On a functional inequality related to the stability problem for the Gołąb–Schinzel equation

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Abstract. We determine all unbounded continuous functions satisfying the inequality
\[ |f(x + yf(x)) - f(x)f(y)| \leq \varepsilon \quad \text{for } x, y \in \mathbb{R}, \]
where \( \varepsilon \) is a fixed positive real number. As a consequence we obtain that in the class of continuous functions the Gołąb–Schinzel functional equation is super-stable.

1. Introduction

The Gołąb–Schinzel functional equation
\[ f(x + yf(x)) = f(x)f(y) \quad \text{for } x, y \in \mathbb{R}, \quad (1) \]
where \( f : \mathbb{R} \to \mathbb{R} \) is the unknown function, is one of the most intensively studied equations of the composite type. Some information concerning (1), recent results, applications and numerous references one can find in [1]–[6] and [8]–[12]. At the 38th International Symposium on Functional Equations (2000, Noszvaj, Hungary) R. Ger raised, among others, the problem of Hyers–Ulam stability of (1) (see [7]). Motivated by this problem, we consider the inequality
\[ |f(x + yf(x)) - f(x)f(y)| \leq \varepsilon \quad \text{for } x, y \in \mathbb{R}, \quad (2) \]

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where $\varepsilon$ is a fixed positive real number. We determine all unbounded continuous solutions of (2). As a consequence we obtain that in the class of continuous functions the equation (1) is superstable.

2. Auxiliary results

For the proof of our main results we need few lemmas.

**Lemma 1.** Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies (2). Then:

(i) either $f(0) = 1$ or $f$ is bounded;

(ii) \(|f(x + yf(x)) - f(y + xf(y))| \leq 2\varepsilon \quad \text{for } x, y \in \mathbb{R}; \quad (3)\)

(iii) if $f$ is bounded above then $f$ is bounded.

**Proof.** (i) Putting $y = 0$ in (2), we get $|f(x)||1 - f(0)| \leq \varepsilon$ for $x \in \mathbb{R}$. Whence either $f(0) = 1$ or $f$ is bounded.

(ii) This follows immediately from (2).

(iii) Suppose that $f$ is unbounded. Then there exists a sequence $(x_n : n \in \mathbb{N})$ of real numbers such that $\lim_{n \to \infty} |f(x_n)| = \infty$. Using (2) we obtain that $f(x_n + x_n f(x_n)) \geq f(x_n)^2 - \varepsilon$ for $n \in \mathbb{N}$. Consequently $f$ is unbounded above. \qed

**Lemma 2.** Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2). Fix a $z \in \mathbb{R} \setminus \{0\}$ and define the function $\psi_z : \mathbb{R} \to \mathbb{R}$ by

$$\psi_z(x) = x + zf(x) \quad \text{for } x \in \mathbb{R}. \quad (4)$$

(i) If $\psi_z$ is bounded then

$$f(x) = 1 - \frac{x}{z} \quad \text{for } x \in \mathbb{R}. \quad (5)$$

(ii) If $f(z) = 0$ and $\psi_z$ is unbounded below (above), then

$$|f(x)| \leq \varepsilon \quad \text{for } x \in (-\infty, z] \quad (x \in [z, \infty), \text{resp.}). \quad (6)$$

(iii) $\psi_z^{n+1}(z) = \psi_z^n(z) + zf(\psi_z^n(z)) \quad \text{for } n \in \mathbb{N}. \quad (7)$

(iv) If there exists a $q := \lim_{n \to \infty} \psi_z^n(z)^n$, then $f(q) = 0$. 

On a functional inequality related to the stability problem

Proof. (i) Assume that $\psi_z$ is bounded. From (4) it follows that

$$f(x) = \frac{1}{z}(\psi_z(x) - x) \quad \text{for } x \in \mathbb{R},$$

so using (2), one can obtain

$$\frac{1}{z^2} \left| z \psi_z \left( x + \frac{y}{z}(\psi_z(x) - x) \right) - \psi_z(x)\psi_z(y) + x(\psi_z(y) - z) \right| \leq \varepsilon$$

for $x, y \in \mathbb{R}$. Since $\psi_z$ is bounded, this means that $\psi_z(y) - z = 0$ for $y \in \mathbb{R}$, which implies (5).

(ii) Assume that $f(z) = 0$ and $\psi_z$ is unbounded above. Since $\psi_z$ is continuous and $\psi_z(z) = z$, we have $[z, \infty) \subset \psi_z(\mathbb{R})$. Moreover, taking in (2) $y = z$, we obtain $|f(\psi_z(x))| \leq \varepsilon$ for $x \in \mathbb{R}$. Hence we get (6). In the case when $\psi_z$ is unbounded below, the proof is analogous.

(iii) This follows immediately from (4).

(iv) This results at once form (iii).

Lemma 3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2) and $I \in \{(0, \infty), (-\infty, 0)\}$. If there is a $z \in I$ with $f(z) = 0$, then $f_I$ is bounded above.

Proof. We present the proof in the case $I = (0, \infty)$ only. Assume that $f(z) = 0$ for some $z \in (0, \infty)$. Let a function $\psi_z$ be defined by (4). If $\psi_z$ is bounded above (say, by a constant $p$), then from (4) it results that $f(x) \leq \frac{p - x}{z}$ for $x \in \mathbb{R}$. Hence $f_I(0, \infty)$ is bounded above. If $\psi_z$ is unbounded above, then according to Lemma 2(ii), we get $f(x) \leq \varepsilon + \max\{f(t) : t \in [0, z]\}$ for $x \in (0, \infty)$, so again $f_I(0, \infty)$ is bounded above.

Lemma 4. Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2) and $I \in \{(0, \infty), (-\infty, 0)\}$. Then either $f_I$ is bounded above or there exists a $k \in I$ such that

$$f(x) \geq kx \quad \text{for } x \in I. \quad (8)$$

Proof. Similarly as in the proof of the previous lemma, we consider only the case $I = (0, \infty)$. Suppose that $f_I(0, \infty)$ is unbounded above. Then from Lemma 1(i) and Lemma 3 it follows that $f(0) = 1$ and $f(x) \neq 0$ for $(0, \infty)$. Hence, by the continuity of $f$,

$$f(x) > 0 \quad \text{for } x \in (0, \infty). \quad (9)$$
We divide the remaining part of the proof into two steps.

**STEP 1.** We show that

\[ f(x) \geq 1 \quad \text{for } x \in (0, \infty). \]

(10)

Suppose that (10) does not hold. Whence, according to (9), there is a \( z \in (0, \infty) \) such that \( f(z) \in (0, 1) \). Let the function \( \psi_z \) be defined by (4).

Consider a sequence \( (\psi^n_z(z) : n \in \mathbb{N}) \). According to (7) and (9), we obtain that the sequence is strictly increasing. Moreover, it is unbounded. Indeed, if it were bounded, then it would exist \( q := \lim_{n \to \infty} \psi^n_z(z) \). Hence, by Lemma 2(iv), \( f(q) = 0 \), which contradicts to (9).

Now, we define a sequence of intervals \( (I_n : n \in \mathbb{N} \cup \{0\}) \) as follows:

\[ I_0 := [0, z], \quad I_n := [\psi^n_z^{-1}(z), \psi^n_z(z)] \quad \text{for } n \in \mathbb{N}. \]

Since the sequence \( (\psi^n_z(z) : n \in \mathbb{N}) \) is unbounded, we get

\[ \bigcup_{n=1}^{\infty} I_n = (0, \infty). \]

(11)

Furthermore, for every \( n \in \mathbb{N} \cup \{0\} \), we have

\[ f(x) \leq M f(z)^n + \varepsilon \sum_{i=0}^{n-1} f(z)^i \quad \text{for } x \in I_n, \]

(12)

where \( M := \sup\{f(x) : x \in [0, z]\} \). In fact, for \( n = 0 \) (12) trivially holds (we adopt the convention \( \sum_{i=0}^{-1} = 0 \)). If (12) occurs for a \( n \in \mathbb{N} \cup \{0\} \), then taking an \( x \in I_{n+1} = [\psi^n_z(z), \psi^{n+1}_z(z)] \) and using the continuity of \( \psi_z \), we obtain that \( x = \psi_z(t) \) for some \( t \in I_n \). Whence, in view of (2) and (12) (for \( n \)), we obtain

\[ f(x) = f(\psi_z(t)) = f(t + zf(t)) \leq f(t)f(z) + \varepsilon \]

\[ \leq M f(z)^n + \varepsilon \sum_{i=0}^{n} f(z)^i. \]

Now, using (12), for every \( n \in \mathbb{N} \cup \{0\} \), we have

\[ f(x) \leq M f(z)^n + \varepsilon \sum_{i=0}^{\infty} f(z)^i \leq M + \frac{\varepsilon}{1 - f(z)} \quad \text{for } x \in I_n. \]

Thus, in view of (11), \( f_{|[0,\infty)} \) is bounded above, which yields a contradiction.
On a functional inequality related to the stability problem 203

Step 2. Since \( f_{(0, \infty)} \) is unbounded above, there is a \( p \in (0, \infty) \) with \( f(p) > 1 + \varepsilon \). Define the function \( h_p : [0, \infty) \to \mathbb{R} \) by \( h_p(x) = p + x f(p) \) for \( x \in [0, \infty) \). Consider a sequence \( (h^n_p(p) : n \in \mathbb{N}) \) and note that

\[
h^n_p(p) = p \sum_{i=0}^{n} f(p)^i \quad \text{for } n \in \mathbb{N}.
\]  

(13)

Hence, the sequence \( (h^n_p(p) : n \in \mathbb{N}) \) is strictly increasing and unbounded. Let \( I_0 := [0, p] \) and \( I_n := [h^{n-1}_p(p), h^n_p(p)] \) for \( n \in \mathbb{N} \). Then (11) occurs. Furthermore, using (10), similarly as in the previous step, one can show that for every \( n \in \mathbb{N} \cup \{0\} \)

\[
f(x) \geq f(p)^n - \varepsilon \sum_{i=0}^{n-1} f(p)^i \quad \text{for } x \in I_n.
\]  

(14)

Fix an \( x \in (0, \infty) \). In view of (11), \( x \in I_n \) for some \( n \in \mathbb{N} \cup \{0\} \). Hence \( x \leq h^n_p(p) \), so according to (13) and (14), we get

\[
\frac{f(x)}{x} \geq \frac{f(p)^n - \varepsilon \sum_{i=0}^{n-1} f(p)^i}{h^n_p(p)} \geq \frac{1 - \varepsilon \sum_{i=1}^{\infty} f(p)^{-i}}{pf(p) } \\
= \frac{f(p) - (1 + \varepsilon)}{pf(p)} > 0.
\]

Therefore (8) holds with \( k := \frac{f(p) - (1 + \varepsilon)}{pf(p)} > 0 \). \Box

Lemma 5. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is an unbounded continuous function satisfying (2). Then either

\[
f(x) \leq M \quad \text{for } x \in (-\infty, 0]
\]  

(15)

and

\[
f(x) \geq kx \quad \text{for } x \in (0, \infty)
\]  

(16)

with some \( M \in \mathbb{R} \) and \( k \in (0, \infty) \); or

\[
f(x) \geq sx \quad \text{for } x \in (-\infty, 0)
\]  

(17)

and

\[
f(x) \leq M \quad \text{for } x \in [0, \infty)
\]  

(18)

with some \( M \in \mathbb{R} \) and \( s \in (-\infty, 0) \).
Proof. According to Lemma 4, it is enough to show that exactly one of functions $f_{(-\infty,0)}$ and $f_{(0,\infty)}$ is unbounded above. From Lemma 1(iii), it follows that at least one of them is unbounded above. Suppose that both $f_{(-\infty,0)}$ and $f_{(0,\infty)}$ are unbounded above. Then, on account of Lemma 4, there exist $k \in (0,\infty)$ and $s \in (-\infty,0)$ such that (16) and (17) occur. Moreover, in virtue of Lemma 1(i), $f(0) = 1$. Since $f$ is continuous, it implies that there is a $d > 0$ such that $f(x) \geq d$ for $x \in \mathbb{R}$. Fix an $x_0 \in \mathbb{R}$ with $f(x_0) > \frac{1+\varepsilon}{d}$. Then $f(x_0) f\left(-\frac{x_0}{f(x_0)}\right) > 1 + \varepsilon$. On the other hand, in view of (2), we get

\[
\left| 1 - f(x_0) f\left(-\frac{x_0}{f(x_0)}\right) \right| = \left| f(0) - f(x_0) f\left(-\frac{x_0}{f(x_0)}\right) \right|
= \left| f\left(x_0 + \left(-\frac{x_0}{f(x_0)}\right) f(x_0)\right) - f(x_0) f\left(-\frac{x_0}{f(x_0)}\right) \right| \leq \varepsilon,
\]

which yields a contradiction. \hfill \Box

Lemma 6. Assume that $f : \mathbb{R} \to \mathbb{R}$ is an unbounded continuous function satisfying (2). Then there exists a $p \in \mathbb{R}$ such that $f(p) = 0$.

Proof. Suppose that $f(x) \neq 0$ for $x \in \mathbb{R}$. Since $f$ is continuous and, in view of Lemma 1(i), $f(0) = 1$, this implies that $f(x) > 0$ for $x \in \mathbb{R}$. According to Lemma 5, either (15) and (16); or (17) and (18) hold. Since the proof in both cases is similar, assume that (15) and (16) occur. Then, on account of (16), we have $x - \frac{x_0}{f(x_0)} f(x) \leq 0$ for $x \in (0,\infty)$. Hence, in view of (15) $f(x - \frac{x_0}{f(x_0)} f(x)) \leq M$ for $x \in (0,\infty)$. On the other hand, from (16) it follows that $f\left(-\frac{x}{k} + x f\left(-\frac{1}{k}\right)\right) \geq -1 + k f\left(-\frac{1}{k}\right) x$ for $x > \frac{1}{k f\left(-\frac{1}{k}\right)}$. Thus $\lim_{x \to \infty} |f\left(-\frac{x}{k} + x f\left(-\frac{1}{k}\right)\right) - f\left(x - \frac{x_0}{f(x_0)} f(x)\right)| = \infty$, which contradicts (3). \hfill \Box

3. Main results

Theorem 1. A function $f : \mathbb{R} \to \mathbb{R}$ is an unbounded continuous solution of (2) if and only if there exists a non-zero real constant $a$ such that either

\[ f(x) = 1 + ax \quad \text{for } x \in \mathbb{R} \] (19)
or

\[ f(x) = \max\{1 + ax, 0\} \quad \text{for } x \in \mathbb{R}. \]  \hfill (20)

**Proof.** It is obvious that for every non-zero real constant \( a \), the function \( f \) given by (19) or (20), is an unbounded continuous solution of (2). Assume that \( f \) is an unbounded continuous function satisfying (2). Then, according to Lemma 1(i) and Lemma 6, \( f(0) = 1 \) and there is a \( p \in \mathbb{R} \setminus \{0\} \) such that \( f(p) = 0 \). Assume that \( p < 0 \) (if \( p > 0 \), the proof is similar). Then, in view of Lemma 3 and 5, we have (15) and (16). Let \( z := \max\{x \in (-\infty, 0] : f(x) = 0\} \) and \( \psi_z \) be given by (4). Then \( z < 0 \) and

\[ f(x) > 0 \quad \text{for } x \in (z, 0). \]  \hfill (21)

If \( \psi_z \) is bounded then, in virtue of Lemma 1(iv), \( f \) has the form (19) with \( a := -\frac{1}{z} \). Assume that \( \psi_z \) were unbounded above, then in virtue of Lemma 2(ii), we would have \(|f(x)| \leq \varepsilon \) for \( x \in [z, \infty) \), which contradicts to (16). Whence \( \psi_z \) is unbounded below and bounded above (say, by a constant \( w \)). Consequently, in view of (4) and Lemma 2(ii), we have

\[ f(x) \geq \frac{w - x}{z} \quad \text{for } x \in \mathbb{R} \]  \hfill (22)

and

\[ |f(x)| \leq \varepsilon \quad \text{for } x \in (-\infty, z]. \]  \hfill (23)

We divide the remaining part of the proof into three steps.

**Step 1.** We prove that

\[ \lim_{x \to \infty} \frac{f(x)}{x} = -\frac{1}{z}. \]  \hfill (24)

Suppose that (24) does not hold. Then, according to (22), there are a constant \( t > 0 \) and a sequence \((x_n : n \in \mathbb{N})\) of positive real numbers such that \( \lim_{n \to \infty} x_n = \infty \) and

\[ \frac{f(x_n)}{x_n} > -\frac{1}{z} + t \quad \text{for } n \in \mathbb{N}. \]  \hfill (25)

Since \( z < \frac{1}{1-tz} < 0 \), according to (21), we get \( f \left( \frac{z}{1-tz} \right) > 0 \). Thus

\[ \lim_{n \to \infty} \left( \frac{z}{1-tz} + x_n f \left( \frac{z}{1-tz} \right) \right) = \infty, \]
so in virtue of (16), we obtain \( \lim_{n \to \infty} f \left( \frac{z}{1-tz} + x_n f \left( \frac{z}{1-tz} \right) \right) = \infty \). On
the other hand, in view of (25), we have
\[
x_n + \frac{z}{1-tz} f(x_n) < x_n + \frac{z}{1-tz} \left( \frac{1}{z} + t \right) x_n = 0 \quad \text{for } n \in \mathbb{N}.
\]
Hence, using (15), we get \( f \left( x_n + \frac{z}{1-tz} f(x_n) \right) \leq M \) for \( n \in \mathbb{N} \). Consequently,
\[
\lim_{n \to \infty} \left| f \left( \frac{z}{1-tz} + x_n f \left( \frac{z}{1-tz} \right) \right) - f \left( x_n + \frac{z}{1-tz} f(x_n) \right) \right| = \infty,
\]
which contradicts to (3).

**Step 2.** We show that
\[
f(x) = \frac{1 - x}{z} \quad \text{for } x \in (z, \infty). \quad (26)
\]
Fix a \( y \in (z, \infty) \). From (2) and (24) it follows that
\[
\lim_{x \to \infty} \frac{f(x + yf(x))}{x} = \lim_{x \to \infty} \frac{f(x)}{x} f(y) = -\frac{1}{z} f(y). \quad (27)
\]
and
\[
\lim_{x \to \infty} \left( 1 + y \frac{f(x)}{x} \right) = 1 - \frac{y}{z} \neq 0.
\]
Thus \( \lim_{x \to \infty} x \left( 1 + y \frac{f(x)}{x} \right) = \lim_{x \to \infty} (x + yf(x)) = \infty \), so according to (24) and (27), we obtain
\[
-\frac{1}{z} = \lim_{x \to \infty} \frac{f(x + yf(x))}{x + yf(x)} = \lim_{x \to \infty} \frac{f(x+yf(x))}{x} = \frac{f(y)}{1 + y \frac{f(x)}{x}} = \frac{f(y)}{y - z}.
\]
Hence \( f(y) = 1 - \frac{y}{z} \), which proves (26).

**Step 3.** We prove that
\[
f(x) = 0 \quad \text{for } x \in (-\infty, z]. \quad (28)
\]
For \( x = z \) (28) trivially occurs. Fix a \( y \in (-\infty, z) \). According to (2) and (23), we have \( f(x + yf(x)) \leq \varepsilon + x^2 \) for \( x \in (-\infty, z] \). Moreover, using (26), we get
\[
x + yf(x) = x + y \left( 1 - \frac{x}{z} \right) = \left( 1 - \frac{y}{z} \right) x + y < \left( 1 - \frac{y}{z} \right) z + y = z < 0
\]
On a functional inequality related to the stability problem

for $x \in (z, \infty)$. Hence, in view of (15), $f(x + yf(x)) \leq M$ for $x \in (z, \infty)$. Consequently, $f(x + yf(x)) \leq \max\{\varepsilon + \varepsilon^2, M\}$ for $x \in \mathbb{R}$, so taking into account (3), we obtain that $f(y + xf(y)) \leq \max\{3\varepsilon + \varepsilon^2, M + 2\varepsilon\}$ for $x \in \mathbb{R}$. Now, if $f(y)$ were different from 0, we would have that $f$ is bounded above, which contradicts to Lemma 1(iii). Therefore $f(y) = 0$, which proves (28).

Finally, from (26) and (28) it follows that $f$ has the form (20) with $a := -\frac{1}{z}$, which completes the proof. □

It is easy to check that for every non-zero real constant $a$, the function $f$ given by (19) or (20) is a continuous solution of (2). Therefore, we can reformulate Theorem 1 in the following way:

**Theorem 2.** If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (2), then either $f$ is bounded or $f$ is a solution of (1).

**Remark 1.** Note that the idea of the introduction of the function $\psi_z$ (cf. (4)) to a given solution $f$ of (1), as well as the idea of the determination of the set of all possible zeroes of $f$ have already been used in the study of the Gołąb–Schinzel equation (cf. e.g. [5], [10], [11]).

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