Title: Two terms with known prime divisors adding to a power

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Let $c$ be a positive odd integer, and $R$ a set of $n$ primes coprime with $c$. We consider equations $X + Y = c^2$ in three integer unknowns $X, Y, z$, where $z > 0$, $Y > X > 0$, and the primes dividing $XY$ are precisely those in $R$. We consider $N$, the number of solutions of such an equation. Given a solution $(X, Y, z)$, let $D$ be the least positive integer such that $(XY/D)^{1/2}$ is an integer. Further, let $\omega$ be the number of distinct primes dividing $c$. Standard elementary approaches use an upper bound of $2^n$ for the number of possible $D$, and an upper bound of $2^{\omega - 1}$ for the number of ideal factorizations of $c$ in the field $\mathbb{Q}(\sqrt{-D})$ which can correspond (in a standard designated way) to a solution in which $(XY/D)^{1/2} \in \mathbb{Z}$, and obtain $N \leq 2^{n + \omega - 1}$. Here we improve this by finding an inverse proportionality relationship between a bound on the number of $D$ which can occur in solutions and a bound (independent of $D$) on the number of ideal factorizations of $c$ which can correspond to solutions for a given $D$. We obtain $N \leq 2^{n - 1} + 1$. The bound is precise for $n < 4$: there are several cases with exactly $2^{n - 1} + 1$ solutions.

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