On the derived length of Lie solvable group algebras

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Dedicated to Professor Adalbert Bovdi on his 70th birthday

Abstract. Let $G$ be a nilpotent group with cyclic commutator subgroup of order $p^n$ and let $F$ be a field of characteristic $p$. It is shown here that the Lie derived length of the group algebra $FG$ is at most $\lceil \log_2(p^n + 1) \rceil$. Furthermore, this bound is achieved if and only if one of the following conditions is satisfied: (i) $p$ is odd; (ii) $p = 2$ and $n \leq 2$; (iii) $p = 2$, $n \geq 3$ and the nilpotency class of $G$ is at most $n$.

1. Introduction

Let $G$ be a group and $F$ a field. The group algebra $FG$ may be considered as a Lie algebra, with the usual bracket operation. Define the Lie derived series and the strong Lie derived series of the group algebra $FG$ respectively as follows: let $\delta^{(0)}(FG) = \delta^{(0)}(FG) = FG$ and

$\delta^{[n+1]}(FG) = [\delta^{[n]}(FG), \delta^{[n]}(FG)]$,

$\delta^{(n+1)}(FG) = [\delta^{(n)}(FG), \delta^{(n)}(FG)] FG$,

where $[X, Y]$ is the additive subgroup generated by all Lie commutators $[x, y] = xy - yx$ with $x \in X$ and $y \in Y$. We say that $FG$ is Lie solvable if there exists $m \in \mathbb{N}$ such that $\delta^{[m]}(FG) = 0$ and the number

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$dl_L(FG) = \min\{m \in \mathbb{N} \mid \delta^{[m]}(FG) = 0\}$ is called the Lie derived length of $FG$. Similarly, $FG$ is said to be strongly Lie solvable of derived length $dl_L(FG) = m$ if $\delta^{(m)}(FG) = 0$ and $\delta^{(m-1)}(FG) \neq 0$. According to a result of Passi, Passman and Sehgal \cite{6} a group algebra $FG$ is Lie solvable if and only if one of the following conditions holds: (i) $G$ is abelian; (ii) $\text{char}(F) = p$ and the commutator subgroup $G'$ of $G$ is a finite $p$-group; (iii) $\text{char}(F) = 2$ and $G$ has a subgroup $H$ of index 2 whose commutator subgroup $H'$ is a finite 2-group. It is easy to check that a group algebra $FG$ is strongly Lie solvable if either $G$ is abelian or $\text{char}(F) = p$ and $G'$ is a finite $p$-group.

Let $G$ be a group with commutator subgroup of order $p^n$ and $\text{char}(F) = p$. Shalev \cite{8} showed that

$$dl_L(FG) \leq \lceil \log_2(2t(G')) \rceil,$$

where $t(G')$ denotes the nilpotent index of the augmentation ideal of $FG'$ and $\lceil r \rceil$ the upper integral part of a real number $r$. Moreover, Lemma 2.2 in \cite{8} states that if $G$ is nilpotent of class 2 then $dl_L(FG) \leq \lceil \log_2(t(G') + 1) \rceil$.

In particular, according to Proposition 2.3 in \cite{8}, if $G$ is nilpotent of class 2 and $G'$ is cyclic of order $p^n$, then

$$dl_L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

In this paper our goal is to generalize the above results of Shalev for the case when the nilpotency class of $G$ is not necessary 2. We obtain the following.

**Theorem 1.** Let $G$ be a nilpotent group with cyclic commutator subgroup of order $p^n$ and let $F$ be a field of characteristic $p$. Then $dl_L(FG) \leq \lceil \log_2(p^n + 1) \rceil$ with equality if and only if one of the following conditions holds:

(i) $p$ is odd;
(ii) $p = 2$ and $G'$ is of order less than 8;
(iii) $p = 2$, $n \geq 3$ and $G$ has nilpotency class at most $n$.

Moreover, if $\text{char}(F) = 2$ we can extend our result as follows.
Corollary 1. Let $G$ be a nilpotent group with commutator subgroup of order $2^n$ and let $F$ be a field of characteristic 2. Then $\text{dl}_L(FG) = n + 1$ if and only if one of the following conditions holds:

(i) $G'$ is the noncyclic group of order 4 and $\gamma_3(G) \neq 1$;
(ii) $G'$ is cyclic of order less than 8;
(iii) $G'$ is cyclic, $n \geq 3$ and $G$ has nilpotency class at most $n$.

In this paper $\omega(FG)$ denotes the augmentation ideal of $FG$; for a normal subgroup $H \subseteq G$ we understand by $I(H)$ the ideal $FG \cdot \omega(FH)$. For $x, y, x_1, x_2, \ldots, x_n \in G$ let $x^y = y^{-1}xy$, $(x, y) = x^{-1}x^y$, and the commutator $(x_1, x_2, \ldots, x_n)$ is defined inductively to be $((x_1, x_2, \ldots, x_{n-1}), x_n)$. By $\zeta(G)$ we mean the center of the group $G$, by $\gamma_n(G)$ the $n$-th term of the lower central series of $G$ with $\gamma_1(G) = G$. Furthermore, denote by $C_n$ the cyclic group of order $n$. The $n$-th term of the upper Lie power series of $FG$ is denoted by $(FG)^{(n)}$ which is the associative ideal generated by all Lie commutators $[x, y]$ with $x \in FG^{(n-1)}$ and $y \in FG$, where $FG^{(1)} = FG$.

We shall use freely the identities

$$[x, yz] = [x, y]z + y[x, z], \quad [xy, z] = x[y, z] + [x, z]y,$$

and for units $a, b, c$ the commutator identities

$$(a, bc) = (a, c)(a, b)^c = (a, c)(a, b)(a, b, c);$$

$$(ab, c) = (a, c)^b(b, c) = (a, c)(a, c, b)(b, c),$$

and that $[a, b] = ba((a, b) - 1)$.

2. Preliminaries

We begin with a statement of independent interest about the strong Lie derived length of group algebras which generalizes the Corollary 4 of Bagiński’s paper [1].

Proposition 1. Let $G$ be a nilpotent group whose commutator subgroup $G'$ is a finite $p$-group and let $\text{char}(F) = p$. If $\gamma_3(G) \subseteq (G')^p$ then

$$\text{dl}_L(FG) = \lceil \log_2(t(G') + 1) \rceil.$$
Proof. We show by induction on \( n \) that
\[
\delta^{(n)}(FG) \subseteq (FG)^{(2^n)} \quad \text{for all } n \geq 0.
\]
Evidently, \( \delta^{(0)}(FG) = (FG)^{(1)} \) and assume that \( \delta^{(n)}(FG) \subseteq (FG)^{(2^n)} \) for some \( n \). By elementary properties of upper Lie power series,
\[
\delta^{(n+1)}(FG) = [\delta^{(n)}(FG), \delta^{(n)}(FG)] \subseteq [(FG)^{(2^n)}, (FG)^{(2^n)}] \subseteq (FG)^{(2^{n+1})} = (FG)^{(2^{n+1})}.
\]

In view of \( \gamma_3(G) \subseteq (G')^p \), Theorem 3.1(i) from [3] states that \( (FG)^{(2^n)} = \mathcal{J}(G')^{2^n-1} \). Furthermore, Lemma 2.2 in [7] asserts \( \mathcal{J}(G')^{2^n-1} \subseteq \delta^{(n)}(FG) \) for all \( n \geq 1 \) and we have \( \delta^{(n)}(FG) = \mathcal{J}(G')^{2^n-1} \). It is easy to see that \( \delta^{(n)}(FG) = 0 \) if and only if \( 2^n - 1 \geq t(G') \), therefore \( n \geq \log_2(t(G') + 1) \), which implies the statement. \( \square \)

Remark 1. (i) Since \( \delta^{[n]}(FG) \subseteq \delta^{(n)}(FG) \) for all \( n \), Proposition 1 yields an upper bound on the Lie derived length. Furthermore, if \( G \) is nilpotent with cyclic commutator subgroup of order \( p^n \), then the condition \( \gamma_3(G) \subseteq (G')^p \) holds and thus
\[
dl_L(FG) \leq \lceil \log_2(p^n + 1) \rceil.
\]

But, as we will see, the equality does not always hold.

(ii) As the following examples show, Proposition 1 breaks down without the condition \( \gamma_3(G) \subseteq (G')^p \):

- Let \( G \) be a group with \( G' = C_2 \times C_2 \) such that \( \gamma_3(G) \neq 1 \) and let \( \text{char}(F) = 2 \). Then \( \gamma_3(G) \not\subseteq (G')^2 \) and, by Theorem 3 and Theorem 6 from [5], \( \text{dl}_L(FG) > 2 \). So \( \text{dl}_L(FG) \neq \lceil \log_2(t(G') + 1) \rceil \), because now \( \lceil \log_2(t(G') + 1) \rceil = 2 \).

- Let \( G \) be a group with \( G' = C_3 \times C_3 \times C_3 \) such that \( \gamma_3(G) \neq 1 \) and let \( \text{char}(F) = 3 \). Then \( \gamma_3(G) \not\subseteq (G')^3 \) and, by Theorem 2.3 from [7], \( \text{dl}_L(FG) > 3 \). It follows that \( \text{dl}_L(FG) \neq \lceil \log_2(t(G') + 1) \rceil \), because \( \lceil \log_2(t(G') + 1) \rceil = 3 \).
The next lemma will be used in the proof of the theorem.

**Lemma 1.** Let $G$ be a nilpotent group with cyclic commutator subgroup of order $p^n$ and let $\text{char}(F) = p$. Then for all $m, k \geq 1$

(i) $[\omega^m(FG'), \omega(FG)] \subseteq \mathfrak{I}(G')^{m+p-1}$;

(ii) $[\mathfrak{I}(G')^m, \mathfrak{I}(G')^k] \subseteq \mathfrak{I}(G')^{m+k+1}$.

**Proof.** (i) We use induction on $m$. For every $y \in G'$ and $g \in G$ we have

$[y - 1, g - 1] = [y, g] = gy((y, g) - 1) \in \mathfrak{I}(\gamma_3(G)) \subseteq \mathfrak{I}(G')^p$.

This shows that the statement (i) holds for $m = 1$, because the elements of the form $g - 1$ with $1 \neq g \in G$ constitute an $F$-basis of $\omega(FG)$.

Now, assume that $[\omega^m(FG'), \omega(FG)] \subseteq \mathfrak{I}(G')^{m+p-1}$ for some $m$. Then

$[\omega^{m+1}(FG'), \omega(FG)]$

$\subseteq \omega^m(FG') [\omega(FG'), \omega(FG)] + [\omega^m(FG'), \omega(FG)] \omega(FG')$

$\subseteq \omega^m(FG') \mathfrak{I}(G')^p + \mathfrak{I}(G')^{m+p-1} \omega(FG') \subseteq \mathfrak{I}(G')^{m+p},$

and the proof of (i) is complete.

(ii) The statement (ii) is a consequence of (i), because

$\mathfrak{I}(G') = \omega(FG)\omega(FG') + \omega(FG')$.

Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1 \rangle$, where $n \geq 3$. It is well known that the automorphism group $\text{aut}(G')$ of $G'$ is a direct product of the cyclic group $\langle \alpha \rangle$ of order 2 and the cyclic group $\langle \beta \rangle$ of order $2^{n-2}$ where the action of these automorphisms on $G'$ is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. For $g \in G$, let $\tau_g$ denote the restriction to $G'$ of the inner automorphism $h \mapsto h^g$ of $G$. The map $G \to \text{aut}(G)$, $g \mapsto \tau_g$ is a homomorphism whose kernel coincides with the centralizer $C = C_G(G')$. Clearly, the map $\varphi : G/C \to \text{aut}(G')$ given by $\varphi(gC) = \tau_g$ is a monomorphism.

The subset

$G_\beta = \{g \in G \mid \varphi(gC) \in \langle \beta \rangle \}$

of $G$ will play an important role in the sequel. It is easy to check that $G_\beta$ is a subgroup of index not greater than 2 and $g \in G_\beta$ if and only if $x^g = x^{5^i}$ for some $i \in \mathbb{Z}$. 
Lemma 2. Let $G$ be a group with cyclic commutator subgroup of order $2^n$, where $n \geq 3$ and let $\text{char}(F) = 2$. Then

(i) $(y,g) \in (G')^4$ for all $y \in G'$ and $g \in G_{\beta}$;

(ii) $[\omega^m(FG'), \omega(FG_{\beta})] \subseteq I(G')^m + 3$.

Proof. Let $g \in G_{\beta}$ and $y \in G'$.

(i) Clearly, $(y,g) = y^{-1}y^g = y^{-1} + 5i$ for some $i \geq 0$ and $-1 + 5i \equiv 0 \pmod{4}$. Therefore, $(y,g) \in (G')^4$.

(ii) Using (i) we have that $[y^{-1}, g^{-1}] = [y, g] = gy((y, g) - 1) \in I(G')^4$, from which (ii) follows for $m = 1$. One can now finish the proof by induction, as in Lemma 1(i). \hfill \Box

Lemma 3. Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1 \rangle$, where $n \geq 3$. Then the following are equivalent:

(i) $G_{\beta} = G$.

(ii) $G$ has nilpotency class at most $n$.

Proof. First of all, note that $G$ is a nilpotent group of class at most $n + 1$.

(i) $\Rightarrow$ (ii) By Lemma 2(i), $\gamma_3(G) \subseteq (G')^4$, so $|\gamma_2(G)/\gamma_3(G)| \geq 4$ and the class of $G$ is at most $n$.

(ii) $\Rightarrow$ (i) Suppose that $G$ has nilpotency class at most $n$, but $G_{\beta} \neq G$. We claim that $x^{2k-2} \in \gamma_k(G)$ for all $k \geq 2$. Indeed, this is clear for $k = 2$ and assume its truth for some $k \geq 2$. If $g \in G \setminus G_{\beta}$ then $(x^{2k-2}, g) \in \gamma_{k+1}(G)$ and $(x^{2k-2}, g) = x^{2k-2}(-1 + 5i) = (x^{2k-1})^j$ with some $i$ and odd $j$. This means that $x^{2k-1} \in \gamma_{k+1}(G)$, as desired. Therefore, $\gamma_{n+1}(G) \neq 1$, which is a contradiction. \hfill \Box

Lemma 4. Let $G$ be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1 \rangle$, where $n \geq 3$. If $G$ has nilpotency class $n + 1$ then $(g, h) \in (G')^2$ for all $g, h \in G_{\beta}$.

Proof. If the lemma were not true we could choose the elements $g, h \in G_{\beta}$ so that $(g, h) = x$. By definition of $G_{\beta}$ we may additionally
assume that \((g, x) = 1\). Lemma 3 states that \(G \setminus G_\beta \neq \emptyset\); let \(y\) be in \(G \setminus G_\beta\). Evidently, \((g, y) = x^i\) for some \(i\). Using the equalities
\[
g^h = gx, \quad g^{h-1} = g(x^{-1})^{h-1}, \quad g^y = gx^i, \quad g^{y^{-1}} = g(x^{-i})y^{-1}
\]
it is easy to check
\[
g = g^{(h, y)} = g^{h^{-1}y^{-1}hy} = (g(x^{-1})^{h^{-1}}y^{-1}hy = g^{y^{-1}hy}x^{-1}
\]
which is a contradiction. Indeed, keeping in mind that \((x, y) = x^{-1-5j}\in \langle x^2 \rangle \setminus \langle x^4 \rangle\) and \((x^{-i}, h^y) = x^{i(1-5j)}\in \langle x^4 \rangle\),
thus \((x, y)(x^{-i}, h^y) \neq 1\).

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**Lemma 5.** Let \(G\) be a group with commutator subgroup \(G' = \langle x \mid x^{2^n} = 1 \rangle\), where \(n \geq 3\) and let \(\text{char}(F) = 2\). If \(G\) has nilpotency class \(n + 1\) then \(\text{dl}_L(FG) \leq n\).

**Proof.** Clearly, the set of the Lie commutators \([a, b]\) with \(a, b \in G\) spans the \(F\)-space \(\delta^{[1]}(FG)\). Since \([a, b] = g^h + g\) with \(g = ba\) and \(h = b\),
while of course \(g^h + g = [a, b]\) with \(a = h^{-1}g\) and \(b = h\) whenever \(g, h \in G\),
this spanning set for \(\delta^{[1]}(FG)\) can also be described as the set of the elements \(g^h + g\) with \(g, h \in G\). It follows that the Lie commutators
\([g_1^{h_1} + g_1, g_2^{h_2} + g_2]\), where \(g_1, g_2, h_1, h_2 \in G\), span \(\delta^{[2]}(FG)\). We shall compute these Lie commutators. It is easy to check that
\[
[g_1^{h_1} + g_1, g_2^{h_2} + g_2] = g_2g_1\left( ((g_1, g_2) + 1)((g_2, h_2) + 1)((g_1, h_1) + 1) + (g_2, h_2)((g_2, h_2, g_1) + 1)((g_1, h_1) + 1) + (g_1, g_2)((g_1, h_1, g_2) + 1)((g_2, h_2) + 1) \right) \quad (1)
\]

Firstly, if neither \(g_1\) nor \(g_2\) are in \(G_\beta\) then
\[
[g_1^{h_1} + g_2, g_2^{h_2} + g_2] = b_{g_3} \quad (2)
\]
for some $b \in G_\beta$ and $g_3 \in \omega^3(FG')$. Indeed, it is clear from the definition of $G_\beta$ that then $g_2 g_1 \in G_\beta$. Furthermore, the second factor on the right-hand side of (1) always belongs to $\omega^3(FG')$, because $\gamma_3(G) \subseteq (G')^2$.

Secondly, if $g_1$ or $g_2$, say $g_1$, belongs to $G_\beta$, then we claim that

$$[g_1^{h_1} + g_1 g_2^{h_2} + g_2] = g_4$$

for some $g_4 \in \omega^4(FG')$ and $g \in G$.

For $g_1 \in G_\beta$, Lemma 2(i) asserts $(g_2, h_1, g_1) \in (G')^4$, therefore the right-hand side of (1) can be written as

$$[g_1^{h_1} + g_1 g_2^{h_2} + g_2] = g_2 g_1 \left( ((g_1, g_2) + 1) ((g_1, h_1) + 1) + (g_1, g_2)(g_1, h_1) (g_1, h_1, g_2) + 1 \right) + g_2 g_1 g_4$$

for some $g_4 \in \omega^4(FG')$. In order to prove (3) it will be sufficient to show that the element for some $g_4 \in \omega^4(FG')$. In order to prove (3) it will be sufficient to show that the element

$$\theta = ((g_1, g_2) + 1) ((g_1, h_1) + 1) + (g_1, g_2)(g_1, h_1) (g_1, h_1, g_2) + 1$$

from the right-hand side of (4) belongs to $\omega^3(FG')$.

This is clear if $g_2$ also belongs to $G_\beta$, because then by Lemma 4 and Lemma 2(i) both summands of $\theta$ are in $\omega^3(FG')$. Furthermore, if $g_2 \notin G_\beta$, then $x^{g_2} = x^{-g_1}$ for some $l$ and we distinguish the following three cases:

Case 1: $(g_1, h_1) \in (G')^2$. Then $(g_1, h_1, g_2) = (g_1, h_1)^{-1}-5^{g_1} \in (G')^4$ and $\theta \in \omega^3(FG')$.

Case 2: $(g_1, g_2) \in (G')^2$. By the well-known Hall–Witt identity,

$$(g_1, h_1, g_2)^{h_1^{-1}} (g_1^{-1}, g_2^{-1}, g_1)^{g_2} (g_2, g_1^{-1}, h_1^{-1}) g_1 = 1.$$}

Lemma 2(i) ensures that the second factor on the left-hand side belongs to $(G')^4$ and this is true for the last factor too, because

$$ (g_2, g_1^{-1}, h_1^{-1}) = \left( (g_1, g_2)^{g_1^{-1}}, h_1^{-1} \right) = \left( (g_1, g_2)^{g_1^{-1}} \right)^{-1} (g_1, g_2) g_1^{-1} h_1^{-1} = (g_1, g_2)^{2^l}.$$
for some $i$. This means that $(g_1, h_1, g_2) \in (G')^4$, which proves $\vartheta \in \omega^3(FG')$.

Case 3: $(g_1, h_1) \notin (G')^2$ and $(g_1, g_2) \notin (G')^2$. Then $\langle (g_1, h_1) \rangle = \langle (g_1, g_2) \rangle = G'$ and $(g_1, g_2) = (g_1, h_1)^k$ for some odd $k$. With the notation $y = (g_1, h_1)$ $\vartheta$ can be written as

$$\vartheta = (y^k + 1)(y + 1) + y^{k+1}(y^{-5^l} - 1) = y^{k-5^l} + 1 + y^{y-1 + 1}.$$  

Of course, if $k \equiv 1 \pmod{4}$ then $y^{-5^l} - 1 + 1$ and $y^{y-1 + 1}$ are in $\omega^4(FG')$, therefore $\vartheta \in \omega^4(FG')$. Otherwise, if $k \equiv 3 \pmod{4}$ then $y^{y-1 + 1} \in \omega^4(FG')$ which implies that

$$y(y^{k-1} + 1) = y((y^{k-3} + 1)(y^2 + 1) + (y^{k-3} + 1) + (y^2 + 1))$$

$$\equiv y(y^2 + 1) \equiv y^2 + 1 \pmod{\omega^3(FG')}.$$  

Similarly, we can obtain that

$$y^{k-5^l} + 1 = (y^{k-5^l} - 2 + 1)(y^2 + 1) + (y^{k-5^l} - 2 + 1) + (y^2 + 1)$$

$$\equiv y^2 + 1 \pmod{\omega^3(FG')}.$$  

Hence

$$\vartheta = y^{k-5^l} + 1 + y^{y-1 + 1} \equiv 2(y^2 + 1) \equiv 0 \pmod{\omega^3(FG')}$$

which completes the checking of (3).

Let $S$ be the additive subgroup generated by all elements of the form $g_2 \eta_4$ and $b \eta_3$, where $g \in G$, $b \in G$ and $\eta_3 \in \omega^3(FG')$, $\eta_4 \in \omega^4(FG')$. We claim that $[S, S] \subseteq \mathcal{Z}(G')^8$. Indeed, the additive subgroup $[S, S]$ can be spanned by some Lie commutators of the forms $[g \eta_3, h \eta_3]$ and $[b \eta_3, b_2 \eta_3]$ with $g \in G, b_1, b_2 \in G$, $\eta_3, \eta_3 \in \omega^3(FG')$, $\eta_4 \in \omega^4(FG')$. Furthermore, by Lemma 1(i),

$$[g \eta_3, h \eta_3] = g[\eta_3, h \eta_3] + [g, h \eta_3] \eta_3$$

$$= g[\eta_3, h + 1] \eta_3 + h g((g, h) + 1) \eta_3 \eta_4 + h[g + 1, \eta_3] \eta_4 \in \mathcal{Z}(G')^8,$$

and by Lemma 2(ii) and Lemma 4,

$$[b \eta_3, b_2 \eta_3] = b_1[b \eta_3, b_2 \eta_3] + [b_1, b_2 \eta_3] \eta_3$$

$$= b_1[b \eta_3, b_2 + 1] \eta_3 + b_2[b_1 + 1, \eta_3] \eta_3 + b_1 b_2((b_1, b_2) + 1) \eta_3 \eta_3.$$
also belongs to $\mathcal{I}(G')^8$. Therefore, $[S, S] \subseteq \mathcal{I}(G')^8$.

From (2) and (3) we get $\delta^{[2]}(FG) \subseteq S$, so we have

$$\delta^{[3]}(FG) = \left[ \delta^{[2]}(FG), \delta^{[2]}(FG) \right] \subseteq [S, S] \subseteq \mathcal{I}(G')^8.$$ 

Now, we use induction on $k$ to show that

$$\delta^{[k]}(FG) \subseteq \mathcal{I}(G')^{2^k} \quad \text{for all} \quad k \geq 3. \quad (5)$$

Indeed, assuming the validity of (5) for some $k \geq 3$ we have

$$\delta^{[k+1]}(FG) = \left[ \delta^{[k]}(FG), \delta^{[k]}(FG) \right] \subseteq \left[ \mathcal{I}(G')^{2^k}, \mathcal{I}(G')^{2^k} \right] \subseteq \mathcal{I}(G')^{2^{k+1}}$$

and this proves the truth of (5) for every $k \geq 3$.

Keeping in mind that $G'$ has order $2^n$, (5) implies that $\delta^{[n]}(FG) = 0$.

Hence $dl_L(FG) \leq n$ and the proof is complete. \hfill $\square$

**Lemma 6.** Let $G$ be a nilpotent group with commutator subgroup $G' = \langle x \mid x^{p^n} = 1 \rangle$, $\text{char}(F) = p$ and assume that one of the following conditions holds:

(i) $p = 2$, $n \geq 3$ and $G$ has nilpotency class at most $n$;

(ii) $p$ is odd.

Then $dl_L(FG) = \lceil \log_2(p^n + 1) \rceil$.

**Proof.** Since $G'$ is cyclic of order $p^n$, we can choose $a, b \in G$ such that $(a, b) = x$. First of all, we claim that

$$[b^l a^m, b^s a^t] \equiv (ms - lt) b^{l+s} a^{m+t} (x - 1) \pmod{\mathcal{I}(G')^2} \quad (6)$$

for every $l, s, m, t \in \mathbb{Z}$. Indeed, an easy computation yields

$$[b^l a^m, b^s a^t] = b^s a^t b^l a^m ((b^l a^m, b^s a^t) - 1)$$

$$= b^{l+s} a^{m+t} (a^t, b^l)^m ((b^l a^m, b^s a^t) - 1) \quad (7)$$

and

$$\equiv b^{l+s} a^{m+t} ((b^l a^m, b^s a^t) - 1) \pmod{\mathcal{I}(G')^2},$$

and

$$\equiv x^{ms-lt} \pmod{(G')^p},$$

$$\equiv x^{ms-lt} \pmod{(G')^p}.$$
because $\gamma_3(G) \subseteq (G')^p$. Thus $(b^la^m, b^sa^t) = x^{ms-lt+pi}$ for some $i$. In view of the identity $uv - 1 = (u - 1)(v - 1) + (u - 1) + (v - 1)$, we have

$$(b^la^m, b^sa^t) - 1 \equiv (ms - lt + pi)(x - 1) \equiv (ms - lt)(x - 1) \pmod{(G')^2}$$

and putting this into (7) we obtain (6).

Now, let $k \geq 1$, $l, m, s, t \in \mathbb{Z}$, $z_1, z_2 \in \mathcal{I}(G')^{2k}$ and set

$$f_k(l, m, s, t, z_1, z_2) = [b^la^m(x - 1)^{2k-1} + z_1, b^sa^t(x - 1)^{2k-1} + z_2].$$

We shall show that

$$f_k(l, m, s, t, z_1, z_2) \equiv (ms - lt)b^{l+s}a^{m+t}(x - 1)^{2k+1-1} \pmod{(G')^{2k+1}}. \quad (8)$$

Lemma 1(ii) ensures that the elements $[b^la^m(x - 1)^{2k-1}, z_2], [z_1, z_2]$ and $[z_1, b^sa^t(x - 1)^{2k-1}]$ belong to $\mathcal{I}(G')^{2k+1}$, thus

$$f_k(l, m, s, t, z_1, z_2) \equiv [b^la^m(x - 1)^{2k-1}, b^sa^t(x - 1)^{2k-1}] \pmod{(G')^{2k+1}}.$$ 

In the case $p = 2$, Lemma 3 forces $b^la^m, b^sa^t \in G_\beta$, so we may apply Lemma 2(ii) to obtain that

$$(b^la^m(x - 1)^{2k-1}], [(x - 1)^{2k-1}, b^sa^t] \in \mathcal{I}(G')^{2k+1}.$$ 

Furthermore, for $p > 2$ the above inclusion follows from Lemma 1(i). This implies that

$$f_k(l, m, s, t, z_1, z_2) \equiv [b^la^m, b^sa^t](x - 1)^{2k+1-2} \pmod{(G')^{2k+1}},$$

which, together with (6), proves (8).

Define the following three series inductively by:

$$u_0 = a, \quad v_0 = b, \quad w_0 = b^{-1}a^{-1},$$
and, for \( k > 0 \),
\[
u_{k+1} = [u_k, v_k], \quad v_{k+1} = [u_k, w_k], \quad w_{k+1} = [w_k, v_k].
\]
Obviously, the \( k \)-th elements of these series belong to \( \delta^{[k]}(FG) \). By induction on \( k \) we show for odd \( k \) that
\[
u_k \equiv \pm ba(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}};
\]
\[
v_k \equiv \pm b^{-1}(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}};
\]
\[
w_k \equiv \pm a^{-1}(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}},\]
(9)
and if \( k \) is even then
\[
u_k \equiv \pm a(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}};
\]
\[
v_k \equiv \pm b(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}};
\]
\[
w_k \equiv \pm b^{-1}a^{-1}(x - 1)^{2^k - 1} \pmod{\mathcal{I}(G')^{2^k}}.\]
(10)
Evidently, \( u_1 = [a, b] = ba(x - 1) \), and by (6) we have
\[
v_1 = [a, b^{-1}a^{-1}] \equiv -b^{-1}(x - 1) \pmod{\mathcal{I}(G')^2},
\]
and \( w_1 = [b^{-1}a^{-1}, b] \equiv -a^{-1}(x - 1) \pmod{\mathcal{I}(G')^2} \). Therefore (9) holds for \( k = 1 \).

Now, assume that (9) is true for some odd \( k \). According to (8) the congruences
\[
u_{k+1} = \pm f_k(1, 1, -1, 0, u_k', v_k')
\equiv \pm (-1)a(x - 1)^{2^{k+1} - 1} \pmod{\mathcal{I}(G')^{2^{k+1}}};
\]
\[
v_{k+1} = \pm f_k(1, 1, 0, -1, u_k', v_k')
\equiv \pm b(x - 1)^{2^{k+1} - 1} \pmod{\mathcal{I}(G')^{2^{k+1}}};
\]
\[
w_{k+1} = \pm f_k(0, -1, -1, 0, u_k', v_k')
\equiv \pm b^{-1}a^{-1}(x - 1)^{2^{k+1} - 1} \pmod{\mathcal{I}(G')^{2^{k+1}}}
\]
hold, where \( u_k', v_k', w_k' \) are suitable elements from \( \mathcal{I}(G')^{2^k} \). Similarly,
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supposing the truth of (10) for some even \( k \) we see

\[
\begin{align*}
  u_{k+1} &= \pm f_k(0, 1, 1, 0, u_k', v_k') \\
  &\equiv \pm ba(x - 1)^{2^{k+1}-1} \pmod{\mathfrak{J}(G')^{2^{k+1}}}; \\
  v_{k+1} &= \pm f_k(0, 1, -1, -1, u_k', v_k') \\
  &\equiv \pm (-1)b^{-1}(x - 1)^{2^{k+1}-1} \pmod{\mathfrak{J}(G')^{2^{k+1}}}; \\
  w_{k+1} &= \pm f_k(-1, -1, 1, 0, u_k', v_k') \\
  &\equiv \pm (-1)a^{-1}(x - 1)^{2^{k+1}-1} \pmod{\mathfrak{J}(G')^{2^{k+1}}}.
\end{align*}
\]

So, (9) and (10) are valid for any \( k > 0 \).

Assume that \( k < \lceil \log_2(p^n + 1) \rceil \). Then \( 2^k - 1 < p^n \) and the elements \( u_k, v_k, w_k \) are nonzero in \( \delta[k](FG) \), thus \( dL(FG) \geq \lceil \log_2(p^n + 1) \rceil \).

At the same time, Remark 1(i) says that \( dL(FG) \leq \lceil \log_2(p^n + 1) \rceil \). \( \square \)

3. Proofs of Theorem 1 and Corollary 1

**Proof of Theorem 1.** For \( p = 2 \) and \( n < 3 \) the statement is a consequence of Remark 1(i) and Theorem 3 in [5]. In the other cases Lemma 5 and Lemma 6 state the required result. The proof is complete. \( \square \)

**Proof of Corollary 1.** Clearly, if \( G' \) is cyclic the statement immediately follows from Theorem 1. Now, assume that \( G' \) is noncyclic and \( \delta[n](FG) \neq 0 \). We know from [2] that \( FG \) is Lie nilpotent, and as we have already seen, \( \delta[n](FG) \subseteq (FG)^{(2^n)} \). Thus \( (FG)^{(2^n)} \neq 0 \) and Theorem 1 of [4] states that \( G' = C_2 \times C_2 \) and \( \gamma_3(G) \neq 1 \). Conversely, if \( G' = C_2 \times C_2 \) then \( t(G') = 3 \) and \( dL(FG) \leq \lceil \log_2(2 \cdot 3) \rceil = 3 \). Furthermore, when \( \gamma_3(G) \neq 1 \), Theorem 3 in [5] says that \( dL(FG) \neq 2 \). Therefore \( dL(FG) = 3 \) and the corollary is proved. \( \square \)

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