Soluble groups with many 2-generator torsion-by-nilpotent subgroups

By NADIR TRABELSI (Sétif)

Abstract. We prove in this paper that a finitely generated soluble group in which every infinite subset contains a pair of distinct elements $x, y$ such that $\langle x, y \rangle$ is torsion-by-nilpotent (respectively, $\langle x, x^y \rangle$ is Chernikov-by-nilpotent), is itself torsion-by-nilpotent (respectively, finite-by-nilpotent).

1. Introduction and results

Following a question of Erdős, B. H. Neumann proved in [18] that a group is centre-by-finite if, and only if, every infinite subset contains a commuting pair of distinct elements. Since this result, problems of similar nature have been the object of many papers (for example [1]–[7], [10], [15]–[17], [21]–[23]). In particular, in [15] Lennox and Wiegold considered the class $(\Omega, \infty)$ of groups in which every infinite subset contains two distinct elements generating an $\Omega$-group, where $\Omega$ is a given class of groups. They characterised finitely generated soluble groups which belong to $(\Omega, \infty)$ when $\Omega$ is the class of polycyclic, or nilpotent, or coherent groups. Here we will consider the class $(\Omega, \infty)$, when $\Omega$ is the class $\mathcal{TN}$ of torsion-by-nilpotent groups, or the class $\mathcal{CN}$ of Chernikov-by-nilpotent groups, and we will prove the following results:

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**Theorem 1.** Let $G$ be a finitely generated soluble group in the class $(T\mathcal{N}, \infty)$. Then $G$ is torsion-by-nilpotent.

Let $k$ be a positive integer and let $\mathcal{N}_k$ be the class of nilpotent groups of class at most $k$. In [2], Abdollahi and Taeri proved that a finitely generated metabelian group $G$ is in $(\mathcal{N}_k, \infty)$ if, and only if, $G/Z_k(G)$ is finite; and a finitely generated soluble group $G$ is in the class $(\mathcal{N}_k, \infty)$, if and only if, $G$ belongs to $\mathcal{F}\mathcal{N}^{(2)}_k$, where $\mathcal{F}$ is the class of finite groups and $\mathcal{N}^{(2)}_k$ denotes the class of groups whose 2-generated subgroups are nilpotent of class at most $k$. Also let $\mathcal{E}_k$ be the class of $k$-Engel groups. In [16], Longobardi proved that if $G$ is a finitely generated locally graded group in the class $(\mathcal{E}_k, \infty)$, then $G$ belongs to $\mathcal{F}\mathcal{E}_k$. Combining the results of [2], [16], and Theorem 1, we shall obtain the following consequences.

**Corollary 2.** Let $k$ be a positive integer.

(i) A finitely generated soluble group $G$ is in the class $(T\mathcal{N}_k, \infty)$ if and only if $G$ belongs to $T\mathcal{N}^{(2)}_k$.

(ii) A finitely generated metabelian group $G$ is in the class $(T\mathcal{N}_k, \infty)$ if and only if $G$ belongs to $T\mathcal{N}_k$.

(iii) A finitely generated soluble group $G$ is in the class $(T\mathcal{E}_k, \infty)$ if and only if $G$ belongs to $T\mathcal{E}_k$.

In the Chernikov-by-nilpotent case, we weaken the hypothesis by considering the class $(\mathcal{C}\mathcal{N}, \infty)^*$ of groups in which every infinite subset contains two distinct elements $x, y$ such that $\langle x, x^y \rangle$ is in $\mathcal{C}\mathcal{N}$. More precisely, we will prove the following result:

**Theorem 3.** Let $G$ be a finitely generated soluble group in the class $(\mathcal{C}\mathcal{N}, \infty)^*$. Then $G$ is finite-by-nilpotent.

Note that Theorem 3 improves the result of [22, Proposition 2], where it is proved that a finitely generated soluble group in the class $(\mathcal{F}\mathcal{N}, \infty)$ is finite-by-nilpotent.

Let $k$ be a positive integer and let $\mathcal{E}_k(\infty)$ be the class of groups in which every infinite subset contains two distinct elements $x, y$ such that $[x, k, y] = 1$. In [1], Abdollahi proved that a finitely generated metabelian group $G$ is in $\mathcal{E}_k(\infty)$ if, and only if, $G/Z_k(G)$ is finite, and if $G$ is a finitely
generated soluble group in the class $\mathcal{E}_k(\infty)$, then there exists an integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite. Note that $(\mathcal{N}_k, \infty)^*$ is contained in $\mathcal{E}_{k+1}(\infty)$. Combining the results of [1], [2], [16] and Theorem 3, we shall obtain the following consequences.

**Corollary 4.** Let $k$ be a positive integer.
(i) If $G$ is a finitely generated soluble group in the class $(\mathcal{CN}_k, \infty)^*$, then there is an integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite.
(ii) A finitely generated metabelian group is in the class $(\mathcal{CN}_k, \infty)^*$ if and only if $G/Z_{k+1}(G)$ is finite.

**Corollary 5.** Let $k$ be a positive integer.
(i) A finitely generated soluble group $G$ is in the class $(\mathcal{CN}_k, \infty)$ if and only if $G$ belongs to $\mathcal{F}\mathcal{N}_k^{(2)}$.
(ii) A finitely generated metabelian group $G$ is in the class $(\mathcal{CN}_k, \infty)$ if and only if $G/Z_k(G)$ is finite.
(iii) A finitely generated soluble group $G$ is in the class $(\mathcal{CE}_k, \infty)$ if and only if $G$ belongs to $\mathcal{F}\mathcal{E}_k$.

**2. Proof of the results**

To prove our theorems, we will use recent results of **Endimioni** and **Traustasson** [9] on torsion-by-nilpotent groups.

**Lemma 6.** Let $c > 0$ be an integer and let $G$ be a group in $\mathcal{N}_c T$. If $G$ belongs to $(T\mathcal{N}_c, \infty)$ then it is in $(T\mathcal{N}_c, \infty)$.

**Proof.** Let $x, y \in G$ such that $\langle x, y \rangle \in T\mathcal{N}$. Clearly $\langle x, y \rangle$ belongs also to $\mathcal{N}_c T$ and the set of its torsion elements is a subgroup $T$. Hence $\langle x, y \rangle / T$ is a torsion-free nilpotent group which belongs to $\mathcal{N}_c T$. It follows from [19, Lemma 6.33] that $\langle x, y \rangle / T \in \mathcal{N}_c$, so $\langle x, y \rangle \in T\mathcal{N}_c$. Consequently, if $G$ belongs to $(T\mathcal{N}_c, \infty)$, then it is in $(T\mathcal{N}_c, \infty)$. \hfill $\Box$

**Lemma 7.** Let $G$ be a soluble group in the class $(T\mathcal{N}, \infty)$. If $G$ is abelian-by-torsion then it is torsion-by-abelian.
Proof. By Lemma 6, $G$ belongs to $(\mathcal{T}A, \infty)$, where $\mathcal{A}$ denotes the class of abelian groups. First of all, we show that the set of torsion elements of $G$ is a subgroup. Let $x, y \in G$ be two elements of finite order. Then $H = \langle x, y \rangle$ is a finitely generated soluble group which belongs to $\mathcal{AT}$, so it is abelian-by-finite. Clearly we may assume $H$ infinite. Therefore $H$ has a torsion-free normal abelian subgroup $A$ of finite index. Let $1 \neq a \in A$ and let $h \in H$, then the subset $\{a^ih : i > 0\}$ is infinite. By the property $(\mathcal{T}A, \infty)$, there are two distinct positive integers $i, j$ such that $\langle a^ih, a^j h \rangle \in \mathcal{T}A$, so $\langle a^{i-j}, a^j h \rangle \in \mathcal{T}A$. Hence $\langle a^{i-j}, a^j h \rangle^m = 1$ for some positive integer $m$. Since $A$ is abelian and normal in $H$ we obtain $[a, h] = 1$, and this gives $[a, h] = 1$ as $A$ is torsion-free. It follows that $A$ is contained in the centre of $H$. So $H$ is a centre-by-finite group. Thus, by a result of Schur [19, Theorem 4.12], $H'$ is finite and therefore $H$ is a finitely generated finite-by-abelian group. This contradicts the fact that $H$ is infinite. Consequently, $H$ is a finite group, so $xy^{-1}$ is of finite order. This means that the elements of finite order in $G$ form a subgroup $T$, as claimed. Now $G/T$ is a torsion-free group in the class $(\mathcal{T}A, \infty)$. So $G/T$ belongs to $(\mathcal{A}, \infty)$. It follows by the result of B. H. Neumann [18] that $G/T$ is centre-by-finite. Thus $G/T$ is finite-by-abelian and, therefore, $G$ is torsion-by-abelian, as required.

Lemma 8. Let $G$ be a finitely generated abelian-by-nilpotent group with abelian Fitting subgroup $A$ and let $x \in G$. Suppose that for each $a \in A$, there are integers $n \geq 0$, $m_1 > 0$ and $m_2 > 0$ such that $[a, x^{m_1} x^{m_2}] = 1$. Then there is a positive integer $d$, depending only on $G$, such that $x^d \in A$.

Proof. Since $G$ is a finitely generated abelian-by-nilpotent group, we may therefore apply a result of Lennox and Roseblade [14, Theorem B], which asserts that in a finitely generated abelian-by-nilpotent group $G$, there is a positive integer $d$, depending only on $G$, such that for all $i > 0$ and for all $g$ in $G$ the inclusion $C_G(g^i) \leq C_G(g^d)$ holds. We firstly show by induction on $n$ that if $a$ is an element of $A$ satisfying the hypothesis of the lemma, then $[a, x^n x^d] = 1$. If $n = 0$, then we have $[a, x^{m_1}] = 1$ hence $[a, x^d] = 1$, as desired. Now assume that $n > 0$ and $[a, x^{m_1}, x^{m_2}] = 1$. So we obtain $[a, x^{m_1}, x^{m_2}, x^d] = 1$. Now $\langle a, x \rangle$ being metabelian, it is easy to see that $[a, x^i, x^j] = [a, x^j, x^i]$ for any integers $i$, $j$. Thus we get

\[ [a, x^i, x^j] = [a, x^j, x^i] \]
that \( [a, x^d, x^{m_1} x^{m_2}] = 1 \), and by the inductive hypothesis we obtain \( [a, n+1 x^d] = 1 \), as required.

Now consider the subgroup \( K = \langle A, x \rangle \). Since \( G/A \) is nilpotent, \( K \) is subnormal in \( G \). For every \( y \in K \), there exist \( a \in A \) and an integer \( r \) such that \( y = x^r a \). As we have just shown, there is a positive integer \( d \) such that \( [a, n+1 x^d] = 1 \) for some non-negative integer \( n \), so \( y^n x^d \) and \( x^r a, n+1 x^d \) are left Engel elements of \( K \). Since \( K \) is soluble, the set of its left Engel elements coincides with its Hirsch–Plotkin radical \( A_1 \) [19, Theorem 7.34], so \( x^d \in A_1 \). Since \( K \) is subnormal in \( G \), \( A_1 \) is a subnormal locally nilpotent subgroup in \( G \). So \( A_1 \) is contained in the Hirsch–Plotkin radical of \( G \) [20, 12.1.4]. Now \( G \) is a finitely generated abelian-by-nilpotent group, so it satisfies the maximal condition on normal subgroups [12]. Therefore the Hirsch–Plotkin radical of \( G \) coincides with its Fitting subgroup, hence \( x^d \in A \) as claimed. \( \square \)

**Proof of Theorem 1.** Let \( G \) be a finitely generated soluble group in the class \((TN, \infty)\). To prove that \( G \) is torsion-by-nilpotent, we proceed by induction on the derived length \( d \) of \( G \). If \( d = 1 \) there is nothing to prove, so we can assume \( d > 1 \). By the inductive hypothesis, \( G/G^{(d-1)} \) is torsion-by-nilpotent. Thus \( G \) is in the class \((AT)N\), and by Lemma 7 it belongs to \( T(AN) \). Therefore, we may suppose \( G \) abelian-by-nilpotent, so \( G \) satisfies the maximal condition on normal subgroups [12] and \((TN, \infty)\) is a quotien closed class, we may assume that \( G \) is a just-non-(torsion-by-nilpotent) group, that is, \( G \notin TN \) but every proper quotient of \( G \) is torsion-by-nilpotent. In [9, Corollary 1.3], it is proved that if \( H \) is a normal subgroup of a locally soluble group \( G \) such that \( H \) and \( G/H' \) are torsion-by-nilpotent, then \( G \) is torsion-by-nilpotent. It follows that every normal torsion-by-nilpotent subgroup of \( G \) is abelian. In particular, the Fitting subgroup \( A \) of \( G \), is abelian. Moreover, it is easy to see that any normal torsion subgroup of \( G \) must be trivial. Thus \( A \) is torsion-free. Let \( 1 \neq a \in A \) and let \( xA \) be an element of infinite order in \( G/A \). Then the subset \( \{ x^i a : i > 0 \} \) is infinite. Hence there exist two positive integers \( i, j \) such that \( \langle x^i a, x^j a \rangle \) is torsion-by-nilpotent. So \( \langle x^i a, x^{i-j} \rangle \) is torsion-by-nilpotent. Then there is an integer \( n \geq 0 \) such that \( \gamma_{n+1}(\langle x^i a, x^{i-j} \rangle) \) is a torsion group. If \( n = 0 \), then \( \langle x^i a, x^{i-j} \rangle \) is a torsion group. So \( (x^i a)^m = 1 \) for some positive integer \( m \). Hence \( x^{im} \in A \), this is a contradiction and so
Thus there is a positive integer \( m \) such that \( [a, x^{i-j}]^m = 1 \). Hence \([a, x^{i-j}] = 1\) as \( A \) is torsion-free. It follows by Lemma 8 that there exists a positive integer \( d \) such that \( x^d \in A \), this is a contradiction and so \( G/A \) is a torsion group. Therefore \( G \) is abelian-by-finite, so by Lemma 7 \( G \) is torsion-by-abelian, a contradiction which completes the proof.

**Proof of Corollary 2.** Let \( k \) be a positive integer.

(i) If \( G \) is a finitely generated soluble group in \((TN_k, \infty)\), then from Theorem 1, \( G \) is torsion-nilpotent. Thus \( G \) has a torsion subgroup \( T \). Clearly \( G/T \) is in \((TN_k, \infty)\), hence \( G/T \) being torsion-free is in \((N_k, \infty)\). So by [2], \( G/T \in FN^{(2)}_k \). Consequently, \( G \in TN^{(2)}_k \), as required. It is easy to see that if \( G \) is in \( TN^{(2)}_k \), then it belongs to \((TN_k, \infty)\).

(ii) If \( G \) is a finitely generated metabelian group in \((TN_k, \infty)\), then as in (i) there is a torsion normal subgroup \( T \) such that \( G/T \) is a finitely generated metabelian group in \((N_k, \infty)\). So by [2], \( G/T \in FN_k \). Thus \( G \in TN_k \), as required. The converse is obvious.

(iii) Let \( G \) be a finitely generated soluble group in the class \((T \mathcal{E}_k, \infty)\). Since soluble Engel groups are locally nilpotent [20, 12.3.3], \( G \) belongs to \((TN, \infty)\). It follows, by Theorem 1, that \( G \) is torsion-nilpotent. Let \( T \) be the torsion subgroup of \( G \). So \( G/T \) is a torsion-free group in the class \((T \mathcal{E}_k, \infty)\). We deduce that \( G/T \) is in \((\mathcal{E}_k, \infty)\). It follows, from [16], that \( G/T \) is in \( \mathcal{F}E_k \). Thus \( G \) is in \( T \mathcal{E}_k \). The converse is obvious.

**Lemma 9.** Let \( G \) be a finitely generated soluble group in the class \((CN, \infty)^*\). Then \( G \) is nilpotent-by-finite.

**Proof.** Let \( G \) be a finitely generated soluble group in the class \((CN, \infty)^*\). By [8, Corollary 2] \( G \) is nilpotent-by-finite if, and only if, for each 2-generator subgroup \( H \), the factor group \( H/H'' \) is nilpotent-by-finite. It follows that we may assume \( G \) metabelian. Since \((CN, \infty)^*\) is a quotient closed class of groups and finitely generated nilpotent-by-finite groups are finitely presented, it follows, by [19, Lemma 6.17], that we may suppose that \( G \) is a just-non-(nilpotent-by-finite) group. In [13, Lemma 2.1] it is proved that the fitting subgroup \( A \) of \( G \) is therefore abelian and either \( A \) is torsion-free, or it is an elementary abelian \( p \)-group of infinite rank for some prime \( p \). Let \( 1 \neq a \in A \) and let \( xA \) be an element of infinite order in \( G/A \). Then the subset \( \{x^ia : i > 0\} \) is infinite. Hence there exist two positive
There exists a positive integer \( m \) in finite. So there are two positive integers \( i, j \) such that \( \langle (x^i a)^{x^j a}, x^i a \rangle = \langle [x^j a, x^i a], x^i a \rangle \) is Chernikov-by-nilpotent. Using the facts that \( A \) is abelian and normal in \( G \) we have \([x^j a, x^i a] = [x^j, a][a, x^i] = [a, x^{-j}]^{x^j}[a, x^i] = [a, x^i x^{-j}]^{x^j} = [a^{x^j}, x^{-j}]\). Set \( H = \langle [a^{x^j}, x^{-j}], x^i a \rangle \), then there is an integer \( n \geq 0 \) such that \( \gamma_{n+1}(H) \) is a Chernikov group. On the other hand \( \gamma_2(H) \) is contained in \( A \) as \( G \) is metabelian. If \( n = 0 \), then \( H \) is finite since Chernikov groups are locally finite. So \((x^i a)^m = 1\) for some positive integer \( m \). Hence \( x^m \in A \), this is a contradiction and so \( n > 0 \). It follows that \( \gamma_{n+1}(H) \) is a Chernikov subgroup of \( A \).

Suppose that \( A \) is torsion-free. Then \( \gamma_{n+1}(H) = 1 \) and hence \([a^{x^j}, x^{-j}, n, x^i a] = 1\), so \([a, x^{-j}, n, x^i] = 1\). By Lemma 8 there is, therefore, a positive integer \( d \) such that \( x^d \in A \), and this contradicts the fact that \( xA \) is of infinite order.

It follows that we may assume that \( A \) is an elementary abelian \( p \)-group. So \( \gamma_{n+1}(H) \) is a Chernikov and an elementary abelian \( p \)-group, hence finite. Thus \( H \) is finite-by-nilpotent, so \( H \) is nilpotent-by-finite. Therefore there exists a positive integer \( m \) such that \([a^{x^j}, x^{-j}, n+1, (x^i a)^m] = 1\), so \([a, x^{-j}, n+1, x^m] = 1\). This gives, by Lemma 8, that \( x^d \in A \), for some positive integer \( d \), a contradiction which completes the proof.

**Corollary 10.** Let \( G \) be a finitely generated soluble group. Then, \( G \in (CN, \infty)^* \) if and only if \( G \in (FN, \infty)^* \).

**Proof.** Let \( G \) be a finitely generated soluble group in the class \((CN, \infty)^*\). By Lemma 9, \( G \) is nilpotent-by-finite. So \( G \) satisfies max, the maximal condition on subgroups. Since Chernikov groups are locally finite, it follows that \( G \) is in the class \((FN, \infty)^*\). □

**Lemma 11.** Let \( G \) be a finitely generated abelian-by-finite group in the class \((FN, \infty)^*\). Then \( G \) is finite-by-nilpotent.

**Proof.** Let \( A \) be a normal abelian subgroup of finite index in \( G \). Since \( G \) is finitely generated, we may assume that \( A \) is torsion-free. Let \( x \in G \) and let \( a \in A \) of infinite order. Then the subset \( \{a^i x : i > 0\} \) is infinite. So there are two positive integers \( i, j \) such that \( \langle [a^j x, a^j x], a^j x \rangle \in FN \). Hence \( \langle [a^{-j}, x]^x, a^j x \rangle \in FN \), and therefore \( \langle [a^{-j}, x], xa^i \rangle \in FN \). Thus there exist two positive integers \( m, n \) such that \( [a^{j-i}, x, na^i]^m = [a, x, na^i]^{(j-i)m} = [a, x, na]^{(j-i)m} = 1 \). Since \( A \) is torsion-free, we obtain
\[ a_{n+1}x \] = 1. It follows that \( a \) is a right Engel element of \( G \). Since \( G \) satisfies max, the set of its right Engel elements coincides with a term of the upper central series \([20, 12.3.7]\). Hence \( A \leq Z_k(G) \) for some integer \( k > 0 \). So \( G/Z_k(G) \) is finite and this gives that \( G \) is finite-by-nilpotent \([11]\). □

**Proof of Theorem 3.** Let \( G \) be a finitely generated soluble group in the class \((CN, \infty)^*\). It follows, from Lemma 9 and Corollary 10, that \( G \) is a nilpotent-by-finite group in the class \((FN, \infty)^*\). Then \( G \) satisfies max. It is proved in \([9, \text{Theorem 1.1}]\) that if \( \Omega \) is a class of groups which is closed under taking subgroups and quotients and if all metabelian groups of \( \Omega \) are torsion-nilpotent, then all soluble groups of \( \Omega \) are torsion-nilpotent.

So, by taking \( \Omega \) to be the class of groups in \((FN, \infty)^*\) which satisfy max, we may assume \( G \) metabelian. Since \( G \) is a finitely generated nilpotent-by-finite group, there is a normal torsion-free subgroup \( H \) such that \( H \in \mathcal{N}_c \) and \( |G/H| = d \) for some positive integers \( c, d \). We prove that \( G \in FN \) by induction on \( c \). From Lemma 11, this is true if \( c = 1 \). Assume that \( c > 1 \). Clearly \( G/\gamma_c(H) \in \mathcal{N}_{c-1}\mathcal{F} \), so by the inductive hypothesis we have that \( G/\gamma_c(H) \in \mathcal{FN} \). Thus there are two positive integers \( m, n \) such that \((\gamma_{n+1}(G))^m \leq \gamma_c(H)\), so \([\gamma_{n+1}(G)]^m, H] = 1 \). Now \( \gamma_{n+1}(G) \) is abelian as \( G \) is metabelian. Hence \([\gamma_{n+1}(G)]^m, H] = [\gamma_{n+1}(G), H]^m = 1 \), and this gives \( \gamma_{n+1}(G), H] = 1 \) since \( H \) is torsion-free. It follows that \( [H, G] \leq \gamma_c(H) \). It is proved in \([9, \text{Lemma 2.1}]\) that if \( H, K \) are normal subgroups of a group \( G \) and if for some integer \( n > 0 \) we have \([H, G] \leq K\), then for any integer \( c > 0 \) we have \( \gamma_c(H)_{c(\gamma_{n-1}+1)} \leq [K, c^{-1}H] \). By taking \( K = \gamma_c(H) \), we obtain \( [\gamma_c(H)_{c(\gamma_{n-1}+1)} \leq [\gamma_c(H)_{c^{-1}H} \leq \gamma_{c+1}(H) = 1 \). It follows that \( [\gamma_c(H)_{c(\gamma_{n-1}+1)} = 1 \), and this means that \( \gamma_c(H) \leq Z_{c(\gamma_{n-1})+1} \). Since \( G/\gamma_c(H) \in \mathcal{FN} \), then \( G/Z_{c(\gamma_{n-1}+1)}(G) \in \mathcal{FN} \), which implies that \( G \in \mathcal{FN} \), as required.

**Proof of Corollary 4.** Let \( k \) be a positive integer and let \( G \) be a finitely generated soluble group in \((CN_k, \infty)^*\). From Theorem 3, \( G \) is finite-by-nilpotent. Thus \( G \) contains a normal finite subgroup \( H \) such that \( G/H \) is nilpotent and finitely generated, so its torsion subgroup \( T/H \) is finite, and consequently \( T \) is finite. Clearly \( G/T \) is in \((CN_k, \infty)^*\), so \( G/T \), being torsion-free, is in \((N_k, \infty)^*\). Since \((N_k, \infty)^*\) is contained in \( \mathcal{E}_{k+1}(\infty) \), we can deduce that:

(i) \( G/T \) is a finitely generated soluble group in \( \mathcal{E}_{k+1}(\infty) \), so by \([1,
Theorem 3], there exists an integer \( c = c(k) \), depending only on \( k \), such that \( (G/T)/Z_c(G/T) \) is finite. So, by [11, Theorem 1] we obtain that \( \gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T \) is finite. Since \( T \) is finite, it follows that \( \gamma_{c+1}(G) \) is finite. Thus by [11, 1.5] we get that \( G/Z_c(G) \) is finite.

(ii) \( G/T \) is a finitely generated metabelian group in \( \mathcal{E}_{k+1}(\infty) \), so by [1, Theorem 2], \( (G/T)/Z_{k+1}(G/T) \) is finite. Hence by [11, Theorem 1] we obtain that \( \gamma_{k+2}(G/T) = \gamma_{k+2}(G)T/T \) is finite. Since \( T \) is finite, it follows that \( \gamma_{k+2}(G) \) is finite. So by [11, 1.5] we deduce that \( G/Z_{k+1}(G) \) is finite.

**Proof of Corollary 5.** Note that if \( G \) is a finitely generated soluble group in the class \( (\mathcal{CN}, \infty) \), then by Theorem 3 it satisfies max. Therefore Corollary 5 follows from Corollary 2 and the fact that finitely generated torsion soluble groups are finite.

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NADIR TRABELSI
DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES
UNIVERSITÉ FERHAT ABBAS
SETIF 19000
ALGÉRIE

E-mail: trabelsi.]@yahoo.fr

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