On some global and local geometric properties of Musielak–Orlicz spaces

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Abstract. It is well known that any SU-point is an exposed point and any LUR-point is a strongly exposed point. Conditions which complete exposed points and strongly exposed points to SU-points and LUR points (respectively) are found. Criteria for SU-points in Musielak–Orlicz spaces with the Luxemburg and the Orlicz norms are given. Moreover, criteria for compact local uniform rotundity of Musielak–Orlicz sequence spaces are established.

1. Introduction

Denote by \( \mathbb{N} \) and \( \mathbb{R} \) the sets of natural and real numbers, respectively. Let \( (X, \| \|) \) be a real Banach space and \( X^* \) be its dual space. By \( S(X) \) we denote the unit sphere of \( X \).

A point \( x \in S(X) \) is said to be an extreme point if for every \( y, z \in S(X) \) such that \( x = \frac{y + z}{2} \), we have \( z = y = x \). We say that \( x^* \in X^* \) is a support functional at \( x \in X \setminus \{0\} \) if \( \|x^*\| = 1 \) and \( x^*(x) = \|x\| \). The set of all support functionals at \( x \in X \setminus \{0\} \) is denoted by \( \text{Grad}(x) \). A point \( x \in S(X) \) is said to be an exposed point if there exists \( x^* \in \text{Grad}(x) \) such that \( x^* \notin \text{Grad}(y) \), whenever \( y \in S(X) \) and \( y \neq x \). It is obvious that every exposed point is an extreme point.

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A point \( x \in S(X) \) is said to be a strongly exposed point if there exists \( x^* \in \operatorname{Grad}(x) \) such that for every sequence \( (x_n) \subset S(X) \) the condition \( x^*(x_n - x) \to 0 \) implies \( \|x_n - x\| \to 0 \).

A point \( x \in S(X) \) is called a strong U-point (SU-point for short) if for any \( y \in S(X) \) with \( \|x + y\| = 2 \), we have \( x = y \).

A point \( x \in S(X) \) is said to be a point of compact local uniform rotundity (local uniform rotundity) (CLU-point, (LUR) for short) if for every \( (x_n)_{n=1}^{\infty} \) in \( S(X) \) such that \( \|x_n + x\| \to 2 \) we have that \( (x_n) \) is a relatively compact set in \( S(X) \) (resp. \( \|x_n - x\| \to 0 \)). If every \( x \in S(X) \) is a CLU-point, then we say that \( X \) is compactly locally uniformly rotund (CLU) for short). If every \( x \in S(X) \) is a LUR-point, then we say that \( X \) is locally uniformly rotund (LUR) for short). The importance of the notion of SU-point follows from the fact that \( x \in S(X) \) is a CLUR-point if and only if \( x \) is a SU-point as well as that “SU-point” cannot be replaced in this equivalence by “extreme point” (see [3]). Moreover, a point \( x \in S(X) \) is an SU-point if and only if any \( x^* \in \operatorname{Grad}(x) \) exposes \( x \), that is, if \( y \in S(X) \) and \( x^*(y) = 1 \), then \( x = y \) (see [10], Remark 2). In consequence any SU-point of \( S(X) \) is an exposed point. It is also known that in Köthe function (or sequence) spaces, if \( x \in S(X) \) is an SU-point, then \( |x| \) is a point of upper monotonicity as well as a point of lower monotonicity (see [10], Lemma 7). For example, the point \((1,1)\) is an extreme point of \( S(l_{\infty}^0) \), but it is not a point of lower monotonicity because \( \|(1,0)\|_{\infty} = \|(1,1)\|_{\infty} = 1 \) and \((1,0) \preceq (1,1), (1,0) \neq (1,1)\).

A Banach space \( X \) is said to have the Kadec–Klee property \((X \text{ has the } H\text{-property)} \) or \( X \in (H) \) for short if the weak convergence and the convergence in norm of sequences in \( S(X) \) to a limit element from \( S(X) \) coincide. It is known that the CLUR property implies the H-property.

A sequence \( \Phi = (\Phi_i)_{i=1}^{\infty} \) is called a Musielak–Orlicz function, if for any \( i \in \mathbb{N} \), the function \( \Phi_i : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\} \) is convex, even, vanishing at 0, \( \lim_{u \to \infty} \Phi_i(u) = \infty \) and \( \Phi_i(u_i) < \infty \) for some \( u_i \in (0,\infty) \). The function \( \Psi = (\Psi_i)_{i=1}^{\infty} \) with \( \Phi_i(v) = \sup_{u \geq 0} \{u|v| - \Phi_i(u)\} \) for any \( i \in \mathbb{N} \) and \( v \in \mathbb{R} \) is called the function complementary to \( \Phi \) in the sense of Young. The function \( \Psi \) is again a Musielak–Orlicz function.

Denote by \( l_0 \) the space of all real sequences and define \( e_n = \chi_{\{n\}} \) for
any \( n \in \mathbb{N} \). Given a Musielak–Orlicz function \( \Phi \), we define the Musielak–Orlicz sequence space \( l_\Phi \), by

\[
l_\Phi = \{ x = (x_i)_{i=1}^\infty \in l_0 : I_\Phi(\lambda x) < \infty \text{ for some } \lambda > 0 \},
\]

where

\[
l_\Phi(x) = \sum_{i=1}^\infty \Phi_i(x_i).
\]

This space equipped with the Luxemburg norm

\[
\|x\|_\Phi = \inf \{ \lambda > 0 : I_\Phi(x/\lambda) \leq 1 \}
\]

or with the equivalent one

\[
\|x\|_{\Phi}^0 = \inf_{k>0} \frac{1}{k} (1 + I_\Phi(kx))
\]

called the Orlicz (or the Amemiya) norm is a Banach space. The spaces \((l_\Phi, \|\cdot\|_\Phi)\) and \((l_\Phi, \|\cdot\|_{\Phi}^0)\) will be denoted shortly by \(l_\Phi\) and \(l_\Phi^0\), respectively.

Moreover, we define the space

\[
\{ x = (x_i)_{i=1}^\infty \in l_0 : \forall \lambda > 0 \exists i_\lambda \in \mathbb{N} \text{ s.t. } \sum_{i=i_\lambda}^{\infty} \Phi_i(\lambda|x(i)|) < \infty \},
\]

and it will be denoted shortly by \(h_\Phi\) and \(h_\Phi^0\), respectively. For \( x \in l_\Phi \) we denote \( \theta(x) = \inf \{ \lambda > 0 : \sum_{i=i_\lambda}^{\infty} \Phi_i(\lambda|x(i)|) < \infty \text{ for some } i_\lambda \in \mathbb{N} \text{ depending on } \lambda \} \).

Let \( p_i^-(u), p_i^+(u) \) (\( q_i^-(u), q_i^+(u) \) respectively) denote the left and right derivatives of \( \Phi_i \) (\( \Psi_i \) respectively) at \( u \). We have the Young inequality

\[
uv \leq \Phi_i(u) + \Psi_i(v)
\]

for all \( u, v \geq 0 \); and

\[
uv = \Phi_i(u) + \Psi_i(v) \iff p_i^-(u) \leq v \leq p_i^+(u) \quad \text{or} \quad q_i^-(v) \leq u \leq q_i^+(v).
\]

For any \( x \in l_\Phi^0 \), we define

\[
k^*(x) = \inf \{ k \geq 0 : I_\Phi(p^+(k|x|)) \geq 1 \};
\]

\[
k^{**}(x) = \sup \{ k \geq 0 : I_\Phi(p^+(k|x|)) \leq 1 \};
\]
\[
K(x) = \begin{cases}
[k^*(x), k^{**}(x)], & \text{if } k^{**}(x) < \infty; \\
[k^*(x), \infty) & \text{if } k^*(x) < \infty \text{ and } k^{**}(x) = \infty.
\end{cases}
\]

We usually will write \(k^*, k^{**}\) instead of \(k^*(x), k^{**}(x)\) if it is clear which \(x\) is considered. It is known that \(\|x\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx))\) if and only if \(k \in K(x)\) for any \(x \in l_\Phi\) (see [21]).

The dual space of \(l_\Phi\) is well known. Namely, we have \((l_\Phi)^* = l_\Psi \oplus S\), that is, any \(x^* \in (l_\Phi)^*\) is uniquely represented in the form

\[x^* = \xi_v + \phi,\]

where \(v \in l_\Psi\) and \(\xi_v\) is the order continuous functional on \(l_\Phi\) generated by \(v\), that is,

\[\xi_v(x) = \sum_{i=1}^{\infty} v_i x_i \quad (\forall x \in l_\Phi)\]

and \(\phi \in S\) is a linear singular functional on \(l_\Phi\), that is, \(\phi(x) = 0\) for any \(x \in h_\Phi\) (order continuous functionals are also called regular functionals).

We denote by \(R \text{Grad}(x), (S \text{Grad}(y))\), respectively) the set of all regular (singular, respectively) support functionals at \(x\).

Let \(N_0\) be a subset of \(\mathbb{N}\). We say that a Musielak–Orlicz function \(\Phi : \mathbb{R} \to \mathbb{R}_+ \cup \{\infty\}\) is upper (lower) affine at \(w \in \mathbb{R}\) if there exists \(v \in \mathbb{R}\) with \(|v| > |w|\) (\(|v| < |w|\)) such that \(\Phi^i\) is affine on the intervals \([|w|, |v|]\) and \([-|v|, -|w|]\) (respectively on the intervals \([|v|, |w|]\) and \([-|w|, -|v|]\)). We introduce the following notations:

\[A_{u}^{i} = \{z \in \mathbb{R} : \Phi_{i} \text{ is upper affine at } z\},\]
On some global and local geometric properties...

\[ A^1_i = \{ z \in \mathbb{R} : \Phi_i \text{ is lower affine at } z \} \]

\[ AS^0_u = \{ z \in A^1_u : p_i^-(z) = p_i^+(z) \}, \quad AS^i = \{ z \in A^1_i : p_i^-(z) = p_i^+(z) \} \]

\[ ANS^0_u = A^1_u \setminus AS^0_u, \quad ANS^i = A^1_i \setminus AS^i \]

\[ A_u(x) = \{ i \in \mathbb{N} : x_i \in A^1_u \}, \quad A_l(x) = \{ i \in \mathbb{N} : x_i \in A^1_l \}, \]

\[ A^a_u(x) = \{ i \in A_u(x) : x_i \in AS^0_u \}, \quad A^a_l(x) = \{ i \in A_l(x) : x_i \in AS^i \}, \]

\[ A^{as}_u(x) = A_u(x) \setminus A^a_u(x), \quad A^{as}_l(x) = A_l(x) \setminus A^a_l(x). \]

Moreover, for every \( i \in \mathbb{N} \) we denote \( \text{Smooth}(\Phi_i) = \{ u \in \mathbb{R} : p_i^+(u) = p_i^-(u) \} \), \( a(\Phi_i) = \sup\{ u \geq 0 : \Phi_i(u) = 0 \} \), \( b(\Phi_i) = \sup\{ u \geq 0 : \Phi_i(u) < \infty \} \), \( \text{Ext}(\Phi_i) = \{ u \in \mathbb{R} : \Phi_i \text{ is strictly convex at } u \} \), \( \partial \Phi_i(u) = [p_i^-(u), p_i^+(u)] \).

If \( a(\Phi_i) = 0 \) \( (b(\Phi_i) = +\infty, \text{ respectively}) \) for all \( i \in \mathbb{N} \), then we will write \( \Phi > 0 \) \( (\Phi < \infty, \text{ respectively}) \).

We will use a few known results that we present below.

**Theorem 1.1** ([3]). Let \( X \) be a Banach space and \( x \in S(X) \). If \( x \) is an SU-point, then \( x \) is an exposed point.

**Theorem 1.2** ([1]). A point \( x \in S(l_\Phi) \) is exposed if and only if:

(i) \( I_\Phi(x) = 1 \),

(ii) \( \text{card}(\{ i \in \mathbb{N} : a(\Phi_i) > 0, x_i \in \text{Smooth}(\Phi_i) \}) = 0 \),

(iii) \( \text{card}(\{ i \in \mathbb{N} : x_i \notin \text{Ext}(\Phi_i) \}) \leq 1 \),

(iv) \( (p_i^-(|x_i|))_{i \in \mathbb{N}} \in l_\Phi \),

(v) there are no pair \( i, j \in \mathbb{N} \) with \( i \neq j \) such that \( x_i \in AS^0_u \) and \( x_j \in AS^0_l \).

**Theorem 1.3** ([6]). A Musielak–Orlicz space \( l_\Phi^p \) has property \( H \) if and only if \( \Phi \in \delta_2 \) or \( \sum_{i=1}^\infty \Psi_i(c_i) \leq 1 \), where

\[
c_i = \begin{cases} b(\Phi_i), & \text{if } \Psi_i(b(\Phi_i)) < 1; \\ \Psi_i^{-1}(1), & \text{otherwise}. \end{cases}
\]

**Theorem 1.4** ([4]). Let \( X \) be a reflexive Banach space. Then both \( X \) and \( X^* \) have the CLUR-property if and only if the both \( X \) and \( X^* \) have the \( H \)-property.
Lemma 1.1 (see [20], [14]). Let $\Phi < \infty$, $x \in S(l_\Phi)$ and $I_\Phi(x) = 1$. Then $x^* \in R\text{Grad}(x)$ if and only if it is of the form

$$x^*(y) = \sum_{i=1}^{\infty} \eta_i y_i / \sum_{i=1}^{\infty} \eta_i x_i \quad (y \in l_\Phi),$$

where $\eta = (\eta_i) \in l_\Psi$ is such that $\eta_i \in \partial \Phi_i(x_i) = [p_i^-(|x_i|), p_i^+(|x_i|)]$ for any $i \in \mathbb{N}$.

Lemma 1.2 ([20]). If $\Psi \notin \bar{\delta}_2$, then there exists a sequence $(I_n) \subset \mathbb{N}$ ($0 = I_0 < I_1 < I_2 < \ldots$) and a family of sequences $(u^n_i)$ in $\mathbb{R}_+$ ($n = 1, 2, \ldots$; $i = I_{n-1} + 1, \ldots, I_n$) such that

$$\sum_{i=I_{n-1}+1}^{I_n} \Phi_i(u^n_i) > 1 \quad \text{for any } n \in \mathbb{N},$$

$$\Phi_i(u^n_i) \leq \frac{1}{n}, \quad \Phi_i\left(\frac{u^n_i}{2}\right) > \left(1 - \frac{1}{n}\right) \frac{\Phi_i(u^n_i)}{2}$$

for any $n \in \mathbb{N}$ and any $i \in \{I_{n-1} + 1, \ldots, I_n\}$.

Theorem 1.5 ([12]). Let $x \in S(l_\Phi^0)$ and $k \in K(x)$. Then

(i) if $k^* < k^{**}$, then there exists exactly one support functional at $x$ and it is generated by $w \text{sgn}(x)$, where $w_i = p_i^+(k^*x_i) = p_i^-(k^{**}x_i)$ for any $i \in \mathbb{N}$;

(ii) if $K(x) = \{k\}$, then

(a) $I_\Psi(p^-(kx)) \leq 1$ and $\theta(kx) \leq 1$;

(b) if $\theta(kx) < 1$, then $I_\Psi(p^+(kx)) \geq 1$. Moreover, each support functional at $x$ belongs to $l_\Psi$ and it is of the form : $w \text{sgn}(x)$, where $p_i^-(kx_i) \leq w_i \leq p_i^+(kx_i)$ for any $i \in \mathbb{N}$ and $I_\Psi(w) = 1$;

(c) if $\theta(kx) = 1$, then each support functional at $x$ is of the form $w \text{sgn}(x) + \phi$, where $p_i^- (k|x_i|) \leq w_i \leq p_i^+ (k|x_i|)$ for any $i \in \mathbb{N}$, $I_\Psi(w) + a = 1$, $\phi$ is a singular functional, $\phi(x) = a/k$ and $a \in [0, 1 - I_\Psi(p^-(kx))]$. 

2. Results

We start with two auxiliary lemmas and two remarks.

**Lemma 2.1.** In the Musielak–Orlicz space $h_\Phi$, the condition $I_\Phi(x) = 1$ is equivalent to the condition $\|x\|_\Phi = 1$ if and only if $\Phi_i(b(\Phi_i)) \geq 1$ for any $i \in \mathbb{N}$.

**Proof.** *Necessity.* Suppose that there exists $i_0 \in \mathbb{N}$ such that $\Phi_i_0(b(\Phi_i_0)) < 1$. Defining $x = b(\Phi_i_0)e_{i_0}$ we have $I_\Phi(x) < 1$ and $I_\Phi(\lambda x) = \infty$ for any $\lambda > 1$, which implies $\|x\|_\Phi = 1$.

*Sufficiency.* It suffices to show that $\|x\|_\Phi = 1$ implies $I_\Phi(x) = 1$ when $\Phi_i(b(\Phi_i)) \geq 1$ for any $i \in \mathbb{N}$. Assume that there exists $x \in h_\Phi$ such that $I_\Phi(x) < 1$. The function $f(\lambda) = I_\Phi(\lambda x)$ is convex. It follows from the definition of the space $h_\Phi$ that there exists $i_0 \in \mathbb{N}$ such that $\sum_{i=i_0}^{\infty} \Phi_i(2x_i) < \infty$. Moreover, there exists $\lambda_0 \in (1,2)$ such that $\sum_{i=1}^{i_0-1} \Phi_i(\lambda_0 x_i) < 1$, because of $I_\Phi(x) < 1$. Therefore, the function $f$ is real-valued in the interval $(0,\lambda_0)$, so it is continuous on this interval, by its convexity. Since $f(1) = I_\Phi(x) < 1$ and $f$ is continuous in a right-neighbourhood of 1, there exists $\lambda_1 > 1$ such that $I_\Phi(\lambda_1 x) < 1$, whence $\|\lambda_1 x\|_\Phi \leq 1$, which gives $\|x\|_\Phi \leq \frac{1}{\lambda_1} < 1$. This contradiction finishes the proof. \hfill \Box

Using similar techniques as in [11], we get

**Lemma 2.2.** Assume that $\Phi = (\Phi_i)$ is a Musielak–Orlicz function such that for any $i \in \mathbb{N}$ there exists $u_i \in \mathbb{R}$ satisfying $\Phi_i(u_i) = 1$. Let $\Phi \in \delta_2$, $x \in l_\Phi$, $(x_n) \subset l_\Phi$, $x_n \to x$ coordinatewise and $\|x_n\|_\Phi \to \|x\|_\Phi$. Then $\|x_n - x\|_\Phi \to 0$.

**Remark 2.1.** For a Banach space $X$ and $x \in S(X)$ define the following condition

(A) \hspace{1cm} If $y \in S(X)$ and $\|x + y\| = 2$, then Grad($x$) = Grad($y$).

Then the point $x$ is an SU-point if and only if $x$ is an exposed point and condition (A) holds.

**Proof.** $\Leftarrow$ Let $x,y \in S(X)$ and $\|x + y\| = 2$. Then condition (A) gives Grad($y$) = Grad($x$). Using the fact that $x$ is an exposed point we get $x = y$. 

\hfill \Box
First we will show that any SU-point in $S(X)$ is a nonexpansive point. Let $y \in S(X)$ and $x^*(x) = x^*(y) = 1$ for $x^* \in S(X^*)$. Then $x^*(x + y) = 2$. Therefore
\[ 2 = \|x^*\|\|x + y\| \geq |x^*(x + y)| = 2, \]
whence $\|x + y\| = 2$. By the assumption that $x$ is an SU-point we get that $x = y$, which means $x$ is an exposed point. Now assume that $x \in S(X)$ is a SU-point, $y \in S(X)$, $\|x + y\| = 2$. Then $x = y$ and so $\text{Grad}(y) = \text{Grad}(x)$. Therefore we have obtained condition $(A)$. \hfill \Box

Remark 2.2. For a Banach space $X$ and $x \in S(X)$ define the following condition
\[(B) \quad \text{If } (x_n) \subset S(X), x^* \in \text{Grad}(x) \text{ and } \|x + x_n\| \to 2, \text{ then } x^*(x_n) \to 1. \]

Then $x$ is a LUR-point if and only if $x$ is a strongly exposed point and condition (B) holds.

**Proof.** $\Rightarrow$ It is well known that a LUR-point is a strongly exposed point. Let $x$ be a LUR-point and $\|x + x_n\| \to 2$. Then $\|x - x_n\| \to 0$, which implies that $x_n \to x$, i.e. $x^*(x_n) \to x^*(x) = 1$. Therefore we have obtained condition (B).

$\Leftarrow$ Let $\|x_n + x\| \to 2$ and $x^* \in S(X^*)$ be a functional which exposes $x$ strongly. Then $x^* \in \text{Grad}(x)$ and by condition (B) we get $x^*(x_n) \to 1 = x^*(x)$, i.e. $x^*(x_n - x) \to 0$ for any $x^* \in \text{Grad}(x)$. Since $x^*$ strongly exposes $x$, so $\|x_n - x\| \to 0$. \hfill \Box

**Theorem 2.1.** Let $\Phi < \infty$ and $x \in S(l_\Phi)$. Then $x$ is an SU-point if and only if:

(i) $I_\Phi(x) = 1$,
(ii) $\text{card}(\{i \in \mathbb{N} : a(\Phi_i) > 0, x_i \in \text{Smooth} (\Phi_i)\}) = 0$,
(iii) $\text{card}(\{i \in \mathbb{N} : x_i \notin \text{Ext}(\Phi_i)\}) \leq 1$,
(iv) there are no $i, j \in \mathbb{N}$, $i \neq j$, such that $x_i \in A_i^j$ and $x_j \in A_j^i$,
(v) $\theta(x) < 1$.

**Proof.** **Necessity.** Assume, without loss of generality that $x \geq 0$ and $x$ is an SU-point of $S(l_\Phi)$. By Theorem 1.1 and Theorem 1.2 we only need to show the necessity of conditions (v) and (iv). Suppose $\theta(x) \geq 1$. Since
\[
x \in S(I_\Phi) \text{ and } \theta(x) \leq \|x\|_\Phi, \text{ we have } \theta(x) = 1. \text{ Then } I_\Phi(\lambda x) = +\infty \text{ for any } \lambda > 1. \text{ Let } i_0 \in \mathbb{N} \text{ be such that } x_{i_0} \neq 0 \text{ and let } y = \{y_i\}, \text{ where }
\]
\[
y_i = \begin{cases} x_i, & i \neq i_0; \\
0, & i = i_0. 
\end{cases}
\]

Then \( I_\Phi(y) \leq 1 \), whence \( \|y\|_\Phi \leq 1 \), and \( I_\Phi(\lambda y) = +\infty \) for any \( \lambda > 1 \), which implies \( \|y\|_\Phi \geq 1 \). Consequently \( \|y\|_\Phi = 1 \). Therefore \( \|x + y\|_\Phi \leq 2 \).

Moreover, \( |y| \leq |x| \) and \( |y| \neq |x| \), which implies \( 2 = 2\|y\|_\Phi \leq \|x + y\|_\Phi \), whence we get \( \|x + y\|_\Phi = 2 \). This contradicts the assumption that \( x \) is an SU-point and gives the necessity of condition (v). Now we will show that condition (iv) is necessary. By Theorem 1.2 we need to consider only two cases: I) there exist \( i, j \in \mathbb{N}, i \neq j \), such that \( x_i \in AS_i^I \) and \( x_j \in ANS_j^I \); II) there exist \( i, j \in \mathbb{N}, i \neq j \), such that \( x_i \in ANS_i^I \) and \( x_j \in ANS_j^I \). We will prove only case I), because the proof of the second one is nearly the same. Assume that there exist \( i, j \in \mathbb{N}, i \neq j \), such that \( x_i \in AS_i^I \) and \( x_j \in ANS_j^I \). We choose \( a, b \in \mathbb{R}_+ \) such that
\[
\Phi_i(x_i) + \Phi_j(x_j) = \Phi_i(x_i + a) + \Phi_j(x_j - b)
\]
and define
\[
y = x\chi_{\mathbb{N}\setminus\{i,j\}} + (x_i + a)e_i + (x_j - b)e_j.
\]

Then \( I_\Phi(y) = I_\Phi(x) = 1 \), by condition (i). Let \( \eta_i = p_i^+(x_i) = p_i(x_i) \), \( \eta_j = p_j^-(x_j) \) and \( \eta_n = p_n(x_n) \) for \( n \notin \{i,j\} \). Then, by Lemma 1.1, the support functional \( x^* \) generated by \( \eta \) belongs to \( \text{Grad}(x) \cap \text{Grad}(y) \). So
\[
2 = x^*(x + y) \leq \|x + y\|_\Phi \leq 2,
\]
whence \( \|x + y\|_\Phi = 2 \). This contradicts the assumption that \( x \) is an SU-point.

In the same way we consider the case: \( x_i \in ANS_i^I \) and \( x_j \in ANS_j^I \) for some natural \( i \neq j \).

**Sufficiency.** Assume that conditions (i)–(v) hold and \( \|x\|_\Phi = \|y\|_\Phi = \|\frac{x+y}{2}\|_\Phi = 1 \). In the same way as in the proof of Theorem 5 in [3] we show that \( I_\Phi(y) = I_\Phi(\frac{x+y}{2}) = 1 \) and the coordinates \( x_i, y_i \) belong to the same interval of the affinity of the functions \( \Phi_i \), for any \( i \in \mathbb{N} \). If there exists \( i_0 \notin \text{Ext}(\Phi_i) \) (condition (iii)), then condition (ii) yields \( \Phi_i(x_{i_0}) \neq 0 \). Since \( x_{i_0} \) and \( y_{i_0} \) belong to the same interval of affinity of the function \( \Phi_i \) and
\[ I_\Phi(x) = I_\Phi(y) = 1, \] so we have \( x_{i_0} = y_{i_0}. \) If \( x_i \in \text{Ext}(\Phi_i) \cap A^i_u \) for some \( i \in \mathbb{N}, \) then, by condition (iv), \( x_j \in A^j_u \) for any natural \( j \neq i. \) Therefore \( x_i \leq y_i \) for any \( i \in \mathbb{N}. \) Now condition (i) implies that \( x_i = y_i \) for any \( i \in \mathbb{N}, \) because of \( I_\Phi(y) = 1. \) Similarly we consider the case \( x_i \in \text{Ext}(\Phi_i) \cap A^i_l \) for some \( i \in \mathbb{N}. \) □

**Theorem 2.2.** Let \( x \in S(L_\Phi) \) and \( b(\Phi(t, \cdot)) = +\infty \) for \( \mu \)-almost every \( t \in T. \) Then \( x \) is SU-point if and only if:

(i) \( I_\Phi(x) = 1, \)
(ii) \( \mu(\{t \in T : a(\Phi(t, \cdot)) > 0, x(t) \in \text{Smooth}(\Phi(t, \cdot))\}) = 0, \)
(iii) \( \mu(\{t \in T : x(t) \notin \text{Ext}(\Phi(t, \cdot))\}) = 0, \)
(iv) \( \mu(A_l(x)) \mu(A_u(x)) = 0, \)
(v) \( \theta(x) < 1. \)

**Proof.** The proof proceeds in the same way as the proof of Theorem 2.1, using Theorem 3 in [1] (which characterizes exposed points in \( L_\Phi \)) instead of Theorem 1.2. □

The next theorem is an analogue of Theorem 2.1, but it concerns the Orlicz norm.

**Theorem 2.3.** A point \( x \in S(l^\Phi) \) is an SU-point if and only if the following conditions hold:

(i) \( K(x) = \{k\} \) for some \( 0 < k < \infty, \)
(ii) \( A^i_l(kx) = A^i_u(kx) = \emptyset, \)
(iii) if \( A^{p+}_l(kx) \neq \emptyset, \) then \( \theta(kx) < 1 \) and \( I_\Phi(p^+(kx)\chi_{\mathbb{N} \setminus \{i\}}) + \Psi_i(p^-_l(kx_i)) < 1 \) for any \( i \in A^{p+}_l(kx). \)
(iv) if \( A^{p-}_u(kx) \neq \emptyset, \) then \( \theta(kx) < 1 \) and \( I_\Phi(p^-(kx)\chi_{\mathbb{N} \setminus \{i\}}) + \Psi_i(p^+_u(kx_i)) > 1 \) for any \( i \in A^{p-}_u(kx). \)

**Proof.** **Necessity.** Condition (i) is necessary in order that \( x \) is an extreme point (see [13] and [5]). Therefore, it is also necessary in order that \( x \) is an SU-point.

(ii) Assume for simplicity (but without loss of generality) that \( x \geq 0. \) Since condition (i) is necessary, we can assume in the remaining part of the proof that \( K(x) = \{k\}. \) Suppose that there exists \( i_0 \in A^i_l(kx). \) Then
the function $\Phi$ is affine in the interval $[kx_0 - b, kx_0]$ for some $b > 0$. We will consider two cases.

1° Let $\theta(kx) < 1$. Define

$$y = x\chi_{\mathbb{N}\setminus\{i_0\}} + \left(x_{i_0} - \text{sgn}(x_{i_0}) \frac{b}{k}\right) e_{i_0}. \quad (1)$$

Then $p_i^+(k y_i) = p_i^+(kx_i)$ for any $i \in \mathbb{N}$. Hence $I_\Psi(p^+(ky)) = I_\Psi(p^+(kx))$. Moreover, for $\alpha \neq k$, we have $I_\Psi(p^+(\alpha y)) \leq I_\Psi(p^+(\alpha x))$, which implies $k = k^*(x) \leq k^*(y)$. Since $0 \leq y \leq x$, we have $\theta(ky) \leq \theta(kx) < 1$. If $I_\Psi(p^+(kx)) < 1$, then by Theorem 1.5(ii)(b), we have that $\text{Grad}(x) = \emptyset$, which contradicts the Hahn–Banach theorem. Hence $I_\Psi(p^+(kx)) \geq 1$.

If $I_\Psi(p^+(kx)) > 1$, then $I_\Psi(p^+(ky)) > 1$, whence $k^*(y) \leq k$ and we get $k^*(y) = k$, by $k^*(y) \geq k$. Consequently $K(y) = \{k\}$, because in the opposite case $I_\Psi(p^+(kx)) = 1$, by Theorem 1.5(i) and the Young inequality. Since $k \in K(y)$ if and only if $k\|y\|^\Phi \in K(y/\|y\|^\Phi)$, so $K(y/\|y\|^\Phi) = \{k\|y\|^\Phi\}$. Moreover

$$p_i^- \left(k\|y\|^\Phi \frac{y_i}{\|y\|^\Phi} \right) = p_i^-(k y_i) = p_i^-(kx_i),$$

$$p_i^+ \left(k\|y\|^\Phi \frac{y_i}{\|y\|^\Phi} \right) = p_i^+(k y_i) = p_i^+(kx_i),$$

for any $i \in \mathbb{N}$, so by Theorem 1.5(ii)(b), $\text{Grad}(y/\|y\|^\Phi) = \text{Grad}(x)$.

If $I_\Psi(p^+(kx)) = 1$, then $I_\Psi(p^+(ky)) = 1$ and $k \in K(y)$, so $k^*(y) = k$, by $k^*(y) \geq k$, which has been already proved. If $k = k^*(y) < k^*(y)$ then, by Theorem 1.5(i), $p^+(kx)$ generates a support functional $x^*$ belonging to $\text{Grad}(x) \cap \text{Grad}(y/\|y\|^\Phi)$. If $k = k^*(y) = k^*(y)$ then, by Theorem 1.5(ii)(b), $p^+(kx)$ generates a support functional $x^*$ that belongs to $\text{Grad}(x) \cap \text{Grad}(y/\|y\|^\Phi)$. For such a functional we have

$$2 = x^* \left(x + \frac{y}{\|y\|^\Phi} \right) \leq \left\|x + \frac{y}{\|y\|^\Phi} \right\|^\Phi \leq 2,$$

whence $\|x + \frac{y}{\|y\|^\Phi} \|^\Phi = 2$, which means that $x$ is not an SU-point. This proves the necessity of the condition $A^\gamma(kx) = \emptyset$ for $k$ satisfying $K(k) = \{k\}$ whenever $\theta(kx) < 1$. 

On some global and local geometric properties... 51
2° Let $\theta(kx) = 1$. Define $y$ as in (1). Then $\theta(ky) = 1$, because in the opposite case there exists $l > k$ such that $I_\Phi(ly) < \infty$. On the other hand, by $I_\Phi(lx) = \infty$, we have

$$I_\Phi(ly) = I_\Phi(lx_n \{i_n\} + \Phi_{i_n}(ly_{i_n}) = I_\Phi(lx) + \Phi_{i_n}(ly_{i_n}) - \Phi_{\{i_n\}}(lx_{i_n}) = +\infty,$$

which gives a contradiction. By Theorem 1.5(ii)(c), the case $I_\Phi(p^+(kx)) \geq 1$ can be considered as the case 1°. So let $I_\Phi(p^+(kx)) < 1$. Then $I_\Phi(p^+(ky)) < 1$. Since $\theta(ky) = 1$, so $I_\Phi(\alpha y) = \infty$ for any $\alpha > k$, whence $k^+(y) = k$ and $K(y) = \{k\}$, by $k^+(y) = k$. It follows from Theorem 1.5(ii)(c) that any $y^* \in \text{Grad} \left( \frac{y}{\|y\|_\Phi} \right)$ is generated by the triple $(v, \phi, a)$, where $v \in l_\Phi$, $\phi$ is a singular functional, $a \geq 0$, $I_\Phi(v) + a = 1$, $\phi(y/\|y\|_\Phi) = a/(k\|y\|_\Phi)$, $a \in [0, 1 - I_\Phi(p^-(ky))]$ and

$$p_i^+(kx_i) = p_i^- \left( k\|y\|_\Phi \frac{y_i}{\|y\|_\Phi} \right) \leq v_i \leq p_i^+ \left( k\|y\|_\Phi \frac{y_i}{\|y\|_\Phi} \right) = p_i^+(kx_i)$$

for any $i \in \mathbb{N}$. Moreover

$$\sum_{i=1}^{\infty} \frac{y_i}{\|y\|_\Phi} \text{sgn}(y_i)v_i = 1 - \frac{a}{k\|y\|_\Phi},$$

because $I_\Phi(v) = 1 - a$, $I_\Phi(ky) = k\|y\|_\Phi - 1$ and

$$k\|y\|_\Phi \sum_{i=1}^{\infty} \frac{y_i}{\|y\|_\Phi} \text{sgn}(y_i)v_i = I_\Phi(ky) + I_\Phi(v).$$

Similarly

$$k \sum_{i=1}^{\infty} x_i \text{sgn}(x_i)v_i = I_\Phi(kx) + I_\Phi(v) = k - 1 + 1 - a = k - a.$$

Hence $\sum_{i=1}^{\infty} x_i \text{sgn}(x_i)v_i = 1 - \frac{a}{k}$. Moreover $x - y \in h_\Phi$, so $\phi(y) = \phi(x) = \frac{a}{k}$. It means that

$$\text{Grad} \left( \frac{y}{\|y\|_\Phi} \right) \subset \text{Grad}(x).$$

Now we can proceed as in the case 1° of the proof, obtaining that $x$ is not an SU-point.

In the same way we can show that $A_n^a(kx) = \emptyset$. 

52 Henryk Hudzik and Wojciech Kowalewski
(iii) By the previous part of the proof we can assume that conditions (i)–(ii) hold. Consider first the case \( \theta(kx) = 1 \). Assume that there exists \( i_0 \in A^\text{T_S}(kx) \). Then the function \( \Phi_{i_0} \) is affine on the interval \([kx_{i_0} - b, kx_{i_0}]\) for some \( b > 0 \). Define \( y \) as in (1). Then, similarly as in the proof of condition the necessity of (ii), we have \( \theta(ky) = 1 \). First we will show that

\[
I_\Psi(p^-(kx)) < 1.
\]  

Suppose for the contrary that \( I_\Psi(p^-(kx)) \geq 1 \). Since \( K(x) = \{k\} \), it follows from Theorem 1.5(ii)(a) that \( I_\Psi(p^-(kx)) \leq 1 \), whence

\[
I_\Psi(p^-(kx)) = 1.
\]  

Consequently, since \( i_0 \in A^\text{T_S}(kx) \), we get

\[
I_\Psi(p^+(kx)) > 1.
\]  

Since \( K(x) = \{k\} \), we get \( I_\Psi(p^+(\alpha x)) < 1 \) for any \( \alpha < k \), because in the opposite case \( k^*(x) \leq \alpha < k \). So by \( i_0 \in A^\text{T_S}(kx) \) and \( y \leq x \), we have

\[
I_\Psi(p^+(\alpha y)) < 1 \quad \text{for any } \alpha < k.
\]  

If \( I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus \{i_0\}}) + \Psi_{i_0}(p^{\text{lo}}(kx_{i_0})) < 1 \), then \( I_\Psi(p^+(kx)) < 1 \), which contradicts the equality \( I_\Psi(p^-(kx)) = 1 \), being a consequence of (3). So we should consider only two subcases.

3° Let \( I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus \{i_0\}}) + \Psi_{i_0}(p^{\text{lo}}(kx_{i_0})) > 1 \). Then \( I_\Psi(p^+(kx)) > 1 \), whence \( k^*(y) \leq k \) and, by (5), \( k^*(y) = k \). If \( k = k^* < k^\ast \), then it follows from Theorem 1.5(i) and from the Young inequality that \( I_\Psi(p^+(kx)) = 1 \), which contradicts the fact that \( I_\Psi(p^+(kx)) > 1 \), whence \( K(y) = \{k\} \). Therefore \( K(y/\|y\|_B) = \{k\} \). Moreover \( I_\Psi(p^-(kx)) = I_\Psi(p^-(kx)) = 1 \), by (3). By Theorem 1.5(ii)(c) we have that the support functional \( x^\ast \), generated by \( p^-(kx) \) belongs to \( \text{Grad}(x) = \text{Grad}(y/\|y\|_B) \). Proceeding as in the case 1° of the proof, we get that \( x \) is not an SU-point.

4° Let \( I_\Psi(p^+(kx)\chi_{\mathbb{N}\setminus \{i_0\}}) + \Psi_{i_0}(p^{\text{lo}}(kx_{i_0})) = 1 \). Note that in this case \( \text{supp}(x) \neq A^\text{T_S}(kx) \), because in the opposite case we have \( I_\Psi(p^-(kx)) < 1 \), which contradicts condition (3). So, in particular, we have \( \text{supp}(x) \neq \{i_0\} \). Moreover \( I_\Psi(p^+(kx)) = 1 \), whence \( k \in K(y) \) and \( I_\Psi(p^+(kx)) = 1 \).
\[ I_\Psi(p^-(ky)) = 1, \text{ by (3) and the fact that } I_\Psi(p^-(ky)) = I_\Psi(p^-(kx)). \] By the assumption that \( \Phi_i(u)/u \to 0 \) as \( u \to 0 \) for any \( i \in \mathbb{N} \), we get
\[ p_i^*(ky_i) = p_i^+(ky_i) \quad \text{for any } i \neq i_0, \ i \in \mathbb{N}. \] Condition (5) and \( I_\Psi(p^+(ky)) = 1 \) imply that \( k^*(y) = k \). By (6) we have \( I_\Psi(p^+(\alpha y)) > I_\Psi(p^+(ky)) = 1 \) for any \( \alpha > k \). Hence \( I_\Psi(p^+(ky)) = 1 \) implies that \( k^{**}(y) = k \). Finally \( K(y) = \{k\} \). Proceeding as in the case 3° of the proof, we get that \( x \) is not an SU-point.

Therefore, we have proved that if \( \theta(kx) = 1 \), then condition (2) is necessary in order that \( x \) is an SU-point and we can assume that this condition holds.

Now once again consider \( y \) as in (1). If \( I_\Psi(p^+(ky)) \geq 1 \), then, as above (cases 3° and 4° of the proof), we have \( k^*(y) = k \). If \( k^{**}(y) = k^* = k \), then, by condition (2) and Theorem 1.5(ii)(c), the support functional \( x^* \in \text{Grad}(x) \), which is generated by the triple \( (w, \phi_1, a_1) \), where \( w = (p_i^*(kx_i))^{y_i} = (p_i^+(ky_i))^{y_i} \) and \( \phi_1 \) satisfy conditions from Theorem 1.5(ii)(c), also belongs to \( \text{Grad}(y/\|y\|_\Psi^*) \). Proceeding with \( x^* \) as in the case 1° of the proof, we get \( \|x + y\|_\Psi^* = 2 \). It means that \( x \) is not an SU-point. If \( k^{**}(y) > k^* = k \), then, by Theorem 1.5(i) and the Young inequality, we get \( I_\Psi(p^+(ky)) = 1 \), whence \( k^{**}(y) \geq k^* \). Since \( \theta(ky) = 1 \), so \( I_\Psi(\alpha y) = \infty \) for any \( \alpha > k \), whence \( k^{**}(y) = k \) and finally \( K(y) = k \). This contradiction shows that the case \( k^{**}(y) > k^* \) is impossible.

If \( I_\Psi(p^+(ky)) < 1 \), then \( k^{**}(y) \geq k \) and as above, using \( \theta(ky) = 1 \), we get \( k^{**}(y) = k \). From Theorem 1.5(i) and the Young inequality, we get that \( K(y) = \{k\} \), because in the opposite case \( I_\Psi(p^-(ky)) = 1 \). Now we proceed as above, obtaining a contradiction.

In order to prove condition (2) in the case \( \theta(kx) < 1 \) together with the assumption that there exists \( i_0 \in A^\Psi_i(kx) \) such that \( I_\Psi(p^+(kx))_{\chi_{N^i_i(i_0)}} + \Psi_{i_0}(p_i^-(kx_i)) \geq 1 \), we proceed exactly in the same way as in the cases 3° and 4° of the proof, using only Theorem 1.5(ii)(b) instead of Theorem 1.5(ii)(c). Then we have \( I_\Psi(p^-(kx)) < 1 \) and \( I_\Psi(p^+(kx)) > 1 \). Defining \( y \) as in (1), we get \( I_\Psi(p^+(ky)) \geq 1 \), whence \( k^*(y) \leq k \) and, by (5), \( k^*(y) = k \). We will consider two cases.

If \( I_\Psi(p^+(ky)) = 1 \), then the support functional \( x^* \), generated by \( p^+(ky) \), belongs to \( \text{Grad}(x) \cap \text{Grad}(y/\|y\|_\Psi^*) \), by Theorem 1.5(ii)(b) and the fact that \( p^-(kx) = p^-(ky) \leq p^+(ky) \leq p^+(kx) \). Therefore, as in
the case 1° of the proof, we have \( \|x + y\|_0^\Phi = 2 \). So, we get that \( x \) is not an SU-point.

If \( I_\Phi(p^+(ky)) > 1 \), then there exists \( A^{ns}(kx) \ni i_1 \neq i_0 \), because in the opposite case \( \text{Grad}(x) = \emptyset \), by Theorem 1.5(ii)(b). If

\[
\sum_{i = 1}^{\infty} \Psi_i(p^+_i(kx_i)) + \Psi_{i_0}(p^-_{i_0}(kx_{i_0})) + \Psi_{i_1}(p^-_{i_1}(kx_{i_1})) = 1,
\]

then the support functional \( x^* \), generated by \( v = (v_i)_i \), where \( v_i = p^+_i(kx_i) \) for \( i \notin \{i_0, i_1\} \), \( v_i = p^-_i(kx_i) \) for \( i \in \{i_0, i_1\} \), belongs to \( \text{Grad}(x) \cap \text{Grad}(y\|y\|_0^\Phi) \). So, as above, \( x \) is not an SU-point.

(iv) This case can be proved similarly to the case (iii), so we omit the proof. \( \square \)

\textbf{Sufficiency.} Assume that \( y \in S(l_k^\Phi) \), \( \|x + y\|_0^\Phi = 2 \). Let \( l \in K(y) \) and conditions (i)–(iv) hold. Then \( \frac{kl}{k+l} \in K(x + y) \) (see [4]) and \( \|x + y\|_0^\Phi = \|x\|_0^\Phi + \|y\|_0^\Phi \), whence

\[
\frac{(k + l) \left(1 + I_\Phi \left( \frac{kl}{k+l}(x + y) \right) \right)}{kl} = \frac{l(1 + I_\Phi(kx)) + k(1 + I_\Phi(ly))}{kl},
\]

\[
(l + k)I_\Phi \left( \frac{kl}{k+l}(x + y) \right) = lI_\Phi(kx) + kI_\Phi(ly),
\]

\[
\Phi_i \left( \frac{kl}{k+l}(x_i + y_i) \right) \leq \frac{l}{k+i} \sum_{i = 1}^{\infty} \Phi_i(kx_i) + \frac{k}{k+l} \sum_{i = 1}^{\infty} \Phi_i(ly_i),
\]

\[
\Phi_i \left( \frac{l}{k+l}kx_i + \frac{k}{k+l}ly_i \right) = \frac{l}{k+l} \Phi_i(kx_i) + \frac{k}{k+l} \Phi_i(ly_i)
\]

for all \( i \in \mathbb{N} \). It means that all functions \( \Phi_i \) are affine on the intervals \([\min\{kx_i, ly_i\}, \max\{kx_i, ly_i\}]\). Suppose that there exist \( i \neq j \) such that \( i \in A^{ns}_0(kx) \) and \( j \in A^{ns}_0(kx) \). Then, by conditions (iii) and (iv), we get

\[
1 > I_\Phi(p^+(kx)\chi_{N \setminus \{i,j\}}) + \Psi_i(p^-_i(kx_i)) + \Psi_j(p^+_j(kx_j))
\]

\[
\geq I_\Phi(p^+(kx)\chi_{N \setminus \{i,j\}}) + \Psi_i(p^-_i(kx_i)) + \Psi_j(p^+_j(kx_j))
\]

\[
= I_\Phi(p^+(kx)\chi_{N \setminus \{i,j\}}) + \Psi_j(p^+_j(kx_j)) > 1,
\]
a contradiction, whence either $N = A_l^{ns}(kx)$ or $N = A_u^{ns}(kx)$. Assume that $N = A_l^{ns}(kx)$. Then $l y_i \leq k x_i$ for any $i \in \mathbb{N}$ and since $x, y \in S(l_k^n)$, so $l \leq k$. By (iii) we get $\theta(kx) < 1$. Suppose that there exists $i_0 \in \mathbb{N}$ such that $l y_{i_0} < k x_{i_0}$. By condition (iii), we have

$$I_\Psi(p^+(kx)\chi_{N\setminus\{i_0\}}) + \Psi_{i_0}(p_{i_0}^-(kx_{i_0})) < 1. \quad (7)$$

Without loss of generality we can assume that $l y_i = k x_i$ for any natural $i \neq i_0$, because condition (7) holds for any $i \in \mathbb{N}$, not only for fixed $i_0$. Moreover $p_{i_0}^+(l y_{i_0}) = p_{i_0}^+(k x_{i_0})$ and $p_{i_0}^-(l y_i) \leq p_{i_0}^+(k x_i)$ for any $i \neq i_0$. Then, by (7), we get $I_\Psi(p^+(l y)) < 1$. Moreover, $l y \leq k x$, whence $\theta(l y) \leq \theta(kx) < 1$. Therefore, it follows from Theorem 1.5(ii)(b) that $\text{Grad}(y) = \emptyset$, which contradicts Hahn–Banach Theorem. Therefore $l y_{i_0} = k x_{i_0}$. It implies that $l y = k x$, and since $x, y \in S(l_k^n)$, so $k = l$ and finally $x = y$.

If $A_l^{ns}(kx) = \mathbb{N}$, then we proceed in the same way as above, using condition (iv) instead of condition (iii).

In a similar way we can prove the following

**Theorem 2.4.** A point $x \in S(L_\Phi^0)$ is an SU-point if and only if the following conditions hold:

(i) $K(x) = \{k\}$ for some $0 < k < \infty$,

(ii) $\mu(A_l^+(kx)) = \mu(A_u^+(kx)) = 0$,

(iii) if $\mu(A_l^{ns}(kx)) > 0$, then $\theta(kx) < 1$ and

$$I_\Psi(p^+(kx)\chi_{T \setminus E}) + I_\Psi(p^-(kx)\chi_E) < 1 \text{ for any } E \subset A_l^{ns}(kx) \text{ such that } \mu(E) > 0,$$

(iv) if $\mu(A_u^{ns}(kx)) > 0$, then $\theta(kx) < 1$ and

$$I_\Psi(p^-(kx)\chi_{T \setminus E}) + I_\Psi(p^+(kx)\chi_E) > 1 \text{ for any } E \subset A_u^{ns}(kx) \text{ such that } \mu(E) > 0.$$

**Theorem 2.5.** If $\Phi$ is a Musielak–Orlicz function such that $\Phi > 0$ and $\Phi_i(1) = 1$ for any $i \in \mathbb{N}$, then the following conditions are equivalent:

(i) $l_\Phi \in \text{(CLUR)},$

(ii) (a) $\Phi \in \delta_2$

(b) $\Psi \in \delta_2$ or $\Phi_i$ is strictly convex on $[0, \Phi_i^{-1}(1)]$ for every $i \in \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii)(a) The implication follows from the facts that $X \in \text{(CLUR)} \Rightarrow X \in (H)$ for any Banach space $X$ and $\Phi \notin \delta_2 \Rightarrow l_\Phi \notin (H)$, because Banach function lattices with property H are order continuous.
(i)⇒ (ii)(b) Assume that condition (ii)(b) does not hold, that is, \( \Psi \notin \partial \Omega \) and there is \( j \in \mathbb{N} \) and an interval \([u, v] \cup [0, \Phi_j^{-1}(1)]\) such that \( \Phi_j \) is affine on \([u, v]\). We may assume without loss of generality that \( j = 1 \) and \( u > 0 \). Denote \( w = (u + v)/2 \) and take \( x \in S(I_\Phi) \) with \( I_\Phi(x) = 1 \) and \( x(1) = v \). Let \((I_n)_{n=1}^\infty \in \mathbb{N} \) and \((u^n_i) \in \mathbb{R} \) be sequences from Lemma 1.2. Denote \( a = \Phi_1(v) - \Phi_1(w) \). Let \( k \in \mathbb{N} \) be the smallest number that satisfies \( \frac{1}{k} \leq a \). Since \( \Phi_i(u^n_l) \leq \frac{1}{n} \) for any \( i, l \in \mathbb{N} \), we have

\[
\frac{1}{k + n - 1} < \sum_{i=2}^{m_n} \Phi_i(u_i^{k+n-1}) \leq a.
\]

Continuing this procedure we can find a sequence \((m_n)_{n=1}^\infty \) with \( I_{k+n-2} + 1 \leq m_n < I_{k+n-1} \) for any \( n \in \mathbb{N} \) such that

\[
\frac{1}{k + n - 1} < \sum_{i=2}^{m_n} \Phi_i(u_i^{k+n-1}) \leq a.
\]

Let \( e_i = \chi_{\{i\}} \) be the basis vector for the space \( c_0 \) and define

\[
x_n = w e_1 + \sum_{i=2}^{I_{k+n-2}} x(i) e_i + \sum_{i=I_{k+n-2}+1}^{m_n} u_i^{k+n-1} e_i.
\]

Since \( \Phi_i(u_i^l) \leq \frac{1}{l} \) for any \( i, l \in \mathbb{N} \), we have

\[
a - \frac{1}{k + n - 1} < \sum_{i=I_{k+n-2}+1}^{m_n} \Phi_i(u_i^{k+n-1}) \leq a.
\]

Consequently

\[
1 - \frac{1}{k + n - 1} < I_\Phi(x_n) \leq 1
\]

and so

\[
1 - \frac{1}{k + n - 1} < \|x_n\|_\Phi \leq 1
\]

for any \( n \in \mathbb{N} \). Moreover

\[
1 \geq I_\Phi \left( \frac{x + x_n}{2} \right) = \Phi_1 \left( \frac{w + v}{2} \right) + \sum_{i=2}^{l_{k+n-2}} \Phi_i(x(i))
\]
+ \sum_{i=I_{k+n-2}+1}^{m_n} \Phi_i \left( \frac{u_i^{k+n-1}}{2} \right) \geq \frac{1}{2} \Phi_1(w) + \frac{1}{2} \Phi_1(v) + \sum_{i=2}^{I_{k+n-2}} \Phi_i(x(i)) \\
+ \frac{1 - \frac{1}{k+n-1}}{2} \left( \sum_{i=I_{k+n-2}+1}^{m_n+1} \Phi_i \left( \frac{u_i^{k+n-1}}{k+n-1} \right) - \frac{1}{k+n-1} \right) \\
= \sum_{i=2}^{I_{k+n-2}} \Phi_i(x(i)) - \frac{1}{2} \left\{ \Phi_1(v) - \Phi_1(w) \right\} \\
+ \frac{1 - \frac{1}{k+n-1}}{2} \left( \Phi_1(v) - \Phi_1(w) - \frac{1}{k+n-1} \right) \\
= \sum_{i=2}^{I_{k+n-2}} \Phi_i(x(i)) - \frac{1}{2(k+n-1)} \left( \Phi_1(v) - \Phi_1(w) \right) \\
- \frac{1 - \frac{1}{k+n-1}}{2(k+n-1)} \rightarrow I_\Phi(x) = 1.

Consequently \( \|\frac{x^m - x^n}{\Phi}\rightarrow 1 \) as \( n \rightarrow \infty \). Moreover

\[
I_\Phi(x_m - x_n) \geq \sum_{i=I_{k+n-2}+1}^{m_n} \Phi_i(u_i^{k+n-1}) \\
= \sum_{i=I_{k+n-2}+1}^{m_n+1} \Phi_i(u_i^{k+n-1}) - \Phi_{m_n+1}(u_i^{k+m_n}) \geq \Phi_1(v) - \Phi_1(w) - \frac{1}{k+m_n}
\]

for all \( m, n \in \mathbb{N} \). Consequently

\[
I_\Phi(x_m - x_n) \geq \frac{1}{2} \left\{ \Phi_1(v) - \Phi_1(w) \right\}
\]

for \( m, n \in \mathbb{N} \) large enough, which yields

\[
\|x_m - x_n\|_\Phi \geq \min \left( 1, \frac{1}{2} \left\{ \Phi_1(v) - \Phi_1(w) \right\} \right) > 0
\]

for \( m, n \in \mathbb{N} \) large enough. This means that sequence \( (x_n) \) has no Cauchy subsequences, that is, \( I_\Phi \notin (CLUR) \).

(ii) \( \Rightarrow \) (i) If \( \Phi \in \delta_2 \) and \( \Psi \in \delta_2 \), then the proof proceeds in the same way as the proof of the analogous fact in the case of the Orlicz spaces (see [3]),
so we omit it. Now assume that \( \Phi \in \delta_2 \), \( \|x\|_\Phi = \|x_n\|_\Phi = \|x+x_n\|_\Phi = 1 \) and \( \Phi_i \) is strictly convex on \([0, \Phi^{-1}(1)]\) for any \( i \in \mathbb{N} \). The last condition implies that \( \text{card}(A_i(x)) = 0 \). We define the sets

\[
A_{i, \varepsilon} = \{ n \in \mathbb{N} : |x_n(i)| \leq |x(i)| \text{ and } |x(i) - x_n(i)| \geq \varepsilon \},
\]

for any \( \varepsilon > 0 \) and any \( i \in \mathbb{N} \). First we will show that

\[
\text{card}(A_{i, \varepsilon}) < \infty \quad \text{for any } \varepsilon > 0 \quad \text{and } i \in \mathbb{N}.
\] (8)

Otherwise, there exists \( \varepsilon_0 > 0 \) and \( i_0 \in \mathbb{N} \) such that \( \text{card}(A_{i_0, \varepsilon_0}) = \infty \), that is, there exists a subsequence \( (x_{n_k}) \) of \( (x_n) \) which satisfies the conditions: \( |x_{n_k}(i_0)| \leq |x(i_0)| \) and \( |x_{n_k}(i_0) - x(i_0)| \geq \varepsilon_0 \). Since \( A_1(x) = \emptyset \), there exists \( \delta > 0 \) such that

\[
\Phi_{i_0} \left( \frac{x(i_0) + x_{n_k}(i_0)}{2} \right) \leq \frac{1 - \delta}{2} [\Phi_{i_0}(x(i_0)) + \Phi_{i_0}(x_{n_k}(i_0))],
\]

for any \( k \in A_{i_0, \varepsilon_0} \). By Lemma 2.1 we have \( I_\Phi(x) = I_\Phi(x_1) = 1 \). Moreover, \( I_\Phi(\frac{x+y}{2}) \to 1 \) as \( n \to \infty \). Then we get

\[
0 - \sum_{i=1}^{\infty} \left[ \frac{\Phi_i(x_{n_k}(i)) + \Phi_i(x(i))}{2} - \Phi_i \left( \frac{x_{n_k}(i) + x(i)}{2} \right) \right]
\]

\[
\geq \frac{\Phi_{i_0}(x_{n_k}(i_0)) + \Phi_{i_0}(x(i_0))}{2} - \Phi_{i_0} \left( \frac{x_{n_k}(i_0) + x(i_0)}{2} \right)
\]

\[
\geq \delta \Phi_{i_0} \left( \frac{x_{n_k}(i_0) + x(i_0)}{2} \right) \geq \delta \Phi_{i_0} \left( \frac{x_{n_k}(i_0) - x(i_0)}{2} \right) \geq \delta \Phi_{i_0} \left( \frac{\varepsilon_0}{2} \right),
\]

a contradiction, which shows that (8) is true. We will show now that \( x_n \to x \) coordinatewise. Suppose the opposite. Then there exist \( \varepsilon_0 > 0 \), \( i_0 \in \mathbb{N} \) and a subsequence \( (x_{n_k}) \) of \( (x_n) \) satisfying

\[
|x_{n_k}(i_0) - x(i_0)| > \varepsilon_0 \quad \text{for any } k \in \mathbb{N}.
\] (9)

By the previous part of the proof we have \( \text{card}(A_{i_0, \varepsilon_0}) < \infty \). By condition (9) and the definition of the set \( A_{i_0, \varepsilon_0} \) we conclude that there exists \( k_0 \in \mathbb{N} \) such that \( |x_{n_k}(i_0)| > |x(i_0)| \) for all \( k > k_0 \), whence we conclude from (9) the existence of \( \varepsilon_1 > 0 \) such that \( \Phi_{i_0}(x_{n_k}(i_0)) - \Phi_{i_0}(x(i_0)) > \varepsilon_1 \) for any \( k > k_0 \). Since \( I_\Phi(x) < \infty \), so there exists a natural number \( i_1 > i_0 \).
such that $\sum_{i \geq n} \Phi_i(x(i)) < \frac{e}{2}$. Moreover, we can assume, without loss of generality, that $x_n(i) \to x(i)$ for any $i \leq i_1, i \neq i_0$. Hence for any $i < i_1, i \neq i_0$, there exists $l_i \in \mathbb{N}$ such that $|\Phi_i(x_n(i)) - \Phi_i(x(i))| < \frac{e}{3}i_1$ for any $n > l_i$. Therefore, in the worst case we have

$$1 = I_\Phi(x_{n_k}) \geq I_{i_0}(|x_{n_k}(i_0)|) + \sum_{i \leq i_1 \atop i \neq i_0} \Phi_i(|x_{n_k}(i)|)$$

$$\geq \Phi_{i_0}(|x(i_0)|) + \varepsilon_1 + \sum_{i \leq i_1 \atop i \neq i_0} \Phi_i(|x(i)|) + \sum_{i \leq i_1 \atop i \neq i_0} [\Phi_i(|x_{n_k}(i)|) - \Phi_i(|x(i)|)]$$

$$> 1 + \frac{\varepsilon_1}{2} + \sum_{i \leq i_1 \atop i \neq i_0} [\Phi_i(|x_{n_k}(i)|) - \Phi_i(|x(i)|)] > 1 + \frac{\varepsilon_1}{2} \geq \frac{\varepsilon_1}{3} = 1 + \frac{\varepsilon_1}{6}$$

for any $k > \max\{k_0, l_1, l_2, \ldots, l_{i_2}\}$. This contradiction proves that $x_n \to x$ coordinatewise. Using Lemma 2.2 we can finish the proof. \hfill \Box

**Remark 2.3.** Let $B = \{i \in \mathbb{N} : b(\Phi_i) < \infty \}$ and $I_\Phi(b(\Phi)\chi_B)) < 1$. Then $l_\Phi \notin (\text{CLUR})$.

**Proof.** Assume first additionally that $\text{card}(B) = \infty$. Then we may assume, without loss of generality, that $b(\Phi_i) < \infty$ for any $i \in \mathbb{N}$. Define

$$x = \sum_{i \in \mathbb{N}} b(\Phi_i)e_i, \quad x_n = \sum_{i=1}^n b(\Phi_i)e_i.$$ 

It is clear that $\|x\|_\Phi = \|x_n\|_\Phi = 1$ for any $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $n > m$, we get $x_n - x_m = \sum_{i=m+1}^n b(\Phi_i)e_i$, so $\|x_n - x_m\|_\Phi = 1$. Moreover

$$\frac{x_n + x}{2} = \sum_{i=1}^n b(\Phi_i)e_i + \sum_{i=n+1}^\infty \frac{b(\Phi_i)}{2}e_i.$$

Therefore $I_\Phi(\frac{x_n + x}{2}) \leq I_\Phi(x) \leq 1$ and consequently, using the triangle inequality, we get $\|\frac{x_n + x}{2}\|_\Phi = 1$. This shows that $l_\Phi \notin (\text{CLUR})$.

Now let $\text{card}(B) < \infty$, $x = \sum_{i \in B} b(\Phi_i)e_i$ and $a = 1 - I_\Phi(x) > 0$. Define $x_n = x$ for $n \in B$ and

$$x_n(i) = \begin{cases} 
  x(i), & \text{for } i \in B; \\
  \Phi_n^{-1}(a), & i = n; \\
  0, & \text{otherwise}
\end{cases}$$

where $\Phi_n^{-1}(a)$ is defined as in Remark 2.2.
for $n \not\in B$. Therefore $I_{\Phi}(x_n) = I_{\Phi}(x) < 1$ for $n \in B$ and $I_{\Phi}(x_n) = I_{\Phi}(x) + a = 1$ for $n \not\in B$. Hence $\|x_n\|_{\Phi} = 1$ for any $n \in \mathbb{N}$. Moreover, $\frac{x_n + x}{2} = x$ for $n \in B$, and

$$\frac{x_n(i) + x(i)}{2} = \begin{cases} x(i), & \text{for } i \in B; \\ \frac{1}{2} \Phi_n^{-1}(a), & \text{for } i = n; \\ 0, & \text{otherwise} \end{cases}$$

for $n \not\in B$. Since $I_{\Phi}(x) \leq 1$ and $I_{\Phi}(x_n) \leq 1$ for any $n \in \mathbb{N}$, by convexity of $I_{\Phi}$, we get $I_{\Phi}(\frac{x_n + x}{2}) \leq 1$ and consequently $\|\frac{x_n + x}{2}\|_{\Phi} \leq 1$ for any $n \in \mathbb{N}$. Moreover, $I_{\Phi}(\lambda \frac{x_n + x}{2}) \geq I_{\Phi}(\lambda x) = \infty$ for any $\lambda > 1$, whence $\|\frac{x_n + x}{2}\|_{\Phi} \geq 1$. This proves that $\|\frac{x_n + x}{2}\|_{\Phi} = 1$ for any $n \in \mathbb{N}$. We also have for $n > m$ and $n, m \not\in B$:

$$x_n(i) - x_m(i) = \begin{cases} -\Phi_m^{-1}(a), & \text{for } i = m; \\ \Phi_n^{-1}(a), & \text{for } i = n; \\ 0, & \text{otherwise} \end{cases}$$

Therefore, $I_{\Phi}(x_n - x_m) = 2a$ for $m, n \not\in B$, $m \neq n$, so $\|x_n - x_m\|_{\Phi} \geq \min\{1, 2a\} > 0$ for $m, n \not\in B$, $m \neq n$. This shows that $l_{\Phi}$ is not CLUR. □

**Theorem 2.6.** Let $\Phi$ be a Musielak–Orlicz function such that $\Phi < \infty$ and $\lim_{u \to 0} \frac{\Phi_i(u)}{u} = 0$ for any $i \in \mathbb{N}$. Then $l_{\Phi}^p$ has property CLUR if and only if $\Phi \in \delta_2$ and $\Psi \in \delta_2$.

**Proof.** Sufficiency. By Theorem 1.3, $l_{\Phi}^p \in (H)$. By $\lim_{u \to 0} \frac{\Phi_i(u)}{u} = 0$ for any $i \in \mathbb{N}$, we get $\Psi < \infty$. Moreover, condition $\Psi \in \delta_2$ is equivalent to $l_{\Phi} \in (H)$ (see [8]). Since our assumptions imply reflexivity of $l_{\Phi}^p$, so by Theorem 1.4 we get $l_{\Phi}^p \in (CLUR)$.

Necessity. Since property CLUR implies property $H$ and property $H$ implies order continuity (see [7]), we have $\Phi \in \delta_2$. Now suppose $\Psi \notin \delta_2$. Then (see [8]) there exists a sequence $(u_n) \subset \mathbb{R}$, $u_n \geq 0$, and sequence of sets $(E_n)$: $E_n \in 2^N$, $E_n \cap E_m = \emptyset$ for $m \neq n$ such that:

(a) $\frac{1}{2^{m+1}} < \sum_{i \in E_n} \Psi_i(u_n(i)) \leq \frac{1}{2^m}$,

(b) $\Psi_i((1 + \frac{1}{2^m})u_n(i)) > 2^{m+1} \Psi_i(u_n(i))$ for any $i \in E_n$. 
Define the sequence \((z_n)\) in \(l_\Psi\) by
\[
  z_n(i) = \begin{cases} 
  u_n(i), & \text{if } i \in E_n; \\
  0, & \text{otherwise}.
  \end{cases}
\]

Then
\[
  I_\Psi(z_n) = \sum_{i=1}^{\infty} \Psi_i(z_n(i)) = \sum_{i \in E_n} \Psi_i(z_n(i)) \leq \frac{1}{2^n}.
\]

Hence \(\|z_n\|_\Psi \leq 1\). If \(\lambda = \frac{2^n}{2^{n+1} + 1}\), then
\[
  I_\Psi\left(\frac{z_n}{\lambda}\right) = \sum_{i \in E_n} \Psi_i \left(\left(1 + \frac{1}{2^n}\right) u_n(i)\right) > \sum_{i \in E_n} 2^{n+1} \Psi_i(u_n(i)) > 1.
\]

Therefore \(\frac{2^n}{2^n + 1} < \|z_n\|_\Psi \leq 1\). Since \(z_n \in h_\Psi\) and \((h_\Psi)^* = l_\Phi\), so for any \(n \in \mathbb{N}\) there exists \(x_n \in S(l_\Phi^0)\) such that
\[
  \frac{2^n}{2^n + 1} < \|z_n\|_\Psi = \langle z_n, x_n \rangle = \sum_{i \in E_n} x_n(i) u_n(i). \tag{10}
\]

Let \(x \in S(l_\Phi^0)\) and \(x \geq 0\). Since \(\Phi \in \delta_2\), so \((l_\Phi^0)^* = l_\Psi\) and there exists \(y \in S(l_\Psi)\) such that \(\langle x, y \rangle = \sum_{i \in N} x_i y_i = \|x\|_\Psi^2\). Define
\[
  y_n = \frac{2^n}{2^n + 1} \left( u_n \chi_{E_n} + y \chi_{\mathbb{N}\setminus E_n} \right).
\]

Then
\[
  I_\Psi(y_n) \leq \frac{2^n}{2^n + 1} (I_\Psi(y) + I_\Psi(z_n)) \leq \frac{2^n}{2^n + 1} \left(1 + \frac{1}{2^n}\right) = 1. \tag{11}
\]

Moreover
\[
  \langle x, y_n \rangle \geq \frac{2^n}{2^n + 1} \sum_{i \in E_n} x_n(i) z_n(i) = \frac{2^n}{2^n + 1} \langle x_n, z_n \rangle \geq \left(\frac{2^n}{2^n + 1}\right)^2. \tag{12}
\]

By the definition of \(y_n\) and the fact that \(x \geq 0\) and \(z_n \geq 0\), we get
\[
  \langle x, y_n \rangle = \frac{2^n}{2^n + 1} \left[ \sum_{i \in E_n} x(i) u_n(i) + \sum_{i \notin E_n} x(i) y(i) \right].
\]
On some global and local geometric properties...

\[\sum_{i \in \mathbb{N}} x(i)y(i) \geq \frac{2^n}{2^{n+1}+1} \left[ (x, y) - \sum_{i \in E_n} x(i)y(i) \right]. \quad (13)\]

Moreover, the Young inequality yields

\[\sum_{i \in E_n} x(i)y(i) \leq I_{\Phi}(x\chi_{E_n}) + I_{\Psi}(y\chi_{E_n}) \to 0. \quad (14)\]

By (12), (13), and by the definition of the Orlicz norm, we have

\[2 \geq \|x + x_n\|_n^\Phi \geq (x + x_n, y_n) \geq \left( \frac{2^n}{2^{n+1}+1} \right)^2 + \frac{2^n}{2^{n+1}+1} \left[ 1 - \sum_{i \in E_n} x(i)y(i) \right] \to 2.\]

Hence \(\|x_n + x\|_n^\Phi \to 2.\) On the other hand, since supports of the elements \(x_n\) are separated, we get \(\|x_n - x_m\|_n^\Phi \geq \|x_n\|_n^\Phi = 1\) for any \(m, n \in \mathbb{N}\), \(m \neq n\). This contradicts the CLUR-property for \(l_\Phi^\circ\).

\[\square\]

References


