A weighted Hermite–Hadamard-type inequality for convex-concave symmetric functions

By PÉTER CZINDER (Gyöngyös)

Abstract. In this paper we give a weighted version of the Hermite–Hadamard inequality

\[
f \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} \int_a^b f(x) \, dx \geq \frac{f(a) + f(b)}{2}.
\]

An extension of that result, applied for convex-concave symmetric functions, will also be provided.

1. Introduction

The so-called Hermite–Hadamard inequality [7] is one of the most investigated classical inequalities concerning convex functions. It reads as follows:

**Theorem 1.** Let \( I \subset \mathbb{R} \) be an interval and \( f : I \to \mathbb{R} \) be a concave (convex) function. Then, for all subinterval \([a, b] \subset I\) with non-empty interior,

\[
f \left( \frac{a+b}{2} \right) \geq \frac{1}{b-a} \int_a^b f(x) \, dx \geq \frac{f(a) + f(b)}{2} \tag{1}
\]
An account on the history of this inequality can be found in [8]. Surveys on various generalizations and developments can be found in [9] and [4]. The description of best possible inequalities of Hadamard–Hermite type are due to Fink [5]. A generalization to higher-order convex function can be found in [1], while [2] offers a generalization for functions that are Beckenbach-convex with respect to a two dimensional linear space of continuous functions.

In this form (1) is valid only for functions that are purely convex or concave on their whole domain. In [3] we proved that under appropriate conditions the same inequalities could be stated for a much larger family of functions. The results, obtained for that situation, could be applied for the investigation of the comparison problem for Gini and Stolarsky means.

In Section 2 we will use another method to extend Theorem 1, replacing the arithmetic mean by more general means, applying weight functions. For a further generalization of Theorem 1, in Section 3 we will introduce the concept of odd and even functions with respect to a point. Finally, in Section 4 we will combine these two directions of the extensions and present a weighted version of the Hermite–Hadamard inequality for convex-concave symmetric functions.

2. The weighted Hermite–Hadamard inequality for convex or concave functions

Given a positive, locally integrable weight function \( \varrho : \mathbb{I} \to \mathbb{R}_+ \), define the \( \varrho \)-mean of \( a \) and \( b \) by

\[
M_\varrho(a, b) := \frac{\int_a^b x \varrho(x) dx}{\int_a^b \varrho(x) dx}.
\]

Then the following statement holds:

**Theorem 2.** Let \( \mathbb{I} \subset \mathbb{R} \) be an interval, \( f : \mathbb{I} \to \mathbb{R} \) be a concave (convex) function and \( \varrho : \mathbb{I} \to \mathbb{R} \) a positive, locally integrable weight
A weighted Hermite–Hadamard-type inequality

function. Then, for all subintervals \([a, b] \subset I\) with non-empty interior,

\[
f(M_\varrho(a, b)) \geq \left(\leq\right) \frac{1}{\int_a^b \varrho(x)dx} \int_a^b f(x) \varrho(x)dx \geq \left(\leq\right) \frac{b - M_\varrho(a, b)}{b - a} f(a) + \frac{M_\varrho(a, b) - a}{b - a} f(b).
\]

(2)

**Proof.** Suppose that \(f\) is concave over \(I\) and let \(e(x) := cx + d\) be a support line of the function \(f\) at the point \(M_\varrho(a, b)\). Let

\[
g(x) = \frac{f(b) - f(a)}{b - a} \cdot x + \frac{bf(a) - af(b)}{b - a}
\]

be the chord of \(f\) from \((a, f(a))\) to \((b, f(b))\). Then, applying the concavity,

\[
e(x) \geq f(x) \geq g(x) \quad (x \in I),
\]

that is,

\[
\frac{\int_a^b e(x) \varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b f(x) \varrho(x)dx}{\int_a^b \varrho(x)dx} \geq \frac{\int_a^b g(x) \varrho(x)dx}{\int_a^b \varrho(x)dx}
\]

(3)

After a calculation, we obtain

\[
\frac{\int_a^b e(x) \varrho(x)dx}{\int_a^b \varrho(x)dx} = \frac{\int_a^b (cx + d) \varrho(x)dx}{\int_a^b \varrho(x)dx} = cM_\varrho(a, b) + d = f(M_\varrho(a, b))
\]

and

\[
\frac{\int_a^b g(x) \varrho(x)dx}{\int_a^b \varrho(x)dx} = \frac{f(b) - f(a)}{b - a} M_\varrho(a, b) + \frac{bf(a) - af(b)}{b - a}
\]

\[
= \frac{b - M_\varrho(a, b)}{b - a} f(a) + \frac{M_\varrho(a, b) - a}{b - a} f(b),
\]

which proves (2).

For convex functions the proof is similar. \(\square\)

(It can immediately be seen that Theorem 1 is a special case of Theorem 2 with \(\varrho(x) \equiv 1\).)
Remark. The primary motivation for the various extension of the Hermite–Hadamard inequality, such as those obtained by Zsolt Páles and the author [3] is to provide inequalities for the Gini and Stolarsky means. (For details about these two parameter, two variable homogeneous means see [6] and [10].)

Theorem 2 can also be applied, for instance, to give an upper and lower bound for the Stolarsky mean $S_{r,s}(\xi,\eta)$. For, suppose that $f(x) = x^{r-s}$, $\varphi(x) := x^{s-1}$. Then $M_{\varphi}(\xi,\eta) = S_{s,s+1}(\xi,\eta)$, while $(\int_{\xi}^{\eta} f(x) \varphi(x) dx) / (\int_{\xi}^{\eta} \varphi(x) dx) = (S_{r,s}(\xi,\eta))^{r-s}$. In this way we can give bounds for the general Stolarsky mean in terms of a more special instance of it, namely, by the one where the difference of the parameters equals 1.

3. Odd and even functions with respect to a point

In the following we will encounter functions showing two kinds of symmetry.

Definition. Let $I$ be a real interval, $m \in I$. We say that the function $f : I \to \mathbb{R}$ is odd with respect to the point $m$, if $t \mapsto f(m + t) - f(m)$ is odd, that is,

$$f(m - t) + f(m + t) = 2f(m) \quad (t \in (I - m) \cap (m - I)), \quad (4)$$

while it is said to be even with respect to the point $m$, if $t \mapsto f(m + t)$ is even, that is,

$$f(m - t) = f(m + t) \quad (t \in (I - m) \cap (m - I)). \quad (5)$$

In a recent paper [3] we proved that (1) is valid for a function $f$ odd with respect to a point $m \in I$ under appropriate convexity conditions:

Theorem 3. Let $f : I \to \mathbb{R}$ be odd with respect to an element $m \in I$ and let $[a, b]$ be a subinterval of $I$ with non-empty interior. If $f$ is convex over the interval $I \cap (-\infty, m]$ and concave over $I \cap [m, \infty)$, then

$$f \left( \frac{a + b}{2} \right) \geq \frac{1}{b - a} \int_{a}^{b} f(x) dx \geq \frac{f(a) + f(b)}{2} \quad (\leq) \quad \text{if } \frac{a + b}{2} \geq m. \quad (6)$$
For the integral of the product of odd and even functions with respect to the midpoint of the same interval, the following statement is true:

**Lemma.** Let $g, h : [\alpha, \beta] \to \mathbb{R}$ be integrable functions over $[\alpha, \beta]$, $g$ be odd and $h$ be even with respect to the point $(\alpha + \beta)/2$. Then

$$\int_{\alpha}^{\beta} g(x)h(x)dx = g\left(\frac{\alpha + \beta}{2}\right)\int_{\alpha}^{\beta} h(x)dx.$$

**Proof.** Let $m$ denote the midpoint of $[\alpha, \beta]$. By splitting the integral at the point $m$ and applying (4) and (5) for $g$ and $h$, respectively, we get

$$\int_{\alpha}^{\beta} g(x)h(x)dx = \int_{\alpha}^{m} g(x)h(x)dx + \int_{m}^{\beta} ((2g(m) - g(2m - x))h(2m - x)dx
= \int_{\alpha}^{m} g(x)h(x)dx - \int_{m}^{\alpha} ((2g(m) - g(y))h(y)dy
= \int_{\alpha}^{m} (g(x)h(x) + 2g(m)h(x) - g(x)h(x))dx
= 2g(m)\int_{\alpha}^{m} h(x)dx = g(m)\int_{\alpha}^{\beta} h(x)dx. \quad \square$$

4. An extension of Theorem 2

**Theorem 4.** Let the function $f : I \to \mathbb{R}$ be odd with respect to the element $m \in I$, $\varrho : I \to \mathbb{R}$ a positive, integrable weight function, which is even with respect to $m$, and let $[a, b]$ be a subinterval of $I$ with non-empty interior. Then the following statement is valid:

If $f$ is convex in the interval $I \cap (-\infty, m]$ and concave in $I \cap [m, \infty)$, then

$$f\left(M_{\varrho}(a, b)\right) \geq \left(\leq\right) \frac{1}{\int_{a}^{b} \varrho(x)dx} \int_{a}^{b} f(x)\varrho(x)dx
\geq \left(\leq\right) \frac{b - M_{\varrho}(a, b)}{b - a}f(a) + \frac{M_{\varrho}(a, b) - a}{b - a}f(b),$$

(7) if \( \frac{a + b}{2} \geq m \).
PROOF. We may restrict ourselves to the proof of (i).
First we shall prove the left hand side inequality.
Suppose that \( m \leq (a + b)/2 \), \( f \) is convex over the interval \( I \cap (-\infty, m] \) and concave over \( I \cap [m, \infty) \). We may assume that \( m > a \). Then, applying the lemma,

\[
\begin{align*}
f(M_\rho(a, b)) &= f \left( \frac{\int_a^{2m-a} x \varrho(x)dx + \int_{2m-a}^b x \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b \varrho(x)dx} \right) \\
&= f \left( \frac{m \int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b x \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx + \int_{2m-a}^b \varrho(x)dx} \right) \\
&= f \left( \frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} \cdot m + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} \cdot \frac{\int_{2m-a}^b \varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx} \right). \\
\end{align*}
\]

Since \( (\int_{2m-a}^b x \varrho(x)dx)/(\int_{2m-a}^b \varrho(x)dx) = M_\rho(2m-a, b) \) – that is, a mean of \( 2m-a \) and \( b \), we get that

\[
b \geq M_\rho(2m-a, b) \geq 2m-a > m.
\]

Therefore, both \( m \) and \( M_\rho(2m-a, b) \) belong to the concavity domain of \( f \). Applying the concavity of \( f \), we conclude that the last expression in (*) is greater than or equal to

\[
\begin{align*}
\frac{\int_a^{2m-a} \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} \cdot f(m) + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_{2m-a}^b \varrho(x)dx} \cdot f(M_\rho(2m-a, b)) \\
&= \frac{f(m) \int_a^{2m-a} \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} \cdot f(M_\rho(2m-a, b)).
\end{align*}
\]

Using the lemma, we replace the numerator of the first expression on the right by \( \int_a^{2m-a} f(x)\varrho(x)dx \). Summarizing the above calculations, we obtain

\[
f(M_\rho(a, b)) \geq \frac{\int_a^{2m-a} f(x)\varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} + \frac{\int_{2m-a}^b \varrho(x)dx}{\int_a^{2m-a} \varrho(x)dx} \cdot f(M_\rho(2m-a, b)). \tag{8}
\]
Since \( f \) is concave over the interval \([2m - a, b]\), we can apply the left hand side inequality of Theorem 2 and get that

\[
f(M_\varrho(2m - a, b)) \geq \frac{1}{\int_{2m-a}^{b} \varrho(x)dx} \int_{2m-a}^{b} f(x)\varrho(x)dx.
\]

Substituting this in (8) we obtain that

\[
f(M_\varrho(a, b)) \geq \frac{\int_{a}^{b} f(x)\varrho(x)dx}{\int_{a}^{b} \varrho(x)dx} + \frac{\int_{2m-a}^{b} \varrho(x)dx}{\int_{2m-a}^{b} \varrho(x)dx} \cdot \frac{1}{\int_{2m-a}^{b} \varrho(x)dx} \int_{2m-a}^{b} f(x)\varrho(x)dx
\]

\[
= \frac{\int_{a}^{b} f(x)\varrho(x)dx}{\int_{a}^{b} \varrho(x)dx},
\]

that is, the proof of the first inequality is complete.

To prove the second inequality in (7), it is enough to prove that

\[
\int_{a}^{b} f(x)\varrho(x)dx \geq \frac{\int_{a}^{b} (b-x)\varrho(x)dx}{b-a} f(a) + \frac{\int_{a}^{b} (x-a)\varrho(x)dx}{b-a} f(b).
\] (9)

We need the following simple statements:

(A) \( f(m) \geq \frac{b-m}{b-a} f(a) + \frac{m-a}{b-a} f(b) \),

(B) \( f(2m - a) \geq \frac{b-2m+a}{b-a} f(a) + \frac{2(m-a)}{b-a} f(b) \).

For (A), observe that \( f \) is concave over the interval \([m, b]\), containing the point \(2m - a\). Thus,

\[
f(2m - a) \geq \frac{b-2m+a}{b-m} f(m) + \frac{m-a}{b-m} f(b).
\] (10)

Substituting \( 2f(m) - f(a) \) for \( f(2m - a) \) in (10), we obtain – after some transformations – (A).

Moreover, if we put in (10) \( f(a) + f(2m - a)\)/2 in place of \( f(m) \), after rearranging the inequality, we get (B).
After these preparations, we are ready to prove (9). First, applying the lemma,
\[
\int_a^b f(x)\varrho(x)\,dx = \int_a^{2m-a} f(x)\varrho(x)\,dx + \int_{2m-a}^b f(x)\varrho(x)\,dx
= f(m)\int_a^{2m-a} \varrho(x)\,dx + \int_{2m-a}^b f(x)\varrho(x)\,dx.
\]

In the first term on the right hand side, we may apply (A) for \(f(m)\). We can apply the right hand side inequality of Theorem 2 to the second term of the last expression since \(f\) is concave in the interval \([2m-a, b]\):
\[
\int_{2m-a}^b f(x)\varrho(x)\,dx \geq \frac{f_{2m-a}(b-x)\varrho(x)\,dx}{b-2m+a}f(2m-a)
+ \frac{f_{2m-a}(x-2m+a)\varrho(x)\,dx}{b-2m+a}f(b).
\]

Applying (A) and (B) to \(f(m)\) and \(f(2m-a)\), we get that
\[
\int_a^b f(x)\varrho(x)\,dx \geq \left(\frac{b-m}{b-a}f(a) + \frac{m-a}{b-a}f(b)\right)\int_a^{2m-a} \varrho(x)\,dx
+ \frac{f_{2m-a}(b-x)\varrho(x)\,dx}{b-2m+a}f(a)
+ \frac{f_{2m-a}(x-2m+a)\varrho(x)\,dx}{b-2m+a}f(b)
\]
\[
= \left[\frac{b-m}{b-a} \int_a^{2m-a} \varrho(x)\,dx + \frac{f_{2m-a}(b-x)\varrho(x)\,dx}{b-a}\right]f(a)
+ \left[\frac{m-a}{b-a} \int_a^{2m-a} \varrho(x)\,dx + \frac{2(m-a)f_{2m-a}(b-x)\varrho(x)\,dx}{b-2m+a}\right]f(b)
+ \frac{f_{2m-a}(x-2m+a)\varrho(x)\,dx}{b-2m+a}f(b).
\]

Finally, we will check that the coefficients of \(f(a)\) and \(f(b)\) are the desired ones.
First, from the lemma we get that \( \int_a^{2m-a} (m-x)\varrho(x)dx = 0 \). Thus,

\[
\frac{b-m}{b-a} \int_a^{2m-a} \varrho(x)dx + \frac{\int_a^b (b-x)\varrho(x)dx}{b-a}
\]

\[
= \frac{1}{b-a} \left( \int_a^{2m-a} (b-m)\varrho(x)dx + \int_{2m-a}^b (b-x)\varrho(x)dx \right)
\]

\[
= \frac{1}{b-a} \left( \int_a^b (b-x)\varrho(x)dx + \int_{2m-a}^b (b-x)\varrho(x)dx \right)
\]

\[
= \frac{1}{b-a} \int_a^b (b-x)\varrho(x)dx.
\]

This accounts for the coefficient of \( f(a) \). Moreover,

\[
\frac{2(m-a)}{b-a} \int_a^{2m-a} (b-x)\varrho(x)dx + \frac{\int_a^b (x-2m+a)\varrho(x)dx}{b-2m+a}
\]

\[
= \int_{2m-a}^b \frac{2(b-x)(m-a) + (x-2m+a)(b-a)}{(b-a)(b-2m+a)}\varrho(x)dx
\]

\[
= \int_{2m-a}^b \frac{x-a}{b-a}\varrho(x)dx,
\]

while, with the lemma, again,

\[
\frac{m-a}{b-a} \int_a^{2m-a} \varrho(x)dx = \int_a^{2m-a} \frac{m-a}{b-a} \varrho(x)dx = \int_a^{2m-a} \frac{x-a}{b-a} \varrho(x)dx.
\]

Therefore, the coefficient of \( f(b) \) equals

\[
\int_a^{2m-a} \frac{x-a}{b-a} \varrho(x)dx + \int_{2m-a}^b \frac{x-a}{b-a} \varrho(x)dx = \frac{1}{b-a} \int_a^b (x-a)\varrho(x)dx,
\]

as required. \( \square \)

References


PÉTER CZINDER
BERZE NAGY JÁNOS GRAMMAR SCHOOL
H-3200 GYÖNGYÖS, KOSSUTH STR. 33
HUNGARY
E-mail: pczinder@berze-nagy.sulinet.hu

(Received November 15, 2004; revised February 1, 2005)