An open problem concerning the diophantine equation $a^x + b^y = c^z$

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Abstract. Let $r$ be an odd integer with $r > 1$, and let $m$ be an even integer with $m \equiv 2 \pmod{4}$. Let $a, b, c$ be positive integers satisfying $(a, b, c) = (|V(r)|, |U(r)|, m^2 + 1)$, where $V(r) + U(r)\sqrt{-1} = (m + \sqrt{-1})^r$. In this paper we prove that if $c$ is a prime and either $r \not\equiv 1 \pmod{8}$ and $m > 2r/\pi$ or $r \equiv 1 \pmod{8}$ and $m > 41r^{3/2}$, then the equation $a^x + b^y = c^z$ has only the positive integer solution $(x, y, z) = (2, 2, r)$.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$ be the sets of all integers and positive integers respectively. Let $a$, $b$, $c$ be fixed positive integers such that $\min(a, b, c) > 1$ and $\gcd(a, b, c) = 1$. Let $r$ be an odd integer with $r > 1$. In this paper we consider the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N}$$

for the case that $a$, $b$ and $c$ satisfy

$$a = |V(r)|, \quad b = |U(r)|, \quad c = m^2 + 1,$$

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where \( m \) is an even integer and
\[
V(r) + U(r)\sqrt{-1} = (m + \sqrt{-1})^r. \tag{3}
\]
We see from (3) that \( V(r) \) and \( U(r) \) are integers satisfying
\[
(V(r))^2 + (U(r))^2 = (m^2 + 1)^r, \quad \gcd(V(r), U(r)) = 1, \quad 2 \mid V(r). \tag{4}
\]
It follows that if (2) holds, then
\[
a^2 + b^2 = c^r \tag{5}
\]
and (1) has a solution \((x, y, z) = (2, 2, r)\). In [1], CAO proposed the following problem.

**Open Problem.** Let \( m \equiv 2 \pmod{4} \) and \( c \) is a prime. It is possible to prove (1) has only the solution \((x, y, z) = (2, 2, r)\) by some elementary methods?

The above mentioned problem is related to a wide conjecture by Terai (see [6], [8]). By the proofs of [1, Corollaries 1 and 2], the answer to the question is “yes” for \( r = 3 \) or 5. In this paper, using some elementary methods, we prove the following theorem.

**Theorem 1.** If (2) holds, \( r \not\equiv 1 \pmod{8} \), \( m \equiv 2 \pmod{4} \), \( m > 2r/\pi \) and \( c \) is a prime, then (1) has only the solution \((x, y, z) = (2, 2, r)\).

On the other hand, using a lower bound for linear forms in two logarithms given by Laurent, Mignotte and Nesterenko [3], we solve the remained cases as follows.

**Theorem 2.** If (2) holds, \( r \equiv 1 \pmod{8} \), \( m \equiv 2 \pmod{4} \), \( m > 41r^{3/2} \) and \( c \) is a prime, then (1) has only the solution \((x, y, z) = (2, 2, r)\).

## 2. Proof of Theorem 1

**Lemma 1** ([7, Formula 1.76]). For any positive integer \( n \) and any complex numbers \( \alpha, \beta \), we have
\[
\alpha^n + \beta^n = \sum_{j=0}^{[n/2]} \binom{n}{j} (\alpha + \beta)^{n-2j} (-\alpha \beta)^j,
\]
where
\[
\left[\frac{n}{j}\right] = \frac{(n-j-1)!n}{(n-2j)!j!}, \quad j = 0, 1, \ldots, \left[\frac{n}{2}\right]
\]
are positive integers.

For any positive integer \(n\), let
\[
V(n) = \frac{1}{2}(\varepsilon^n + \bar{\varepsilon}^n), \quad U(n) = \frac{1}{2\sqrt{-1}}(\varepsilon^n - \bar{\varepsilon}^n), \quad (6)
\]
\[
E(n) = \frac{\varepsilon^n + \bar{\varepsilon}^n}{\varepsilon + \bar{\varepsilon}} , \quad F(n) = \frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}}, \quad (7)
\]
where
\[
\varepsilon = m\sqrt{-1}, \quad \bar{\varepsilon} = m - \sqrt{-1}. \quad (8)
\]
Clearly, \(V(n), U(n)\) and \(F(n)\) are integers for any \(n\), and \(E(n)\) is an integer if \(2 \nmid n\).

**Lemma 2.** If \(m > 2r/\pi\), then \(V(n), U(n), E(n)\) and \(F(n)\) are positive numbers for \(n = 1, 2, \ldots, r\).

**Proof.** Since \(m^2 + 1 = c\), we see from (8) that
\[
\varepsilon = \sqrt{c}e^{\theta\sqrt{-1}}, \quad \bar{\varepsilon} = \sqrt{c}e^{-\theta\sqrt{-1}}, \quad (9)
\]
where \(\theta\) is a unique real number satisfying
\[
\tan \theta = \frac{1}{m}, \quad 0 < \theta < \frac{\pi}{2}. \quad (10)
\]
Substitute (9) into (6) and (7), we get
\[
V(n) = c^{n/2} \cos(n\theta), \quad U(n) = c^{n/2} \sin(n\theta) \quad (11)
\]
and
\[
E(n) = c^{(n-1)/2} \frac{\cos(n\theta)}{\cos \theta}, \quad F(n) = c^{(n-1)/2} \frac{\sin(n\theta)}{\sin \theta}, \quad (12)
\]
respectively. By (10), we get
\[
0 < \theta = \arctan \frac{1}{m} < \frac{1}{m}. \quad (13)
\]
Hence, if \(m > 2r/\pi\), then \(0 < n\theta < n\pi/2r\). It follows that \(0 < n\theta < \pi/2\) if \(n \leq r\). Thus, by (11) and (12), the lemma is proved. \(\square\)
Lemma 3. If $n$ is an odd integer, then we have

(i) $E(n) \equiv (-1)^{(n-1)/2}n \mod m^2$, $E(n) \equiv (-1)^{(n-1)/2}2^{n-1} \mod c$.

(ii) $E(n) \equiv \begin{cases} 
1 \mod 8, & \text{if } m \equiv 2 \mod 4 \text{ and } n \equiv 1, 3 \mod 8 \\
5 \mod 8, & \text{if } m \equiv 2 \mod 4 \text{ and } n \equiv 5, 7 \mod 8 \\
\text{or } m \equiv 0 \mod 4 \text{ and } n \equiv 1, 7 \mod 8,
\end{cases}$

(iii) $F(n) \equiv (-1)^{(n-1)/2} \mod m^2$, $F(n) \equiv (-1)^{(n-1)/2}2^{n-1} \mod c$.

(iv) $F(n) \equiv \begin{cases} 
1 \mod 8, & \text{if } n \equiv 1 \mod 4 \\
3 \mod 8, & \text{if } m \equiv 2 \mod 4 \text{ and } n \equiv 3 \mod 4 \\
7 \mod 8, & \text{if } m \equiv 0 \mod 4 \text{ and } n \equiv 3 \mod 4.
\end{cases}$

(V) $E(n) \equiv -c^{(n-1)/2} \mod E(\ell)$, $E(n) \equiv c^{(n-1)/2} \mod F(\ell)$, where $\ell = (n + (-1)^{(n-1)/2})/2$.

Proof. By (8), we get

\[ \varepsilon + \bar{\varepsilon} = 2m, \quad \varepsilon - \bar{\varepsilon} = 2\sqrt{-1}, \quad \varepsilon\bar{\varepsilon} = c. \]  

(14)

Since $2 \nmid n$, by Lemma 1, we get from (7) that

\[ E(n) = \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i} m^{n-2i-1} = \sum_{i=0}^{(n-1)/2} \binom{n}{i} (2m)^{n-2i-1}(-c)^i, \]  

(15)

\[ F(n) = \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n}{2i+1} m^{n-2i-1} \]

\[ = \sum_{i=0}^{(n-1)/2} \binom{n}{i} (-4m^2)^{(n-1)/2-i}c^i. \]  

(16)

Since

\[ c \equiv \begin{cases} 
1 \mod 8, & \text{if } m \equiv 0 \mod 4 \\
5 \mod 8, & \text{if } m \equiv 2 \mod 4,
\end{cases} \]  

(17)
by (15) and (16), we obtain (i)–(iv) immediately.

On the other hand, we get from (6)–(8) that

\[
E(n) = \begin{cases} 
2U\left(\frac{n-1}{2}\right)E\left(\frac{n+1}{2}\right) - c^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}, \\
2U\left(\frac{n+1}{2}\right)E\left(\frac{n-1}{2}\right) - c^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4},
\end{cases} \tag{18}
\]

\[
F(n) = \begin{cases} 
-4F\left(\frac{n+1}{2}\right)\left(\frac{\varepsilon^{(n+1)/2} - \bar{\varepsilon}^{(n+1)/2}}{\varepsilon^2 - \bar{\varepsilon}^2}\right) + c^{(n-1)/2}, & \text{if } n \equiv 1 \pmod{4}, \\
-4F\left(\frac{n-1}{2}\right)\left(\frac{\varepsilon^{(n-1)/2} - \bar{\varepsilon}^{(n-1)/2}}{\varepsilon^2 - \bar{\varepsilon}^2}\right) + c^{(n-1)/2}, & \text{if } n \equiv 3 \pmod{4},
\end{cases} \tag{19}
\]

where

\[
\frac{\varepsilon^{(n-(-1)^{(n-1)/2})/2} - \bar{\varepsilon}^{(n-(-1)^{(n-1)/2})/2}}{\varepsilon^2 - \bar{\varepsilon}^2}
\]

is an integer. Thus, by (18) and (19), we obtain (v). The lemma is proved. \qed

**Lemma 4** ([1, Lemma 3]). If (2) holds an \( m \equiv 2 \pmod{4} \), then we have \( (a/c) = -1 \) and \( (b/c) = 1 \), where \( */* \) denotes the Jacobi symbol. Therefore, then the solutions \( (x, y, z) \) of (1) satisfy \( 2 \mid x \).

**Lemma 5.** If (2) holds, \( m \equiv 2 \pmod{4} \) and \( m > 2r/\pi \), then we have

\[
\frac{F(r)}{E(r)} = \begin{cases} \ 1, & \text{if } r \equiv 1,3 \pmod{8}, \\
\ -1, & \text{if } r \equiv 5,7 \pmod{8}.
\end{cases} \tag{20}
\]

**Proof.** Since \( m > 2r/\pi \), by Lemma 2, \( E(n) \) and \( F(n) \) are positive integers for the odd integers \( n \) with \( 1 \geq n \geq r \). If \( r \equiv 1 \pmod{4} \), then \( (r+1)/2 \) is an odd integer, and by (7), we get

\[
F(r) + E(r) = 2E\left(\frac{r+1}{2}\right)F\left(\frac{r+1}{2}\right). \tag{21}
\]
Hence, by (21), we obtain
\[
\frac{F(r)}{E(r)} = \frac{F(r) + E(r)}{E(r)} = \frac{2}{E(r)} \left( \frac{E(r+1)}{E(r)} \right) \left( \frac{F(r+1)}{E(r)} \right).
\] (22)

On applying Lemma 3 again and again, we get
\[
\frac{2}{E(r)} = \begin{cases} 
1, & \text{if } r \equiv 1 \pmod{8}, \\
-1, & \text{if } r \equiv 5 \pmod{8}, 
\end{cases}
\] (23)

\[
\frac{E(r+1)}{E(r)} = \frac{E(r)}{E(r+1)} = \frac{-c^{(r-1)/2}}{E(r+1)} = \frac{1}{E(r+1)} = 1,
\] (24)

\[
\frac{F(r+1)}{E(r)} = \frac{E(r)}{F(r+1)} = \frac{c^{(r-1)/2}}{F(r+1)} = \frac{1}{F(r+1)} = 1.
\] (25)

The combination (23)–(25) with (22) yields (20) for \( r \equiv 1 \pmod{4} \).

Similarly, if \( r \equiv 3 \pmod{4} \), then \( (r - 1)/2 \) is an odd integer and
\[
F(r) - E(r) = 2cE \left( \frac{r - 1}{2} \right) F \left( \frac{r - 1}{2} \right).
\] (26)

Therefore, we get from (26) that
\[
\frac{F(r)}{E(r)} = \frac{F(r) - E(r)}{E(r)} = \frac{2}{E(r)} \left( \frac{c}{E(r)} \right) \left( \frac{E(r+1)}{E(r)} \right) \left( \frac{F(r+1)}{E(r+1)} \right).
\] (27)

By Lemma 3, we obtain
\[
\frac{2}{E(r)} = \begin{cases} 
1, & \text{if } r \equiv 3 \pmod{8}, \\
-1, & \text{if } r \equiv 7 \pmod{8}, 
\end{cases}
\] (28)

\[
\frac{c}{E(r)} = \frac{E(r)}{c} = \frac{(2m)^{r-1}}{c} = \frac{1}{c} = 1,
\] (29)

\[
\frac{E(r+1)}{E(r)} = \frac{E(r)}{r+1} = \frac{-c^{(r-1)/2}}{E(r+1)} = \frac{c}{E(r+1)}
\]
\[
= \frac{E(r+1)}{c} = \frac{(2m)^{(r-3)/2}}{c} = \frac{1}{c} = 1,
\] (30)
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\[
\left( \frac{F(r-1)}{E(r)} \right) = \left( \frac{E(r)}{F(r-1)} \right) = \left( \frac{-c^{(r-1)/2}}{F(r-1)} \right) = \left( \frac{c}{F(r-1)} \right)
\]

\[
= \left( \frac{F(r-1)}{c} \right) = \left( \frac{(-1)^{(r-3)/2}2^{r-3}}{c} \right) = \left( \frac{1}{c} \right) = 1. \tag{31}
\]

The combination of (28)–(31) with (27) yields (20) for \( r \equiv 3 \pmod{4} \). Thus, the lemma is proved. \( \square \)

**Lemma 6** ([1, Theorem]). If (5) holds, \( b \equiv 3 \pmod{4} \), \( c \equiv 5 \pmod{8} \) and \( c \) is a prime power, then (1) has only the solution \( (x, y, z) = (2, 2, r) \).

**Lemma 7.** Let \( a, b, c \) be fixed positive integers such that \( \min(a, b, c) > 1 \) and \( c \) is an odd prime power, then (1) has at most one solution \( (x, y, z) \) satisfying \( 2 \mid x \) and \( 2 \mid y \).

**Proof.** This lemma follows directly from the proof of [4, Theorem]. \( \square \)

**Proof of Theorem 1.** Since \( m > 2r/\pi \), by Lemma 2, we see from (2), (6) and (7) that
\[
a = mE(r), \quad b = F(r), \quad c = m^2 + 1. \tag{32}
\]

Since \( m = 2 \pmod{4} \), by (17) and (iv) of Lemma 3, we get that if \( r \equiv 3 \pmod{4} \), then \( b \equiv 3 \pmod{4} \) and \( c \equiv 5 \pmod{8} \). Therefore, by Lemma 6, the theorem holds for \( r \equiv 3 \pmod{4} \).

Let \( (x, y, z) \) be a solution of (1) with \( (x, y, z) \neq (2, 2, r) \). Since \( m \equiv 2 \pmod{4} \), by Lemma 4, we have \( 2 \mid x \). On the other hand, if \( 2 \nmid y \), then from (1) and (32) we get
\[
1 = \begin{cases} 
\left( \frac{b}{E(r)} \right), & \text{if } 2 \mid x, \\
\left( \frac{bc}{E(r)} \right), & \text{if } 2 \nmid x.
\end{cases} \tag{33}
\]

Since \( (c/E(r)) = 1 \) by (29), we see from (32) and (33) that
\[
\left( \frac{b}{E(r)} \right) = \left( \frac{F(r)}{E(r)} \right) = 1. \tag{34}
\]
However, by Lemma 5, we get \((F(r)/E(r)) = -1\) if \(r \equiv 5 \pmod{8}\). Therefore, we find from (34) that if \(r \equiv 5 \pmod{8}\), then \(2 \mid y\). But, by Lemma 7, it is impossible, since \((x, y, z) \neq (2, 2, r)\). Thus, if \(r \not\equiv 1 \pmod{8}\), then (1) has only the solution \((x, y, z) = (2, 2, r)\). The theorem is proved. \(\square\)

### 3. Proof of Theorem 2

**Lemma 8** ([2]). Let \(p\) be an odd prime, and let \(u, v\) be coprime positive integers. Then we have either \(\gcd(u + v, (u^p + v^p)/(u + v)) = 1\) or \(\gcd(u + v, (u^p + v^p)/(u + v)) = p\). Moreover, if \(p \mid (u^p + v^p)/(u + v)\) then \(p^2 \not\mid (u^p + v^p)/(u + v)\).

**Lemma 9.** If (32) holds and \(m > 4r/\pi\), then we have \(\max(a, b) < c^{r/2}\) and \(\min(a, b) > c^{(r-1)/2}\).

**Proof.** Since \(a^2 + b^2 = c^r\), it follows that \(\max(a, b) < c^{r/2}\). Since \(m > 4r/\pi\), we get from (13) that

\[
0 < \sin \theta < \sin(r\theta) < r\theta < \frac{r}{m} < \frac{\pi}{4}.
\]  

(35)

Hence, by (12) and (32), we obtain

\[
b = F(r) = c^{(r-1)/2} \frac{\sin(r\theta)}{\sin \theta} > c^{(r-1)/2}.
\]  

(36)

On the other hand, by (11), (32) and (35), we get

\[
a = mE(r) = V(r) = c^{r/2} \cos(r\theta) = c^{r/2} (1 - (\sin(r\theta))^2)^{1/2}
\]

\[
> c^{r/2} \left(1 - \frac{\pi^2}{16}\right)^{1/2} > 0.6c^{r/2} > c^{(r-1)/2}.
\]  

(37)

Thus, by (36) and (37), we obtain \(\min(a, b) > c^{(r-1)/2}\). The lemma is proved. \(\square\)

**Lemma 10.** If (32) holds, \(m \equiv 2 \pmod{4}\), \(m > 4r/\pi\) and \(c\) is a prime, then (1) has no solution \((x, y, z)\) with \(2 \mid z\).
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**Proof.** Under the assumption, by Lemma 4, we have $2 \mid x$. If $2 \mid z$, then from (1) we get

$$c^{z/2} + a^{z/2} = b_1^y, \quad c^{z/2} - a^{z/2} = b_2^y, \quad b = b_1 b_2, \quad b_1 b_2 \in \mathbb{N}. \quad (38)$$

If follows that

$$b_1^y + b_2^y = 2c^{z/2}. \quad (39)$$

By Lemma 7, (1) has only the solution $(x, y, z) = (2, 2, r)$ satisfying $2 \mid x$ and $2 \mid y$. So we have $2 \nmid y$. If $y > 1$, then $y \geq 3$ and $y$ has an odd prime divisor $p$. Since $c$ is a prime, by Lemma 8, we get from (39) that

$$b + 1 \geq b_1 + b_2 \geq 2c^{z/2-1} > 2c \quad (40)$$

and

$$c \geq \frac{b_1^y + b_2^y}{b_1^{y/p} + b_2^{y/p}} \geq \frac{b_1^y + b_2^y}{b_1^{y/3} + b_2^{y/3}} = b_1^{2y/3} - b_1^{y/3} + b_2^{2y/3}$$

$$= \left( b_1^{2y/3} - b_2^{2y/3} \right)^2 + b_1^{y/3} b_2^{y/3} \geq b > 2c - 1 > c, \quad (41)$$

a contradiction. So we have $y = 1$.

If $x = 2$ and $y = 1$, then $z < r$ and $b(b-1) \equiv 0 \pmod{c^z}$ by (1). Since $\gcd(b, c) = 1$, we get $b - 1 \equiv 0 \pmod{c^z}$ and $b > b - 1 \geq c^z = a^2 + b > b$, a contradiction. It follows that $x \geq 4$, since $2 \mid x$. Then, by (38), we get

$$b \geq b_1 = c^{z/2} + a^{z/2} > 2a^{z/2} \geq 2a^2. \quad (42)$$

But, by Lemma 9, we have $b < c^{z/2}$ and $2a^2 > 2c^{z-1} > 2c^{z/2}$, since $r \geq 3$. Thus, (42) is impossible. The lemma is proved. \hfill $\square$

**Lemma 11** ([5, Lemma 5]). Let $a_1, a_2, b_1, b_2$ be positive integers satisfying $\min(a_1, a_2) > 10^3$. Further, let $\Lambda = b_1 \log a_1 - b_2 \log a_2$. If $\Lambda \neq 0$, then we have

$$\log |\Lambda| > -17.61(\log a_1)(\log a_2)(1.7735 + B)^2,$$

where

$$B = \max \left( 8.445, 0.2257 + \log \left( \frac{b_1}{\log a_2} + \frac{b_2}{\log a_1} \right) \right).$$
Lemma 12. Let \((x, y, z)\) be a solution of \((1)\). If \(\min(b, c) > 10^3\), \(x = 2\), \(y \geq 3\) and \(b^3 > a^2\), then we have

\[
y < 1856 \log c. \tag{43}
\]

Proof. Since \(a^2 + b^y = c^z\) and \(b^y > a^2\), we get

\[
z \log c = \log(b^y + a^2) = y \log b + \frac{2a^2}{b^y + c^z} \sum_{k=0}^{\infty} \frac{1}{2k + 1} \left( \frac{a^2}{b^y + c^z} \right)^{2k}\]
\[
< y \log b + \frac{2a^2}{b^y + c^z} \sum_{k=0}^{\infty} \frac{3^{-2k}}{2k + 1} = y \log b + \frac{(3 \log 2) a^2}{b^y + c^z} \tag{44}
\]
\[
< y \log b + \frac{1.04a^2}{b^y}.
\]

Let \(\Lambda = z \log c - y \log b\). Then from \((44)\) we get

\[
\log(1.04a^2) - \log |\Lambda| > y \log b. \tag{45}
\]

Since \(\min(b, c) > 10^3\), by Lemma 11, we have

\[
\log |\Lambda| > -17.61(\log b)(\log c)(1.7735 + B)^2, \tag{46}
\]

where

\[
B = \max \left( 8.445, 0.2257 + \log \left( \frac{z}{\log b} + \frac{y}{\log c} \right) \right). \tag{47}
\]

If \(B = 8.445\), then from \((44)\) and \((47)\) we obtain

\[
\frac{2y}{\log c} < \frac{z}{\log b} + \frac{y}{\log b} \leq e^{8.2193} < 3712, \tag{48}
\]

whence we get \((43)\).

If \(B > 8.445\), then from \((47)\) we get

\[
B = 0.2257 + \log \left( \frac{z}{\log b} + \frac{y}{\log c} \right). \tag{49}
\]

Substitute \((46)\) and \((49)\) into \((45)\), we get

\[
\log \frac{1.04 + 2 \log a}{(\log b)(\log c)} + 17.61 \left( 1.9992 + \log \left( \frac{z}{\log b} + \frac{y}{\log c} \right) \right)^2 > \frac{y}{\log c}. \tag{50}
\]
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Since $b^3 > a^2$ and $\min(b, c) > 10^3$, we have

$$0.44 > \frac{\log 1.04 + 2 \log a}{(\log b)(\log c)}. \quad (51)$$

By (44), we get

$$0.22 + \frac{2y}{\log c} > \frac{z}{\log b} + \frac{y}{\log c}. \quad (52)$$

Thus, by (50)–(52), we obtain

$$0.44 + 17.61 \left(1.9992 + \log \left(0.22 + \frac{2y}{\log c}\right)\right)^2 > \frac{y}{\log c},$$

whence we conclude that (43) holds. The lemma is proved. \qed

**Proof of Theorem 2.** Let $(x, y, z)$ be a solution of (1) with $(x, y, z) \neq (2, 2, r)$. Then, by Lemmas 4, 7 and 10, we have $2 \mid x$, $2 \nmid y$ and $2 \nmid z$, respectively. Since $r \equiv 1 \pmod{8}$ and $m > 41r^{3/2} > 4r/\pi$, we see from (32) and (iv) of Lemma 3 that $r \geq 9$ and $b \equiv 1 \pmod{8}$. Further, since $m \equiv 2 \pmod{4}$ and $c \equiv 5 \pmod{8}$, we get from (1) that $a^x \equiv c^z - b^y \equiv 5 - 1 \equiv 4 \pmod{8}$. It follows that $x = 2$. Furthermore, we find from the proof of Lemma 10 that $y \neq 1$ and $y \geq 3$. Since $m > 4r/\pi$, by Lemma 9, we get $b^3 > c^{3(r-1)/2} > c^r > a^2$. Therefore, by Lemma 12, the solution $(x, y, z)$ satisfies (43).

On the other hand, we get from (15), (16) and (32) that

$$a^2 \equiv r^2 m^2 \pmod{m^4}, \quad b^y \equiv 1 - y \left(\frac{r}{2}\right) m^2 \pmod{m^4}, \quad c^z \equiv 1 + zm^2 \pmod{m^4}. \quad (53)$$

Substitute (53) into (1), we obtain

$$\frac{1}{2}r(r - 1)y + z \equiv r^2 \pmod{m^2}. \quad (54)$$

Since $y \geq 3$, we see from (54) that

$$\frac{1}{2}r(r - 1)y + z \geq r^2 + m^2. \quad (55)$$
Since $a^2 + b^2 = c^r$ and $a^2 + b^y = c^z$, we have
\[
c^ry = (a^2 + b^2)^y > a^{2y} + \left(\frac{y}{2}\right)^2 b^y + b^{2y}
\]
\[
> b^{2y} + 2a^2b^y + a^4
\]
\[
= (a^2 + b^y)^2 = c^{2z}.
\]  
(56)

It follows that $ry > 2x$. Therefore, by (55), we get
\[
r^2 \left(\frac{y}{2} - 1\right) > m^2.
\]  
(57)

The combination of (43) and (57) yields
\[
r^2 > \frac{m^2}{y/2 - 1} > \frac{m^2}{928 \log(m^2 + 1) - 1}.
\]  
(58)

Since $m > 41r^{3/2}$, we get from (58) that
\[
928 \log(1681r^3 + 1) > 1681r + 1.
\]  
(59)

However, (59) is false if $r \geq 9$. Thus, the theorem is proved. □

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