Let $m$ and $n$ be integers such that $1 \leq m \leq n$. By $G_{m,n} = (\gcd(i,j))_{m \leq i, j \leq n}$ we denote the $(n-m+1) \times (n-m+1)$ matrix having $\gcd(i,j)$ as its $i, j$-entry for all integers $i$ and $j$ between $m$ and $n$. Smith showed in 1875 that $\det(G_{1,n}) = \prod_{k=1}^{n} \varphi(k)$, where $\varphi$ is the Euler’s totient function. In 2016, Hong, Hu and Lin proved that if $n \geq 2$ is an integer, then $\det(G_{2,n}) = \left( \prod_{k=1}^{n} \varphi(k) \right) \sum_{k=1}^{n} \frac{1}{\varphi(k)}$. In this paper, we show that if $n \geq 3$ is an integer, then $\det(G_{3,n}) = \left( \sigma_0 \sigma_1 + \frac{1}{2} \sigma_1 \sigma_2 + \frac{1}{2} \sigma_0 \sigma_2 \right) \prod_{k=1}^{n} \varphi(k)$, where for $i = 0, 1$ and 2, one has $\sigma_i := \sum_{\substack{k \text{ is odd squarefree} \atop 1 \leq k \leq n}} \frac{1}{\varphi(k)}$. Further, we calculate the determinants of the matrices $(f(\gcd(x_i, x_j)))_{1 \leq i, j \leq n}$ and $(f(\lcm(x_i, x_j)))_{1 \leq i, j \leq n}$ having $f$ evaluated at $\gcd(x_i, x_j)$ and $\lcm(x_i, x_j)$ as their $(i,j)$-entries, respectively, where $S = \{x_1, ..., x_n\}$ is a set of distinct positive integers such that $x_i > 1$ for any integer $i$ with $1 \leq i \leq n$, and $S \cup \{1,p\}$ is factor closed (that is, $S \cup \{1,p\}$ contains every divisor of $x$ for any $x \in S \cup \{1,p\}$), where $p \notin S$ is a prime number. Our result answers partially an open problem raised by Ligh in 1988.

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