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Reduction theorems of certain Douglas spaces to Berwald spaces

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Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday

Abstract. The notion of Douglas space was proposed by the present authors as a generalization of the notion of Berwald space. Some Finsler spaces of Douglas type are reduced to Berwald spaces. In the present paper we are mainly concerned with Finsler spaces with (α, β)-metric and expect further development.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L(x, y))$ on a smooth n-manifold M^n with a fundamental function $L(x, y)$. Consider $F = L^2/2$ and denote the fundamental tensor by $g_{ij}(x, y) = \dot{\partial}_i \dot{\partial}_j F$. If we define functions $G^i(x, y)$ by $2g_{ij}G^i = (\dot{\partial}_j \partial_r F)y^r - \partial_j F$, then the geodesic curve $x(t)$ of F^n is given by the differential equations

$$d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0,$$

in terms of the arc-length $s = \int L(x(t), dx/dt)dt$ as the parameter. The functions $G^i(x, y)$ are positively homogeneous in y^i of degree two.

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The Berwald connection $B\Gamma = (G^i_j, G^i_{jk}, 0)$ of F^n is defined by $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_k G^i_j$. Then $G^i_{jkh} = \dot{\partial}_h G^i_{jk}$ are components of the hv-curvature tensor of F^n. The h- and v-covariant differentiations with respect to $B\Gamma$ are indicated by $(;,.)$: For a contravariant vector field $X = (X^i)$ we have

$$X^i_{;j} = \delta_j X^i + X^r G^i_{rj}, \quad X^i{}_{.j} = \dot{\partial}_j X^i,$$

where $\delta_j = \partial_j - G^r_j \dot{\partial}_r$.

If $G^i(x,y)$ of F^n are homogeneous polynomials $G^i = G^i_{jk}(x) y^j y^k / 2$ in y^i, then F^n is called *Berwald space* as usual. Thus a Berwald space is characterized by the tensorial equation $G^i_{jkh} = 0$.

The present authors defined the notion of Douglas space [BM,2]: In general, $D^{ij}(x,y) = G^i(x,y)y^j - G^j(x,y)y^i$ are positively homogeneous in y^i of degree three. If $D^{ij}(x,y)$ of F^n are homogeneous polynomials in y^i of degree three, then F^n is called a *Douglas space*. Thus, a Douglas space is characterized by $D^{lm}_{hijk} = \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h D^{lm} = 0$. It is easy to show

$$\begin{aligned} &D^{lr}_{hijr} = (n+1) D^l_{hij}, \\ &D^{lm}_{hijk} = (\dot{\partial}_k D^l_{hij}) y^m + \{D^l_{ijk} \delta^m_h + (h,i,j,k)\} - [l,m], \end{aligned} \tag{1}$$

where (h,i,j,k) denotes cyclic permutation of these subscripts, $[l,m]$ interchange of these superscripts. The tensors D^l_{hij} are components of the Douglas tensor

$$D^l_{hij} = G^l_{hij} - G_{hij} y^l/(n+1) - \{G_{hi} \delta^l_j + (h,i,j)\}/(n+1),$$

where $G_{hi} = G^r_{rhi}$ and $G_{hij} = G_{hi \cdot j}$. Since (1) shows that $D^{lm}_{hijk} = 0$ is equivalent to $D^l_{hij} = 0$, the vanishing of the Douglas tensor characterizes a Douglas space, the origin of this naming.

If we treat the projective invariants

$$Q^i = G^i - G^r_r y^i/(n+1),$$

then we have $D^i_{jkh} = \dot{\partial}_j \dot{\partial}_k \dot{\partial}_h Q^i$, and hence F^n is a Douglas space, if and only if Q^i are homogeneous polynomials in y^i of degree two. If we consider $Q^i_j = \dot{\partial}_j Q^i$ and $Q^i_{jk} = \dot{\partial}_k Q^i_j$, then the latter is a function of the position x alone in a Douglas space.

Let us define

$$Q^i_{jkh} = \partial_h Q^i_{jk} - (\dot{\partial}_r Q^i_{jk}) Q^r_h + Q^r_{jk} Q^i_{rh} - [k, h],$$

and let $Q_{jk} = Q^r_{rjk}$. Then

$$W^i_{jkh} = Q^i_{jkh} + \{\delta^i_k Q_{jh} - [k, h]\}/(n-1)$$

coincide with the components of the Weyl curvature tensor [BM,3]. Consequently, both of the projective invariant tensors, the Douglas tensor D^i_{jkh} and the Weyl tensor W^i_{jkh}, are obtained from the invariants Q^i. For a Douglas space, Q^i_{jk} are functions of the position (x^i) alone, and so are W^i_{jkh}.

In particular, for a two-dimensional Douglas space with the local coordinates (x, y), the equation of a geodesic curve can be written in the form

$$y' = dy/dx,$$

$$dy'/dx = Y_3(y')^3 + Y_2(y')^2 + Y_1 y' + Y_0,$$

where the coefficients Y_0, Y_1, Y_2, Y_3 are functions of (x, y) alone.

Finally, we consider the following sets of special Finsler spaces:

$$\begin{aligned} M(n) &= \{\textit{locally Minkowski spaces of dimension } n\} \\ B(n) &= \{\textit{Berwald spaces of dimension } n\} \\ L(n) &= \{\textit{Landsberg spaces of dimension } n\} \\ S(n) &= \{\textit{spaces of dimension } n \textit{ without stretch curvature}\}. \end{aligned}$$

L. Berwald stated the following inclusion relations at the International Mathematical Congress, Bologna, 1928 [B]:

$$M(n) \subset B(n) \subset L(n) \subset S(n).$$

The reduction theorems of Landsberg spaces to Berwald spaces ([BM,1], [M,3]) are related to $B(n) \subset L(n)$.

If we deal with the set

$$D(n) = \{\textit{Douglas spaces of dimension } n\},$$

then Theorem 1 of [BM,2] states that

$$B(n) = L(n) \bigcap D(n).$$

In terms of the reduction this is expressed as

Theorem 1.1. (1) *If a Landsberg space is of Douglas type then it reduces to a Berwald space.* (2) *If a Douglas space is of Landsberg type then it reduces to a Berwald type.*

2. Randers space and Kropina space

We are concerned with Finsler spaces $F^n = (M^n, L)$ with a special metric $L(\alpha, \beta)$, called (α, β)-metric where α is a Riemannian metric and β is a 1-form in y^i:

$$\alpha^2 = a_{ij}(x)y^i y^j, \quad \beta = b_i(x)y^i.$$

Thus we obtain a Riemannian space $R^n = (M^n, \alpha)$ on M^n, called the associated Riemannian space [AIM].

We treat R^n which is equipped with the Levi–Civita connection $\gamma = (\gamma^i_{jk}(x))$, and denote by (,) the covariant differentiation with respect to γ. We shall use the usual notation:

$$r_{ij} = (b_{i,j} + b_{j,i})/2, \quad s_{ij} = (b_{i,j} - b_{j,i})/2,$$

$$s^i_j = a^{ir} s_{rj}, \quad s_j = b_r s^r_j, \quad b^i = a^{ir} b_r, \quad b^2 = b_r b^r.$$

Let $B\Gamma = (G^i_j, G^i_{jk})$ be the Berwald connection of F^n and consider $2G^i = G^i_j y^j = G^i_{jk} y^j y^k$. Owing to ([M,4], [KAM]) we have that the difference $B^i = G^i - \gamma^i_{00}/2$ is given by

$$B^i = (E/\alpha)y^i + (\alpha L_2/L_1)s^i_0 - (\alpha C^* L_{11}/L_1)(y^i/\alpha - \alpha b^i/\beta)$$

$$E = \beta C^* L_2/L, \quad C^* = \alpha\beta(r_{00}L_1 - 2\alpha s_0 L_2)/2(\beta^2 L_1 + \alpha r^2 L_{11}),$$

where $r^2 = b^2\alpha^2 - \beta^2$, $(L_1, L_2) = (\partial L/\partial\alpha, \partial L/\partial\beta)$ and the subscript 0 denotes the contraction by y^i.

Thus F^n is a Berwald space, if and only if B^i are homogeneous polynomials in y^i of degree two, and it is a Douglas space, if and only if

$$\begin{aligned} B^{ij} &= B^i y^j - B^j y^i \\ &= (\alpha L_2/L_1)(s^i_0 y^j - s^j_0 y^i) + (\alpha^2 C^* L_{11}/\beta L_1)(b^i y^j - b^j y^i), \end{aligned}$$

are homogeneous polynomials in y^i of degree three.

I. Randers space. We first consider a Randers space F^n with $L = \alpha + \beta$. Then we have

$$B^i = (r_{00} - 2\alpha s_0) y^i / 2L + \alpha s^i_0.$$

Owing to ([K], [M,2]), F^n is a Berwald space, if and only if $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i,j} = 0$. Then G^i are reduced to $\gamma^i_{00}/2$.

Next we have

$$B^{ij} = \alpha(s^i_0 y^j - s^j_0 y^i).$$

According to [BM,2], F^n is a Douglas space, if and only if $s_{ij} = 0$, that is, b_i is a gradient vector field. Then $G^i = \gamma^i_{00}/2 + r_{00} y^i/2L$.

Therefore we conclude that *there exist Randers spaces of Douglas type which are not of Berwald type.*

II. Kropina space. We deal with a Kropina space F^n with $L = \alpha^2/\beta$. Then we have $C^* = (\beta r_{00} + \alpha^2 s_0)/2b^2\alpha$ and

$$B^i = 2\alpha C^* (b^i/2\beta - y^i/\alpha^2) - (\alpha^2/2\beta) s^i_0.$$

Thus $b^2 \neq 0$ is assumed [M,4].

Owing to [K], [M,2], F^n is a Berwald space, if and only if there exist functions $f_i(x)$ satisfying

$$\text{(i)} \quad r_{ij} = (f_r b^r) a_{ij}, \qquad \text{(ii)} \quad s_{ij} = b_i f_j - b_j f_i.$$

Then B^i is written as

$$B^i = (\alpha^2/2b^2)(s^i + f_r b^r b^i) - (s_0 + f_r b^r \beta) y^i / b^2.$$

Let us consider (ii). This yields

$$b^i s_{ij} (= s_j) = b^2 f_j - b^i f_i b_j,$$

$$b_i s_j - b_j s_i = b^2(b_i f_j - b_j f_i) = b^2 s_{ij}.$$

Thus (ii) is equivalent to the necessary and sufficient condition $s_{ij} = (b_i s_j - b_j s_i)/b^2$ for F^n to be of Douglas type, according to ([BM,2], [M,4]). The B^{ij} of a Douglas space F^n is written as

$$B^{ij} = (r_{00}/2b^2)(b^i y^j - b^j y^i) + (\alpha^2/2b^2)(s^i y^j - s^j y^i).$$

Consequently, (ii) is the only condition for F^n to be of Douglas type, and hence *there exist Kropina spaces of Douglas type which are not of Berwald type* [BM,4].

3. Generalized Kropina space

We consider an (α, β)-metric of the form

$$L = \alpha^{m+1}\beta^{-m}, \quad m \neq -1, 0.$$

Since the case $m = +1$ is the Kropina metric, this is called *generalized Kropina metric* [HHM]. For this metric we have

$$C^* = \alpha\{(1+m)r_{00}\beta + 2ms_0\alpha^2\}/2(1+m)\{(1-m)\beta^2 + mb^2\alpha^2\},$$

and hence $b^2 = 0$ may be admissible, provided that $m \neq 1$.

Now $\alpha^2 \equiv 0 \pmod{\beta}$ causes a special situation [M,4]: $n = 2$ and $b^2 = 0$. Since there exists a 1-form γ such that $\alpha^2 = \beta\gamma$, the metric L reduces to the 1-form metric $L = \beta^{(1-m)/2}\gamma^{(1+m)/2}$ of product type (Example 3.5.1.2 of [AIM]). Consequently, the space F^2 is a Berwald space (Theorem 3.5.3.1 of [AIM]).

In the ordinary case ($\alpha^2 \not\equiv 0 \pmod{\beta}$ and $b^2 \neq 0$), we have Theorem 1 of [M,4]:

F^n is a Douglas space, if and only if $b_{i,j}$ are given by $b_{i,j} = r_{ij} + s_{ij}$ where there exists a function $k(x)$ satisfying

$$\begin{aligned} r_{ij} &= \{k/m(m+1)\}\{mb^2 a_{ij} + (1-m)b_i b_j\} \\ &\quad + \{(1-m)/(1+m)b^2\}(s_i b_j + s_j b_i), \\ s_{ij} &= (b_i s_j - b_j s_i)/b^2. \end{aligned}$$

If we consider

$$v_i = 4m(s_i/b^2 + kb_i/2m)/(1+m),$$

then r_{ij} and s_{ij} are written in the form

$$r_{ij} = (b^r v_r/2)a_{ij} + \{(1-m)/4m\}(b_i v_j + b_j v_i),$$
$$s_{ij} = \{(1+m)/4m\}(b_i v_j - b_j v_i).$$

Hence we get

$$b_{i,j} = \frac{1}{2}(b_i v_j/m - b_j v_i + b^r v_r a_{ij}),$$

which coincides with the condition (4.5), given by [K] for F^n to be a Berwald space.

In fact, these r_{ij} and s_{ij} give B^i of the form

$$B^i = [\alpha^2(ms^i/b^2 + kb^i/2) - (k\beta + 2s_0 m/b^2)y^i]/(1+m),$$

which are homogeneous polynomials in y^i of degree two. Then F^n is a Berwald space.

Theorem 3.1. *Let F^n be a generalized m-Kropina space which is not a Kropina space. If F^n is a Douglas space, then F^n reduces to a Berwald space.*

4. Matsumoto space and space with $L = \alpha + \beta^2/\alpha$

I. Matsumoto space. The second of the present authors introduced an (α, β)-metric $L = \alpha^2/(\alpha - \beta)$ $[M, 1]$ as a realization of P. Finsler's idea "a slope measure of a mountain with respect to a time measure." A Finsler space with this metric was called *Mastumoto space* by the authors of $[AHY]$. According to them, a Matsumoto space is of Berwald type, if and only if $b_{i,j} = 0$.

On the other hand, [M,4] proved that the space is a Douglas space, if and only if $b_{i,j} = 0$, provided that $\alpha^2 \not\equiv 0$. Therefore

Theorem 4.1. *Let F^n be a Matsumoto space satisfying $\alpha^2 \not\equiv 0$ (mod β). If F^n is a Douglas space, then it reduces to a Berwald space.*

II. Space with $L = \alpha + \beta^2/\alpha$. We are concerned with a Finsler space F^n with $L = \alpha + \beta^2/\alpha$ which was first proposed in $[M, 4]$. This space is of Berwald type, if and only if $b_{i,j} = 0$, provided that $\alpha^2 \not\equiv 0$ (mod β).

On the other hand, the space F^n $(n > 2)$ is of Douglas type, if and only if there exists a function $k(x)$ such that

$$b_{i,j} = k\{(1 + 2b^2)a_{ij} - 3b_i b_j\}, \tag{2}$$

provided that $b^2 \neq 0, 1$. The assumption $b^2 \neq 0$ implies $\alpha^2 \not\equiv 0$ (mod β), by the Lemma of [M,4].

For this space we have

$$B^{ij} = r_{00}\alpha^2(b^i y^j - b^j y^i)/\{(1 + 2b^2)\alpha^2 - 3\beta^2\}.$$

Under (2) we have B^{ij} of the form

$$B^{ij} = k\alpha^2(b^i y^j - b^j y^i),$$

which are certainly homogeneous polynomials in y^i of degree three.

Since $b_{i,j} = 0$ of (2) holds only in the case $k = 0$, we have

Theorem 4.2. *Let F^n be a Finsler space with $L = \alpha + \beta^2/\alpha$ satisfying $b^2 \neq 0, 1$. It is of Douglas type, if and only if there exists a function $k(x)$ such that we have* (2)*. It reduces to a Berwald space, if and only if k vanishes.*

5. On two-dimensional Douglas spaces

From the standpoint of the reduction theorem, we have two interesting theorems on two-dimensional Douglas spaces in [BM,2].

First we recall that a two-dimensional Finsler space F^2 is a Douglas space, if and only if the *main scalar I* satisfies the equation

$$6I_{,1} + \varepsilon J_{;2} + 2IJ = 0, \tag{3}$$

where $J = I_{,1;2} + I_{,2}$ and $\varepsilon = \pm 1$ is the signature of the metric: $h_{ij} = g_{ij} - l_i l_j = \varepsilon m_i m_j$ in the Berwald frame (l_i, m_i).

We are concerned with the T-tensor T_{hijk} of F^2:

$$T_{hijk} = LC_{hij}|_k + l_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}.$$

In the two-dimensional case we have $LT_{hijk} = I_{;2} m_h m_i m_j m_k$. From (3) and $I_{;2} = 0$ it follows that

Theorem 5.1. *If a two-dimensional Douglas space has a vanishing T-tensor, then it reduces to a Berwald space with constant main scalar.*

Next we are concerned with a Finsler space with cubic metric

$$L^3 = a_{ijk}(x) y^i y^j y^k,$$

which has the components of a symmetric covariant tensor $a_{ijk}(x)$ as coefficients. In the two-dimensional case, this metric is characterized by the main scalar I as

$$2I_{;2} + 6\varepsilon I^2 + 3 = 0.$$

From this condition and (3) we can show

Theorem 5.2. *If a two-dimensional Douglas space F^2 is equipped with a cubic metric, then F^2 reduces to a locally Minkowski space, or a Berwald space with $L^3 = \{b_i(x)y^i\}\{c_j(x)y^j\}^2$, $\varepsilon = -1$ and $I^2 = 1/2$.*

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