leftrightarrow x \in \overline{\{y\}}^{\tau_1}$ and $S(X, \tau, \leq) = (X, \tau^\sharp, \tau^\flat)$ respectively. Example 6 in [4] of a completely regular ordered space which is not a strictly completely regular ordered space shows that the image of $S$ is not necessarily the category CR2Top. It was shown in [16] and [17] that the functor $M$ maps into CRTopOrd. In [16] it was proved that $MI = \mathbf{1}$ and $IM \gneqq \mathbf{1}$

A topological ordered space $(X, \tau, \leq)$ is a completely regular ordered space if it satisfies the following two conditions: *(a) Let $a \in X$ and let $V$ be a neighborhood of $a$ in $X$. There exist two continuous functions $f, g : (X, \tau, \leq) \to \mathbb{I}_0$ such that $f$ is order-preserving, $g$ is order-reversing,*

*$f(a) = 1 = g(a)$, and either $f(x) = 0$ or $g(x) = 0$, whenever $x \in X \setminus V$.*
*(b) Let $a, b \in X$ such that $a \nleq b$. Then there exists an order-preserving function $f$ on $X$ such that $f(a) > f(b)$* [11].

The following strengthening of complete regularity was introduced in [7].

Let $(X, \tau, \leq)$ be a topological ordered space. Then $X$ is said to be *strictly completely regular ordered space* if: *(i) the order on $X$ is semiclosed, i.e. $d(a)$ and $i(a)$ are closed, where $d(a)$ ($i(a)$) is the smallest lower (upper respectively) set containing $a$. (ii) $X$ is strongly order convex, i.e. the open upper sets and open lower form a subbasis for the topology. (iii) given a closed lower (upper respectively) set $A$ and a point $x \notin A$, there exists a continuous order-preserving function $f : (X, \tau, \leq) \to \mathbb{I}_0$ such that $f(A) = 0$ and $f(x) = 1$ ($f(A) = 1$ and $f(x) = 0$ respectively).*

In [4] an example of a completely regular ordered space which is not a strictly completely regular ordered space was given. By Proposition 1 in [4], this example shows that the functors $I$ : CRTopOrd $\to$ CR2Top, and $S$ : CRTopOrd $\to$ 2Top are not the same, answering a question raised in [16].

We shall use the definition of a pairwise compact bispace, as defined by Salbany in [15], i.e. a bispace $(X, \tau_1, \tau_2)$ is a pairwise compact bispace if $(X, \tau_1 \vee \tau_2)$ is a compact space. Let $\mathbb{R}_b$ be the real numbers with the upper topology and the lower topology, where the upper topology is the topology which has sets of the form $\{(-\infty, a) : a \in \mathbb{R}\}$ as a base and the lower topology has sets of the form $\{(a, \infty) : a \in \mathbb{R}\}$ as a base. We shall denote the upper topology by $u$ and the lower topology by $l$. The bitopological space $(\mathbb{I}, u, l)$ denotes the closed unit interval $[0, 1]$ with the upper topology and the lower topology. Lane in [6] defines complete regularity in bitopological spaces as follows: *A bitopological space $(X, \tau_1, \tau_2)$ is said to be completely regular if for each point $x$ of $X$ and each $\tau_1$-closed set $F$ not containing $x_0$ there is a bicontinuous function $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ such that $f(x_0) = 1$ and $f(x) = 0$ when $x \in F$; and for each $\tau_2$-closed set $H$ not containing $x_0$ there is a bicontinuous function $g : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ such that $g(x_0) = 0$ and $g(x) = 1$ for $x \in H$.*

We now give an analogue of strictness for completely regular bitopological spaces.

*Definition 1.* A bitopological space $(X, \tau_1, \tau_2)$ is a *strictly completely regular bitopological space* if the following conditions are satisfied:

(a) Every $\tau_1 \vee \tau_2$-closed set which is $\leq_{\tau_1}$-decreasing ($\leq_{\tau_1}$-increasing) is $\tau_1$-closed ($\tau_2$-closed).

(b) given a $\tau_1 \vee \tau_2$-closed $\leq_{\tau_1}$-decreasing ($\leq_{\tau_1}$-increasing) set $A$ and a point $x \notin A$, there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ such that $x \in f^{\leftarrow}(\{1\})$ and $A \subseteq f^{\leftarrow}(\{0\})$ ($x \in f^{\leftarrow}(\{0\})$ and $A \subseteq f^{\leftarrow}(\{1\})$).

After introducing the notion of strict complete regularity in ordered topological spaces, Lawson [7] asked if the notion coincides with complete regularity in ordered spaces as defined by Nachbin [11]. As indicated earlier, the question was answered in the negative by Künzi in [4].

We shall now give examples to show that the notion of strict complete regularity in bitopological spaces as defined above does not coincide with complete regularity as defined by Lane [6]. These are much simpler than the one given in [4] which answered Lawson's question.

*Example 1.* $(\mathbb{R}, u, l)$ is a strictly completely regular bitopological space.

*Example 2.* $(\{0, 1\}, \tau_1, \tau_2)$, where $\tau_1$ is the upper Sierspinki topology and $\tau_2$ is the lower Sierspinki topology is a strictly completely regular bitopological space.

*Example 3.* $(\mathbb{R}, s_1, s_2)$ where $s_1$ has base sets of the form $[a, b)$, $a < b$ and $s_1$ has base sets of the form $(a, b]$, $a < b$, is completely regular but not strictly completetly regular. Note that this space is also quasipseudometrizable and pairwise normal.

*Example 4* ([15] and [16]). Let $C : \text{Top} \to 2\text{Top}$ be the functor given by $C(X, \tau) = (X, \tau, \tau^*\ )$, where $\tau^*$ is the topology having the closed sets of $\tau$ as a base. Let $C_1 : \text{Top}_1 \to 2\text{Top}$ be the restriction to $T_1$-spaces. If $X \in \text{Top}_1$ and $X$ is a non-discrete space (e.g. the cofinite topology on an infinite set) then $C_1X$ is completely regular but not strictly completely regular. For complete regularity see [15] and [16]. Since $\tau \vee \tau^*$ is the discrete topology and $\leq_\tau$ is the discrete order we have that $C_1X = (X, \tau, \tau^*)$ is not strict.

*Example 5.* The bitopological spaces induced by the natural quasi-uniformities in Examples 1, 3, 4, 5 and the Example 3 of [4] are all completely regular but not strictly completely regular.

**Proposition 1.** *Every strictly completely regular bitopological space is a completely regular bitopological space.*

Proof. Let $(X, \tau_1, \tau_2)$ be strictly completely regular. Let $A$ be a $\tau_1$-*closed* set and $x \notin A$. Since the $\tau_1$-*closed* sets are $\leq_{\tau_1}$-decreasing and $\tau_1 \vee \tau_2$-*closed*, by our assumption there exists a bicontinuous function $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ such that $x \in f^{\leftarrow}(\{1\})$ and $A \subseteq f^{\leftarrow}(\{0\})$. A dual argument holds for a $\tau_2$-*closed* set. Therefore $(X, \tau_1, \tau_2)$ is a completely regular bitopological space. $\square$

**Proposition 2.** *For a bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent:*

(a) $(X, \tau_1, \tau_2)$ *is a strictly completely regular bitopological space.*

(b) $(X, \tau_1, \tau_2)$ *is completely regular and $\tau_1$ is the upper topology of $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ and $\tau_2$ is the lower topology of $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$.*

Proof. (a) $\Rightarrow$ (b) It is easy to show that if $U$ is $\tau_1$-open then $U$ is increasing with respect to $\leq_{\tau_1}$. Since $U$ is $\tau_1 \vee \tau_2$-open, we have that $\tau_1 \subseteq (\tau_1 \vee \tau_2)^\sharp$. On the other hand if $U$ is $\tau_1 \vee \tau_2$-open and $\leq_{\tau_1}$-increasing then by strict complete regularity it is $\tau_1$-open. Therefore $(\tau_1 \vee \tau_2)^\sharp \subseteq \tau_1$. Hence $\tau_1 = (\tau_1 \vee \tau_2)^\sharp$. The other case is similar.

(b) $\Rightarrow$ (a) The proof of condition (a) in the definition of strictly completely regular bispace is easy. Condition (b) follows from complete regularity. $\square$

*Remark 1.* In ordered spaces we have that a combination of normality and complete regularity gives strict complete regularity [7]. *The situation is different in bitopological spaces.* In Example 3 above, the bispace is pairwise normal, pairwise completely regular and pairwise quasi-pseudo metrizable but not strictly completely regular.

**Proposition 3.** *If $(X, \tau_1, \tau_2)$ is pairwise compact and pairwise regular then $(X, \tau_1, \tau_2)$ is a strictly completely regular bitopological space.*

Proof. See [14, Proposition 10] or [16, Proposition 7, page 500]. $\square$

### 3. Correspondence between the notion of strictness in bitopological spaces and ordered spaces

In this section we investigate the relationship between the notion of strict complete regularity as introduced by Lawson in ordered spaces [7] and our notion of strictness for bitopological spaces. This will be done via the three functors $M$ : CR2Top $\to$ CRTopOrd, $S$ : CRTopOrd $\to$ 2Top, $I$ : CRTopOrd $\to$ CR2Top given in the preceeding section. It turns out that the two categories are isomorphic, i.e. on strict spaces the functors $S$ and $I$ coincide, and form an isomorphism with the functor $M$.

The straightforward proof of the following proposition will be omitted.

**Proposition 4.** *If $(X, \tau_1, \tau_2)$ is a strictly completely regular bispace then $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ is a strictly completely regular ordered space.*

It was shown in [4] that the functor $M$ : CR2Top $\to$ CRTopOrd does not preserve normality in general. However, the following holds.

**Proposition 5.** *The functor $M$ : CR2Top $\to$ CRTopOrd preserves normality when restricted to strict complete regularity.*

Proof. Suppose $(X, \tau_1, \tau_2)$ is pairwise normal and strictly completely regular. Let $F$ and $H$ be disjoint $\leq_{\tau_1}$-increasing and $\leq_{\tau_1}$-decreasing (respectively) $\tau_1 \vee \tau_2$-closed sets. Since $(X, \tau_1, \tau_2)$ is strictly completely regular bispace, we have that $F$ and $H$ are $\tau_1$-closed and $\tau_2$-closed respectively. By normality of $(X, \tau_1, \tau_2)$ there are $\tau_1$-open and $\tau_2$-open sets $U$ and $V$ such that $F \subset U$ and $H \subset V$ such that $U \cap V = \emptyset$. Since the sets $U$ and $V$ are $\leq_{\tau_1}$-increasing and $\leq_{\tau_1}$-decreasing (respectively), we have that $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$ is a monotonically normal ordered space. $\square$

**Theorem 6.** *Let $(X, \tau, \leq)$ be a completely regular ordered space. The following are equivalent:*

(i) *$(X, \tau, \leq)$ is a strictly completely regular ordered space.*

(ii) *$S(X, \tau, \leq) = (X, \tau^\sharp, \tau^\flat)$ is a strictly completely regular bitopological space.*

(iii) *$I(X, \tau, \leq) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ is a strictly completely regular bitopological space.*

(iv) *$S(X, \tau, \leq) = I(X, \tau, \leq)$.*

Proof. (i) $\Rightarrow$ (ii) Let $(X, \tau, \leq)$ be strictly completely regular and let $A$ be $\tau^\sharp \vee \tau^\flat$-closed and $\leq_\tau^\sharp$-increasing. Since $(X, \tau, \leq)$ is strongly order convex, we have that $A$ is $\tau$-closed and $\leq$-increasing. By definition of $\tau^\sharp$, $A$ is $\tau^\sharp$-*closed*. If $x \notin A$ then the existence of a bicontinuous function $f : (X, \tau^\sharp, \tau^\flat) \to (\mathbb{I}, u, l)$ such that $x \in f^{\leftarrow}(\{1\})$ and $A \subseteq f^{\leftarrow}(\{0\})$ follows from the strict complete regularity of $(X, \tau, \leq)$. A similar argument holds if $A$ is $\tau^\sharp \vee \tau^\flat$-*closed* set and $\leq_{\tau^\sharp}$-decreasing.

(ii) $\Rightarrow$ (i) Let $(X, \tau^\sharp, \tau^\flat)$ be a strictly completely regular bitopological space. Since $X$ is strongly order convex, by Proposition 4 above, we have that $(X, \tau^\sharp \vee \tau^\flat, \leq_{\tau^\sharp}) = (X, \tau, \leq)$ is a strictly completely regular ordered space.

(i) $\Rightarrow$ (iii) Suppose $(X, \tau, \leq)$ is strictly completely regular. Let $A$ be a $\tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$-*closed*, $\leq_{\tau_{\mathcal{A}_0}}$-increasing set and $x \notin A$. Since $\tau = \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}$ and $\leq = \leq_{\tau_{\mathcal{A}_0}}$, there is a continuous order-preserving function $f : (X, \tau, \leq) \to \mathbb{I}_0$ such that $x \in f^{\leftarrow}(\{0\})$ and $A \subseteq f^{\leftarrow}(\{1\})$. By applying the functor $I$ to the continuous order-preserving function$f : (X, \tau, \leq) \to \mathbb{I}_0$, we have that this function is bicontinuous as a function from $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ to $(\mathbb{I}, u, l)$. Thus A is $\tau_{\mathcal{A}_0}$-*closed*. The dual conclusion follows in the same way. Hence $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ is strictly completely regular.

(iii) $\Rightarrow$ (i) If $(X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$ is strictly completely regular then $(X, \tau_{\mathcal{A}_0} \vee \tau_{\mathcal{B}_0}, \leq_{\tau_{\mathcal{A}_0}}) = (X, \tau, \leq)$ is strictly completely regular by Proposition 4 above.

(ii) $\Rightarrow$ (iv) We know that $\tau_{\mathcal{A}_0} \subseteq \tau^\sharp$ and $\tau_{\mathcal{B}_0} \subseteq \tau^\flat$. We show the reverse inclusions. Let $U \in \tau^\sharp$ and $x \in U$. Then $X - U$ is $\tau$-*closed* and $\leq$-increasing. Then there is $A \in \mathcal{A}_0$ such that $x \notin A$ and $X - U \subseteq A$. Thus $x \in X - A \subseteq U$. Therefore $U$ is $\tau_{\mathcal{A}_0}$-*open*. Hence $\tau^\sharp \subseteq \tau_{\mathcal{A}_0}$. Similarly $\tau^\flat \subseteq \tau_{\mathcal{B}_0}$. Therefore $S(X, \tau, \leq) = I(X, \tau, \leq)$.

(iv) $\Rightarrow$ (ii) Suppose $S(X, \tau, \leq) = I(X, \tau, \leq)$ i.e. $(X, \tau^\sharp, \tau^\flat) = (X, \tau_{\mathcal{A}_0}, \tau_{\mathcal{B}_0})$. Then $(X, \tau^\sharp, \tau^\flat)$ is a completely regular bitopological space [6, Proposition 2.9]. Thus $(X, \tau, \leq)$ is a strictly completely regular ordered space [4, Proposition 1]. By (i) $\iff$ (ii) above, $(X, \tau^\sharp, \tau^\flat)$ is a strictly completely regular bispace. □

The notion of a sober space is related to compact ordered spaces as observed by Lawson in [7]. We will now show that this notion is also related to strictness. Recall that a topological space is sober if every irreducible closed set is the closure of a singleton set. A topological space is strongly sober if the adherence set of every ultrafilter is the closure of a singleton.

For a topological space $(X, \tau)$, the cocompact topology is defined as a topology $\tau^k$ which has the compact upper sets as a subbase for closed sets. Part of the following proposition was observed by Lawson in [7].

**Proposition 7.** *Let $(X, \tau)$ be a $T_o$-space. The following are equivalent:*

(i) $(X, \tau)$ *is locally compact and strongly sober.*

(ii) $(X, \tau)$ *is locally compact and sober, and the intersection of two compact upper sets is again compact.*

(iii) *The patch space of* $(X, \tau, \tau^k)$ *is a compact ordered space.*

(iv) $(X, \tau \vee \tau^k, \leq_\tau)$ *is a strictly completely regular ordered space.*

(v) $(X, \tau, \tau^k)$ *is strictly completely regular.*

(vi) $(X, \tau, \tau^k)$ *is a pairwise compact bispace.*

Proof. (i) $\iff$ (ii) $\iff$ (iii) follow from [7, Theorem 25].

(iii) $\iff$ (iv) $\iff$ (v) Follows from (ii) above since $(\tau \vee \tau^k)^\sharp = \tau$ and $(\tau \vee \tau^k)^\flat = \tau^k$.

(iii) $\iff$ (v) Obvious. □

**Lemma 8** [4]**.** *Let $(X, \tau, \leq)$ be an ordered space. Then $f : (X, \tau, \leq) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function if and only if $f : (X, \tau^\sharp, \tau^\flat) \to (\mathbb{I}, (u \vee l)^\sharp, (u \vee l)^\flat)$ is a bicontinuous function.*

**Proposition 9.** *Let $(X, \tau_1, \tau_2)$ be a strictly completely regular bispace. Then $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ is a bicontinuous function if and only if $f : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function.*

Proof. Suppose $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ is bicontinuous. Then $f := Mf : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function (see [16] and [17]). Suppose $f : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function. By the above lemma $f : (X, (\tau_1 \vee \tau_2)^\sharp, (\tau_1 \vee \tau_2)^\flat) \to (\mathbb{I}, (u \vee l)^\sharp, (u \vee l)^\flat)$ is a bicontinuous function. Since $(X, \tau_1, \tau_2)$ and $(\mathbb{I}, u, l)$ are strictly completely regular bispaces, we have that $\tau_1 = (\tau_1 \vee \tau_2)^\sharp$, $\tau_2 = (\tau_1 \vee \tau_2)^\flat$, $u = (u \vee l)^\sharp$ and $l = (u \vee l)^\flat$. Therefore $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ is bicontinuous. □

Let $(X, \tau_1, \tau_2)$ be a bitopological space. If $f : (X, \tau_1, \tau_2) \to (\mathbb{I}, u, l)$ is bicontinuous then we shall call sets of the form $\{x \in X : f(x) \leq 0\}$, *lower-zero* sets of $(X, \tau_1, \tau_2)$ and sets of the form $\{x \in X : f(x) \geq 0\}$, *upper-zero* sets of $(X, \tau_1, \tau_2)$ (see [6], Definition 2.7).

**Corollary 10.** *Let $(X, \tau_1, \tau_2)$ be a strictly completely regular bispace. Then the lower-zero sets and the upper-zero sets of $(X, \tau_1, \tau_2)$ coincide with the decreasing zero sets and the increasing zero sets of $(X, \tau_1 \vee \tau_2, \leq_{\tau_1})$.*

*Remark 2.* If $(X, \tau_1, \tau_2)$ is not a strictly completely regular bispace the above proposition and corollary do not hold. In Examples 3 and 4, the ordered spaces $(R, s_1 \vee s_2, \leq_{s_1})$ and $(X, \tau \vee \tau^*, \leq_\tau)$ have discrete topologies and discrete orderes. Hence every function $f : (R, s_1 \vee s_2, \leq_{s_1}) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function and every function $g : (X, \tau \vee \tau^*, \leq_\tau) \to (\mathbb{I}, u \vee l, \leq_u)$ is a continuous order-preserving function. The corresponding functions $f : (R, s_1, s_2) \to (\mathbb{I}, u, l)$ and $g : (X, \tau, \tau^*) \to (\mathbb{I}, u, l)$ need not be bicontinuous.

**Theorem 11.** *If $(X, \tau_1, \tau_2)$ is a strictly completely regular bitopological space then $IM(X, \tau_1, \tau_2) = (X, \tau_1, \tau_2)$.*

PROOF. We want to show that $I(X, \tau_1 \vee \tau_2, \leq_{\tau_1}) = (X, \tau_1, \tau_2)$. We know that $I(X, \tau_1 \vee \tau_2, \leq_{\tau_1}) = (X, (\tau_1 \vee \tau_2)_{\mathcal{A}_0}, (\tau_1 \vee \tau_2)_{\mathcal{B}_0})$. Let $\mathcal{P}_0$ be the collection of *lower-zero* sets and $\mathcal{Q}_0$ be the collection of *upper-zero* sets. Then $\tau_1 = \tau_{1_{\mathcal{P}_0}}$ and $\tau_2 = \tau_{2_{\mathcal{Q}_0}}$, where $\tau_{1_{\mathcal{P}_0}}$ is the topology having $\mathcal{P}_0$ as a base for closed sets and $\tau_{2_{\mathcal{Q}_0}}$ is the topology having $\mathcal{Q}_0$ as a base for closed sets (see [6]). Since the decreasing zero sets coincide with the lower-zero sets and the increasing zero sets coincide with the upper-zero sets, we have that $\tau_1 \subseteq (\tau_1 \vee \tau_2)_{\mathcal{A}_0}$ and $\tau_2 \subseteq (\tau_1 \vee \tau_2)_{\mathcal{B}_0}$. We now show that $(\tau_1 \vee \tau_2)_{\mathcal{A}_0} \subseteq \tau_1$ and $(\tau_1 \vee \tau_2)_{\mathcal{B}_0} \subseteq \tau_2$. It suffices to show that every member of $\mathcal{A}_0$ is $\tau_1$-*closed* and every member of $\mathcal{B}_0$ is $\tau_2$-*closed*. Let $A \in \mathcal{A}_0$. Then $A = \{x : f(x) \leq 0\}$ where $f : (X, \tau_1 \vee \tau_2, \leq_{\tau_1}) \to \mathbb{R}_0$ is continuous and order-preserving. Then $A$ is $\tau_1 \vee \tau_2$-*closed* and $\leq_{\tau_1}$-increasing. Since $(X, \tau_1, \tau_2)$ is strictly completely regular, we have that $A$ is $\tau_1$-*closed*. Therefore $(\tau_1 \vee \tau_2)_{\mathcal{A}_0} \subseteq \tau_1$. Hence $\tau_1 = (\tau_1 \vee \tau_2)_{\mathcal{A}_0}$. Similarly $\tau_2 = (\tau_1 \vee \tau_2)_{\mathcal{B}_0}$. Therefore $(X, \tau_1, \tau_2) = (X, (\tau_1 \vee \tau_2)_{\mathcal{A}_0}, (\tau_1 \vee \tau_2)_{\mathcal{B}_0})$ and thus $IM(X, \tau_1, \tau_2) = (X, \tau_1, \tau_2)$. $\square$

Let SCR2Top be a full subcategory of CR2Top which consists of strictly completely regular bispaces and SCRTopOrd be a full subcategory of CRTopOrd which consists of strictly completely regular ordered spaces.

**Corollary 12.** *The categories* SCR2Top *and* SCRTopOrd *are isomorphic.*

Proof. Follows from Proposition 11 above and the fact that $MI = \mathbf{1}$, see [16]. $\square$

**Corollary 13.** $MS = \mathbf{1}$ *when restricted to strictly completely regular ordered spaces.*

Proof. By Theorem 6 above the functors $S$ and $I$ coincide on strict spaces. The result follows from the above corollary. $\square$

## 4. Quasi-uniform spaces, and strictly completely regular bitopological spaces

In this section we give conditions on a quasi-uniformity that ensure that it induces a strict complete regular bitopology.

In [2] the notion of a *small set symmetric quasi-uniform space* was introduced. It was shown in [5] that a quasi-uniform space $(X,\mathcal{U})$ is small-set symmetric if and only if $T(\mathcal{U}^{-1}) \subset T(\mathcal{U})$.

A quasi-uniformity $\mathcal{U}$ on a set $X$ is called *point-symmetric* if $T(\mathcal{U}) \subset T(\mathcal{U}^{-1})$ and is a *Lebesgue quasi-uniformity* provided that for each $T(\mathcal{U})$-open cover $\mathcal{H}$ of $X$ there is $U \in \mathcal{U}$ such that the cover $\{U(x) : x \in X\}$ refines $\mathcal{H}$ [1].

**Proposition 14.** *Let $(X,\mathcal{U})$ be a $T_0$-quasi-uniform-space. If each proper $T(\mathcal{U}^*)$-closed lower set of $X$ is the intersection of $T(\mathcal{U}^{-1})$-compact lower sets of $X$ and that each proper $T(\mathcal{U}^*)$-closed upper set of $X$ is the intersection of $T(\mathcal{U})$-compact upper sets of $X$. Then $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular bispace.*

Proof. By Lemma 1 in [3] we have $T(\mathcal{U}) = T(\mathcal{U}^*)^\sharp$ and $T(\mathcal{U}^{-1}) = T(\mathcal{U}^*)^\flat$. By Proposition 2 in Section 2, and Lemma 1 and Proposition 1 in [4], $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is a strictly completely regular bispace. $\square$

**Proposition 15.** *Let $(X,\mathcal{U})$ be a small set symmetric quasi-uniform space. Then the bitopological space $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular if and only if $T(\mathcal{U}^{-1}) = T(\mathcal{U})^\flat$.*

Proof. Suppose $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular. From (i) $\Rightarrow$ (ii) in Proposition 2, we have $T(\mathcal{U}^{-1}) = T(\mathcal{U}^*)^\flat$. Since $(X,\mathcal{U})$

is a small set symmetric quasi-uniform space, we have $T(\mathcal{U}^{-1}) \subset T(\mathcal{U})$ and thus $T(\mathcal{U}) = T(\mathcal{U}^*)$. Therefore $T(\mathcal{U}^{-1}) = T(\mathcal{U})^\flat$. Conversely suppose $T(\mathcal{U}^{-1}) = T(\mathcal{U})^\flat$. Then $T(\mathcal{U}^{-1}) = T(\mathcal{U}^*)^\flat$ since $T(\mathcal{U}) = T(\mathcal{U}^*)$ from small set symmetry. It is easy to show that always $T(\mathcal{U}) \subset T(\mathcal{U}^*)^\sharp$. Then $T(\mathcal{U}) \subset T(\mathcal{U}^*)^\sharp = T(\mathcal{U})^\sharp$. Hence $T(\mathcal{U}) = T(\mathcal{U}^*)^\sharp$. By Proposition 2, $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular. □

**Proposition 16.** *Let $(X,\mathcal{U})$ be a point symmetric quasi-uniform space. Then the bitopological space $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular if and only if $T(\mathcal{U}) = T(\mathcal{U})^\sharp$.*

Proof. Similar to that of Proposition 15. □

**Corollary 17.** *If a quasi-uniform space $(X,\mathcal{U})$ is both small-set symmetric and point symmetric then the bitopological space $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular.*

*Remark 3.* In Example 4 we have that if $(X, \tau)$ is a non discrete $T_1$-*space* then $C_1X$ is a completely regular bitopological space which is not strictly completely regular. We also have that $MC_1X$ is a strictly completely regular ordered space. The behaviour with respect to quasi-uniform spaces is not different. Let $\mathcal{U}$ be the coarsest compatible quasi-uniformity on a locally compact noncompact $T_2$-*space* $X$. Then the completely regular ordered space $(X, T(\mathcal{U}^*), \cap\mathcal{U})$ is strictly completely regular, but the corresponding bitopological space $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is not strictly completely regular since $T(\mathcal{U}^{-1}) \neq T(\mathcal{U}^*)^\flat$, (see [4]).

*Definition 2* [8]. A quasi-uniformity $\mathcal{U}$ on a set $X$ is *pair Lebesgue quasi-uniform* space if for each pair of open cover $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{I}\}$ there is $U \in \mathcal{U}$ such that the cover $\{(U(x), U^{-1}) : x \in X\}$ refines $\{(G_\alpha, H_\alpha) : \alpha \in \mathcal{I}\}$ (i.e. for each $x \in X$ there is $\alpha \in \mathcal{I}$ such that $U(x) \subset G_\alpha$ and $U^{-1}(x) \subset H_\alpha$).

**Proposition 18.** *Let $(X,\mathcal{U})$ be a $T_1$ quasi-uniform space. If $\mathcal{U}$ is a pair Lebesgue quasi-uniformity then $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is strictly completely regular.*

Proof. By Proposition 3.8 in [8] both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are Lebesque quasi-uniformities. Therefore $T(\mathcal{U}) = T(\mathcal{U}^{-1}) = T(\mathcal{U}^*)$. (See Proposition 3.8 in [8].) Hence $(X,\mathcal{U})$ is both small-set symmetric and point symmetric. By Corollary 17, $(X, T(\mathcal{U}), T(\mathcal{U}^{-1}))$ is a strictly completely regular bitopological space. □

*Remark 4.* Although the category of strictly completely regular bitopological spaces is isomorphic to the category of strictly completely regular ordered spaces (Corollary 12), Examples 3 and 4 show that the notions of strictness in the category of completely regular bispaces and in the category of completely regular ordered spaces behave differently. These examples show that a combination of complete regularity and normality in CR2Top does not imply strictness, whereas it implies strictness in CRTopOrd (see [7]). As seen in Proposition 3 we need compactness in CR2Top which is a stronger property than normality in order to have strictness. By Remark 2 the strictness situation with respect to bispaces and ordered spaces induced by quasi-uniformities is not the same.

*Problem.* Characterize those quasi-uniformities which induce a strictly completely regular bitopology?

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