A representation of $CD_w(K)$-spaces

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Abstract. We give a representation of the space $CD_w(K)$ which was defined by Abramovich and Wickstead. We apply this to reprove the Banach–Stone type theorem for $CD_w(K)$ spaces. By using this representations we note that for each compact Hausdorff space $K$ without isolated points there exists compact Hausdorff space $T$ which contains $K$ as a closed subspace such that the Dedekind completion of $C(T)$ is $B(K)$.

For a given non-empty set $K$, $l^\infty_w(K)$ denotes the set all real valued bounded functions $d$ on $K$ satisfying $\{ k \in K : |d(k)| \neq 0 \}$ is countable. As usual, for a given topological space $K$, $C(K)$ is the set of all continuous real valued functions on $K$. Let $K$ be a compact Hausdorff space without isolated points. Then $CD_w(K) = C(K) \oplus l^\infty_w(K)$ is an AM space with order unit 1 under pointwise operations (see [1] and [3]).

For any bounded function $f : S \to \mathbb{R}$, the continuous extension of $f$ to the Stone-Cech compactification $\beta S$ (of discrete space $S$) will be denoted by $f^\ast$. Let $K$ be a compact Hausdorff space and set

$$T_K = \{(k, r) : k \in K, r \in \beta K f(k) = f^\ast(r) \text{ for each } f \in C(K)\}.$$ 

Let $\sim$ be defined by

$$(k_1, r_1) \sim (k_2, r_2) \iff f(k_1) + d^\ast(r_1) = f(k_2) + d^\ast(r_2).$$

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for each \( f \in C(K) \), \( d \in l_\infty^w(K) \). Then \( \sim \) defines an equivalence relation on \( T_K \). For each \((k, r) \in T_K\), let \( [(k, r)] = \{(t, s) \in T_K, (k, r) \sim (t, s)\}\), the equivalence class of \((k, r)\). Define 
\[
[T_K] = \{(k, r) : (k, r) \in T_K\}
\]
and 
\[
[A_T] = \{(k, r) \in [T_K] : d^*(r) = 0 \text{ for each } d \in l_\infty^w(K)\}.
\]

**Lemma 1.** Let \( K \) be a compact Hausdorff space \([T_K]\) and \([A_K]\) be defined as above. Then

i) \([T_K]\) is a compact Hausdorff space under the convergence 
\[
[(k_\alpha, r_\alpha)] \to [(k, r)] \iff f(k_\alpha) \to f(k) \quad \text{and} \quad d^*(r_\alpha) \to d^*(r)
\]
for all \( f \in C(K) \) and \( d \in l_\infty^w(K) \).

ii) \([A_K]\) is a closed subspace of \([T_K]\).

**Proof.** i) It is easy to check the convergence defines a Hausdorff topology on \([T_K]\). Let \([(k_\alpha, r_\alpha)]\) be a net in \([T_K]\). Choose a subnet \((k_{\alpha_\beta})\) of \((k_\alpha)\) and subnet \((r_{\alpha_\beta})\) of \((r_\alpha)\) with \( k_{\alpha_\beta} \to k \) in \( K \) and \( r_{\alpha_\beta} \to r \) in \( \beta K \).

It is clear that \( f(k_{\alpha_\beta}) = f^*(r_\alpha) \) for each \( f \in C(K) \) and \( [(k_{\alpha_\beta}, r_{\alpha_\beta})] \to [(k, r)] \) in \([T_K]\). This shows that \([T_K]\) is compact.

ii) Let \([(k_\alpha, r_\alpha)]\) be net in \([A_K]\) with \([(k_\alpha, r_\alpha)] \to [(k, r)] \) in \([T_K]\). Then \( f(k_\alpha) \to f(k) + d^*(k) \) for each \( f \in C(K) \) and \( d \in l_\infty^w(K) \). If we take \( d = 0 \) then we see that \( f(k_\alpha) \to f(k) \) for each \( f \in C(K) \). This shows that \( d^*(r) = 0 \) for each \( d \in l_\infty^w(K) \). Hence \( [(k, r)] \in [A_K] \), that is \([A_T]\) is closed. \( \square \)

Now we are ready to give a representation of \( CD_w(K) \) as follows:

**Theorem 2.** Let \( K \) be a compact Hausdorff space without isolated points. Then

i) \( CD_w(K) \) is isometric Riesz isomorphic to \( C([T_K]) \).

ii) \( C(K) \) is isometric Riesz isomorphic to \( C([A_T]) \).

iii) \( K \) and \([A_T]\) are homeomorphic spaces.
Proof. i) Let $L : CD_w(K) \to C([T_K])$ be defined by

$$L(f + d)([(k, r)]) = f(k) + d^*(r).$$

It is clear that $L$ is linear. Let $f \in C(K)$, $d \in l^\infty_w(K)$ with $0 \leq f + d$ in $CD_w(K)$. Then $0 \leq (f + d)^*$ in $C(\beta K)$. Let $(k, r) \in T_K$. Then

$$L(f + d)([(k, r)]) = f(k) + d^*(r) = f^*(r) + d^*(r) = (f + d)^*(r) \geq 0$$

so $L$ is positive and clearly $0 \leq f + d$ in $CD_w(K)$ whenever $L(f + d) \geq 0$, since $(k, k) \in T_K$ for each $k \in K$. This shows $L$ is bipositive, so it is Riesz isomorphism into $C([T_K])$. We also have that

$$\|f + d\| = \sup_{k \in K} |f(k) + d(k)| \leq \sup_{[(k, r)] \in [T_K]} |f(k) + d^*(r)|$$

and

$$\|L(f + d)\| = \|L(f + d)\| = \|L(\|f + d\|)\| = \|f + d\|$$

so $\|L(f)\| = \|f\|$ for each $f \in CD_w(K)$. Let $(k_1, r_1) \neq (k_2, r_2)$. Choose $f \in C(K)$ and $d \in l^\infty_w(K)$ with $f(k_1) + d^*(r_1) \neq f(k_2) + d^*(r_2)$, that is, $T(f + d)([(k, r)]) \neq T(f + d)([(k_2, r_2)])$. This shows that $L(CD_w(K))$ separates the points of $[T_K]$. Now it follows from the Stone–Weierstrass theorem that $L$ is also onto since $L(CD_w(K))$ is closed in $C(T_K)$. This proves the first part of the theorem.

ii) Define

$$R : C(K) \to C([A_K]), \quad R(f)([(k, r)]) = f(k).$$

It is clear that $R$ is isometry Riesz isomorphism and $R(C(K))$ separates the points of $[A_K]$. We apply the Stone–Weierstrass Theorem to complete the proof.

iii) Since $C([A_K])$ and $C(K)$ are Riesz isomorphic, from Banach–Stone Theorem $[A_K]$ and $K$ are homeomorphic spaces. □

The proof of the following lemma is clear so we omit its proof.

Lemma 3. Let $K$ and $M$ be compact Hausdorff spaces without isolated points. If $Q$ is an isometric Riesz isomorphism from $CD_w(K)$ onto $CD_w(M)$ then

$$Q(l^\infty_w(K)) = l^\infty_w(M).$$
Theorem 4. Let $K$ and $M$ be compact Hausdorff spaces without isolated points. If $[T_K]$ and $[T_M]$ are homeomorphic then $K$ and $M$ are homeomorphic.

Proof. Let $R: C([T_K]) \to C([T_M])$ be an isometric Riesz isomorphism defined by $R(f) = f \circ \pi^{-1}$ where $\pi: [T_K] \to [T_M]$ is a homeomorphism. For $S \in \{K, M\}$ define $R_S: CD_w(S) \to C([T_S])$ by

$$R_S(f + d)([(k, r)]) = f(s) + d^*(r)$$

for each $f \in C(S)$, $d \in l_\infty^w(S)$. It is enough to show that $\pi([A_K]) \subset [A_M]$. Let $Q = R_K^{-1} \circ R^{-1} \circ R_M$. $Q$ is isometric Riesz isomorphic from $CD_w(M)$ onto $CD_w(K)$. Let

$$\pi([(k, r)]) = [(m, s)], \quad [(k, r)] \in [A_K].$$

Let $d \in l_\infty^w(M)$. Then from the previous lemma

$$Q(d) \in l_\infty^w(K)$$

and

$$0 = (Q(d))^*(r) = R_K \circ Q(d)([(k, r)])$$
$$= R^{-1} \circ R_M(d)([(k, r)])$$
$$= R_M(d) \circ \pi([(k, r)])$$
$$= R_M(d)([(m, s)])$$
$$= d^*(s).$$

So, $[T_K]$ and $[T_M]$ are homeomorphic. From Theorem 2, $K$ and $M$ are homeomorphic. □

We reprove the following theorem which is one of the main results of [2].

Theorem 5. Let $K$ and $M$ be compact Hausdorff spaces without isolated points. If $CD_w(K)$ and $CD_w(M)$ are isometric isomorphic spaces then $K$ and $M$ are homeomorphic.
Proof. Let $R$ be an isometry operator from $CD_w(K)$ onto $CD_w(M)$. Then it is easy to see that $R(\mathbb{1})$ is a unimodular function, so $Q = T(\mathbb{1})^{-1}R$ is an isometry from $CD_w(K)$ onto $CD_w(M)$ and $Q(\mathbb{1}) = \mathbb{1}$. From the following fact
\[ \|f - \|f\|\mathbb{1}\| \leq \|f\| \iff 0 \leq f \]
that $Q$ is also a Riesz isomorphism. So $CD_w(K)$ and $CD_w(M)$ are isometric Riesz isomorphic spaces. Under the assumptions of Theorem 2, $C([T_K])$ and $C([T_M])$ are isometric Riesz isomorphic spaces. From the Banach Stone Theorem $[T_K]$ and $[T_M]$ are homeomorphic spaces. From the previous theorem $K$ and $M$ are homeomorphic spaces. □

If $K$ is Stone–Cech compactification of a discrete space $M$, then the Dedekind completion of $C(K)$ (already is Dedekind complete) is $B(M)$. The following theorem also provides many examples of infinity compact Hausdorff space $A$ such that the Dedekind completion of $C(A)$ is $B(S)$ ($= \text{the set of all real valued bounded functions on } S$) with cardinal number of $S$ is less than the cardinal number of $A$.

Theorem 6. For each compact Hausdorff space $K$ without isolated points there exists another compact Hausdorff space $K'$ which contains $K$ as a closed subspace where the Dedekind completion of $C(K')$ is $B(K)$ and the universal completion is $\mathbb{R}^K$.

Proof. Let $K' = [T_K]$. It follows immediately from the [3] and Theorem 1 that the Dedekind completion of $C(K')$ is $B(K)$ so the universal completion of $C(K')$ is $\mathbb{R}^K$. □

Remark. Let $\alpha$ be an infinity cardinal and let $K$ be a compact Hausdorff space such that the interior of any subset of $K$ with cardinality at most $\alpha$ is empty. Let $l^\infty_{\alpha}(K)$ be the set of all bounded real valued functions $f$ on $K$ with the cardinality of support $f$ at most $\alpha$. Then
\[ CD^\infty_{\alpha}(K) = C(K) \oplus l^\infty_{\alpha}(K) \]
is an AM-space with order unit $\mathbb{1}$. The above theorems can also be proved for $CD^\infty_{\alpha}(K)$-spaces.
References


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