A new formula for the convexity coefficient of Orlicz spaces

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Abstract. In [2], a formula for the convexity coefficient of Orlicz spaces $L_{M,\varepsilon_0}(L_M)$, equipped with the Luxemburg norm, in the case of a non-atomic and infinite measure space, has been given in terms of some parameter depending on the generating Orlicz function $M$. In this paper, we explain this formula in terms of a parameter $\beta(p)$ depending on the right derivative of $M$. We also give a way how to compute the parameter $\beta(p)$, which is more convenient when we look for an Orlicz function $M$ giving concrete value of $\varepsilon_0(L_M)$.

I. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+ = [0, \infty)$. A function $M: \mathbb{R} \to [0, \infty)$ is called an Orlicz function if it is convex, even and vanishing only at zero (see [1]).

Let $p_-$ (resp. $p$) be the left (resp. the right) derivative of $M$. Then $M$ is an Orlicz function if and only if $M(u) = \int_0^u p(t)dt$, where the right derivative $p$ of $M$ is right continuous, nondecreasing on $\mathbb{R}_+$, and $p(u) > 0$ for $u > 0$.

An interval $[a, b)$, where $0 < a < b < \infty$, is called a structural interval of $p$, provided that $p$ is constant on $[a, b)$ and $p$ is not constant on either $[a - \varepsilon, b)$ or $[a, b + \varepsilon)$ for any $\varepsilon > 0$. An interval $[0, b)$, where $0 < b < \infty$, is called a structural interval of $p$, provided that $p$ is constant on $[0, b)$ and

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$p$ is not constant on $[0, b + \varepsilon)$ for any $\varepsilon > 0$. An interval $[a, \infty)$, where $0 < a < \infty$, is called a structural interval of $p$, provided that $p$ is constant on $[a, \infty)$ and $p$ is not constant on $[a - \varepsilon, \infty)$ for any $\varepsilon > 0$. The interval $[0, \infty)$ is called a structural interval of $p$, provided that $p$ is constant on $[0, \infty)$. Let $\{[a_k, b_k]\}_k$ be all structural intervals of $p$. Define
\[
h(p) = \inf_k \frac{a_k}{b_k},
\]
assuming $\frac{a_k}{b_k} = 0$ if $b_k = \infty$, and $h(p) = 1$ if $p$ is strictly increasing on $(0, \infty)$.

For a given Orlicz function $M$ and its right derivative $p$, denote
\[
\alpha(M) = \sup \left\{ a \in (0, 1) : \exists \delta > 0 \forall u > 0 M\left(\frac{u + au}{2}\right) \leq \frac{1 - \delta}{2} [M(u) + M(au)] \right\},
\]
\[
\beta(p) = \sup \left\{ a \in (0, 1) : \sup_{u > 0} \frac{p(au)}{p(u)} < 1 \right\},
\]
assuming $\sup \emptyset := 0$. Given any Orlicz function $M$, the number $\alpha(M)$ is called the convexity characteristic of $M$. For the function $p$ given above, define
\[
h_0(p) = \sup \left\{ a \in (0, 1) : \lim_{u \to 0^+} \frac{p(au)}{p(u)} < 1 \right\},
\]
\[
h_\infty(p) = \sup \left\{ a \in (0, 1) : \lim_{u \to \infty} \frac{p(au)}{p(u)} < 1 \right\},
\]
assuming $\sup \emptyset := 0$, whenever the limits that appear in the definitions of $h_0(p)$ and $h_\infty(p)$ exist.

The convexity coefficient $\varepsilon_0(X)$ of a normed space $X$ (called also the convexity characteristic of $X$) is a very important parameter of $X$ (for the definition of $\varepsilon_0(X)$ see Section III). Namely, $X$ is uniformly rotund if and only if $\varepsilon_0(X) = 0$, $X$ is uniformly non-square if and only if $\varepsilon_0(X) < 2$. Moreover, if $\varepsilon_0(X) < 1$, then $X$ has uniformly normal structure and, in consequence, $X$ has the fixed point property (see [4]). In [2], $\varepsilon_0(L_M)$ has been computed in the case of $L_M$ over a non-atomic infinite measure space and the Luxemburg norm in terms of a convexity characteristic of the generating Orlicz function $M$. In this paper, that parameter is explained in terms of the right derivative $p$ of $M$. This gives an easy possibility to find for any $a \in [0, 2]$ an Orlicz function $M$ such that $\varepsilon_0(L_M) = a.$
II. Convexity characteristic of Orlicz functions in terms of their right derivatives

Theorem 1. Let $M$ be an Orlicz function and $p$ be its right derivative on $\mathbb{R}_+$. Then $\alpha(M) = \beta(p)$.

Proof. Let $a \in (0, 1)$ and $\sup_{u > 0} \frac{p(au)}{p(u)} = 1$. Then

\[
M\left(\frac{u + au}{2}\right) = \frac{1}{2}(M(u) + M(au)) \left[ 1 - \frac{M(u) + M(au) - 2M\left(\frac{u + au}{2}\right)}{M(u) + M(au)} \right]
\]

\[
= \frac{1}{2}[M(u) + M(au)] \left[ 1 - \frac{\left(\left(M(u) - M\left(\frac{u + au}{2}\right)\right) - \left(M\left(\frac{u + au}{2}\right) - M(au)\right)\right)}{M(u) + M(au)} \right]
\]

\[
= \frac{1}{2}[M(u) + M(au)] \left[ 1 - \frac{\int \frac{u + au}{2} p(t)dt - \int \frac{u + au}{2} p(t)dt}{\int_0^u p(t)dt + \int_0^u p(t)dt} \right]
\]

\[
\geq \frac{1}{2}[M(u) + M(au)] \left[ 1 - \frac{p(u)(u - \frac{u + au}{2}) - p(au)(\frac{u + au}{2} - au)}{p(au)(u - au)} \right]
\]

\[
= \frac{1}{2}[M(u) + M(au)] \left[ 1 - \frac{p(u) - p(au)}{p(au)} \cdot \frac{u - \frac{u + au}{2}}{u - au} \right]
\]

\[
= \frac{1}{2}[M(u) + M(au)] \left[ 1 - \frac{1}{2} \left( \frac{p(u)}{p(au)} - 1 \right) \right]
\]

for all $u \in (0, \infty)$. Hence it follows that there is no $\delta > 0$ such that $M\left(\frac{u + au}{2}\right) \leq \frac{1 - \delta}{2} [M(u) + M(au)]$ for all $u > 0$. Therefore

\[
\alpha(M) \leq \beta(p). \tag{1}
\]

In particular,

\[
\beta(p) = 0 \Rightarrow \alpha(M) = 0. \tag{2}
\]

Let $\beta := \beta(p) > 0$ and $b \in (0, \beta)$. Then $\sup_{u > 0} \frac{p(b + u)}{p(u)} = k < 1$. 

From the following inequalities

\[
M(u) + M(bu) - 2M\left(\frac{u + bu}{2}\right) \\
= \left[M(u) - M\left(\frac{u + bu}{2}\right)\right] - \left[M\left(\frac{u + bu}{2}\right) - M(bu)\right] \\
= \int_{u+bu}^{u-\beta-bu} p(t)dt - \int_{u}^{u+bu} p(t)dt + \int_{u}^{u-\beta-bu} p(t)dt - \int_{bu}^{ub+\beta} p(t)dt \\
\geq \int_{u-\beta-bu}^{u} p(t)dt - \int_{bu}^{ub+\beta} p(t)dt \\
= \int_{u-\beta-bu}^{u} \left[p(t) - p\left(t - \left(\frac{b + \beta}{2}\right)\right)\right] dt \\
\geq \int_{u-\beta-bu}^{u} \left[p(t) - p\left(t - \left(\frac{b + \beta}{2}\right)\right)\right] dt \\
\geq \int_{u-\beta-bu}^{u} [p(t) - kp(t)]dt \\
= (1 - k) \left[M(u) - M\left(\frac{u - \beta - b}{2}\right)\right] \\
\geq \frac{1}{4}(1 - k)(\beta - b)[M(u) + M(bu)]
\]

being true for any \(u > 0\), we get

\[
M\left(\frac{u + bu}{2}\right) \leq \frac{1 - \delta}{2}[M(u) + M(bu)]
\]

for any \(u > 0\) with \(\delta = \frac{1}{4}(1 - k)(\beta - b) \in (0, 1)\). Hence

\[
\alpha(M) \geq \beta(p) \quad \text{if} \quad \beta(p) > 0. \quad (3)
\]

Combining (1), (2) and (3), we have \(\alpha(M) = \beta(p)\). \qed
Lemma 2. Let $M$ be an Orlicz function and $p_-$ (resp. $p$) be the left (resp. the right) derivative of $M$. Then $p_-$ is left continuous, nondecreasing and

$$\lim_{t \to u^-} p(t) = p_-(u) \quad \text{for all } u > 0.$$ 

Proof. Since $M$ is convex on $(0, \infty)$, we have

$$p_-(u - h) \leq p(u - h) \leq \frac{M(u) - M(u - h)}{h} \leq p_-(u) \leq p(u)$$

for all $u > 0$ and any $h > 0$ such that $u - h > 0$. Therefore

$$\lim_{t \to u^-} p_-(t) \leq \lim_{t \to u^-} p(t) \leq p_-(u). \quad (4)$$

On the other hand,

$$\lim_{t \to u^-} p(t) \geq \lim_{t \to u^-} p_-(t) = \lim_{t \to u^-} \lim_{h \to 0^+} \frac{M(t) - M(t - h)}{h} = p_-(u). \quad (5)$$

By (4) and (5), we have

$$\lim_{t \to u^-} p(t) = \lim_{t \to u^-} p_-(t) = p_-(u)$$

for all $u > 0$. \hfill \Box

Lemma 3. Let $a \in (0, 1)$ and $0 < c < d < \infty$. If $p_-(au) < p_-(u)$ and $p(au) < p(u)$ for any $u \in [c, d]$, then $\sup_{u \in [c,d]} \frac{p(au)}{p(u)} < 1$.

Proof. If $\sup_{u \in [c,d]} \frac{p(au)}{p(u)} = 1$, then there is a sequence $\{u_n\}$ in $[c, d]$ such that $\lim_{n} \frac{p(au_n)}{p(u_n)} = 1$. Since $\{u_n\}$ is bounded, there is a monotone subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u' \in [c, d]$.

We may assume without loss of generality (passing to a subsequence if necessary) that $u_{n_k} \leq u'$ for all $k \in \mathbb{N}$ or $u_{n_k} \geq u'$ for all $k \in \mathbb{N}$ and that the sequence $\{u_{n_k}\}$ is monotone. If $u_{n_k} \not\to u'$, then by Lemma 2 and by the assumption that $p(au) < p(u)$ for any $u \in [c, d]$, we have $1 = \lim_{n} \frac{p(au_n)}{p(u_n)} = \lim_{k} \frac{p(au_{n_k})}{p(u_{n_k})} = \frac{p_-(au')}{p_-(u')} < 1$, a contradiction. If $u_{n_k} \not\to u'$, then by the right continuity of $p$, we get, $1 = \lim_{n} \frac{p(au_n)}{p(u_n)} = \lim_{k} \frac{p(au_{n_k})}{p(u_{n_k})} = \frac{p(au')}{p(u')} < 1$, a contradiction too. This completes the proof. \hfill \Box
Theorem 4. Assume that the limits $\lim_{u \to 0^+} \frac{p(au)}{p(a)}$ and $\lim_{u \to \infty} \frac{p(au)}{p(a)}$ exist for all $a \in (0,1)$. Then

$$\beta(p) = \min \{ h^{(p)}, h_0^{(p)}, h_{\infty}^{(p)} \}. \tag{6}$$

Proof. Denote $h(p) := \min \{ h^{(p)}, h_0^{(p)}, h_{\infty}^{(p)} \}$. We discuss three cases.

I. $h(p) = 0$. If $h^{(p)} = 0$, then for any $a \in (0,1)$, there is $k_0 \in \mathbb{N}$ such that $\frac{a_k}{b_k} < a$, where $[a_k, b_k)$ is a structural interval of $p$. Take $u_0 = \frac{1}{a} a_k$. Then $u_0 < b_k$ and so $\frac{p(au_0)}{p(u_0)} = 1$, whence it follows that $\beta(p) = 0$. If $h_0^{(p)} = 0$, then $\lim_{u \to 0^+} \frac{p(au)}{p(u)} = 1$ for any $a \in (0,1)$. Then it is obvious that $\sup_{u>0} \frac{p(au)}{p(u)} = 1$, whence, $\beta(p) = 0$. Similarly, we can prove that $h_{\infty}^{(p)} = 0$ implies $\beta(p) = 0$. Hence,

$$\beta(p) = h(p) \text{ if } h(p) = 0. \tag{6}$$

II. $h(p) = 1$. In this case, $h^{(p)} = h_0^{(p)} = h_{\infty}^{(p)} = 1$. This yields that $p$ is strictly increasing on $(0, \infty)$ and for any $a \in (0,1)$,

$$\lim_{t \to 0^+} \frac{p(at)}{p(t)} < 1 \text{ and } \lim_{t \to \infty} \frac{p(at)}{p(t)} < 1. \tag{7}$$

So there exist $u_0$ and $u_1$ with $0 < u_0 < u_1 < \infty$ such that

$$\sup_{u \in (0,u_0)} \frac{p(au)}{p(u)} < 1, \sup_{u \in (u_1,\infty)} \frac{p(au)}{p(u)} < 1 \text{ and } \sup_{u \in [u_0,u_1]} \frac{p(au)}{p(u)} < 1,$$

where the last inequality follows from Lemma 3. Therefore, $\sup_{u>0} \frac{p(au)}{p(u)} < 1$, that is,

$$\beta(p) = h(p) \text{ if } h(p) = 1. \tag{7}$$

III. $0 < h(p) < 1$. Let $a \in (0,h(p))$. Then

$$\lim_{t \to 0^+} \frac{p(at)}{p(t)} < 1, \lim_{t \to \infty} \frac{p(at)}{p(t)} < 1, \frac{p(at)}{p(t)} < 1 \text{ and } \frac{p-(at)}{p-(t)} < 1$$

for any $t \in (0, \infty)$. By Lemma 3, we can prove that $\sup_{u>0} \frac{p(au)}{p(u)} < 1$. Hence

$$\beta(p) \geq h(p) \text{ if } h(p) \in (0,1). \tag{8}$$
A new formula for the convexity coefficient of Orlicz spaces

Let \( a \in (h(p), 1) \). Then

\[
\lim_{t \to 0+} \frac{p(at)}{p(t)} = 1 \quad \text{or} \quad \lim_{t \to \infty} \frac{p(at)}{p(t)} = 1 \quad \text{or} \quad \inf_k \frac{a_k}{b_k} < a.
\]

It is easy to deduce that \( \sup_{a>0} \frac{p(au)}{p(u)} = 1 \). Hence

\[
\beta(p) \leq h(p) \quad \text{if} \quad h(p) \in (0, 1).
\]

Combining (7), (8) and (9), we obtain

\[
\beta(p) = \min \{ h(p), h_0(p), h_\infty(p) \}.
\]

\( \square \)

**Corollary 5.** Assume that the limits \( \lim_{u \to \infty} \frac{p(au)}{p(u)} \), \( \lim_{u \to 0+} \frac{p(au)}{p(u)} \) exist for any \( a \in (0, 1) \). Then \( \beta(p) = 0 \) if and only if one of the following assertions is true:

1) \( \inf_k \frac{a_k}{b_k} = 0 \), where \( \{a_k, b_k\} \) are the structural intervals of \( p \),

2) \( \lim_{u \to \infty} \frac{p(au)}{p(u)} = 1 \) for any \( a \in (0, 1) \),

3) \( \lim_{u \to 0+} \frac{p(au)}{p(u)} = 1 \) for any \( a \in (0, 1) \).

**Corollary 6.** Assume that the limits \( \lim_{u \to \infty} \frac{p(au)}{p(u)} \), \( \lim_{u \to 0+} \frac{p(au)}{p(u)} \) exist for any \( a \in (0, 1) \). Then \( \beta(p) = 1 \) if and only if:

1) \( p \) is strictly increasing on \((0, \infty)\),

2) \( \lim_{u \to \infty} \frac{p(au)}{p(u)} < 1 \) and \( \lim_{u \to 0+} \frac{p(au)}{p(u)} < 1 \) for any \( a \in (0, 1) \).

### III. Some consequences

The convexity coefficient of a Banach space \( X \) is defined by

\[
\varepsilon_0(X) = \sup \{ \varepsilon \in (0, 2) : \delta_X(\varepsilon) = 0 \},
\]

where \( \delta_X : (0, 2] \to [0, 1] \) is the modulus of convexity of \( X \), that is,

\[
\delta_X(\varepsilon) = \inf \left\{ 1 - \| \frac{1}{2}(x + y) \| : \| x \| \leq 1, \| y \| \leq 1, \| x - y \| \geq \varepsilon \right\},
\]

for \( \varepsilon \in (0, 2] \).
Let \((T, \Sigma, \mu)\) be a non-atomic and infinite measure space. Given any Orlicz function \(M\), the Orlicz space \(L^M\) is defined as the set of all (equivalent classes of) \(\Sigma\)-measurable functions \(f : T \to \mathbb{R}\) such that

\[
\varrho_M(af) = \int_T M(|af(t)|)d\mu < \infty
\]

for some \(a > 0\). The space \(L^M\) equipped with the Luxemburg norm \(\| \cdot \|\) defined by

\[
\|f\| = \inf \left\{ a > 0 : \varrho_M\left(\frac{f}{a}\right) \leq 1 \right\}
\]

is a Banach space (see [1]). We say that an Orlicz function \(M\) satisfies the \(\Delta_2\)-condition on the whole \(\mathbb{R}\) \((M \in \Delta_2\) for short) if there is a constant \(K \geq 2\) such that \(M(2u) \leq KM(u)\) for all \(u \in \mathbb{R}\). Then

\[
\varepsilon_0(L^M) = 2(1 - \alpha(M)) + \frac{\alpha(M)}{1 + \alpha(M)}
\]

if \(M \in \Delta_2\), and \(\varepsilon_0(L^M) = 2\) if \(M \notin \Delta_2\) (see [2], [3]).

**Corollary 7.** \(\varepsilon_0(L^M) = 2\) if \(M \notin \Delta_2\), and \(\varepsilon_0(L^M) = \frac{2(1 - \beta(p))}{1 + \beta(p)}\) if \(M \in \Delta_2\).

**Example 1.** Let \(M(u) = (1 + |u|) \ln(1 + |u|) - |u|\). Then \(p(u) = \ln(1 + u)\) for \(u \geq 0\). Since \(\lim_{u \to \infty} \frac{p(au)}{p(u)} = 1\) for any \(a \in (0,1)\), we have \(\alpha(M) = \beta(p) = 0\). It is easy to verify that \(M \in \Delta_2\). By Corollary 7, \(\varepsilon_0(L^M) = 2\).

**Example 2.** Let \(M(u) = \frac{1}{s} |u|^s\) \((s > 1)\). Then \(p(u) = u^{s-1}\) for \(u \geq 0\), so \(p\) is strictly increasing on \(\mathbb{R}_+\) and \(\lim_{u \to 0^+} \frac{p(au)}{p(u)} = a^{s-1} = \lim_{u \to \infty} \frac{p(au)}{p(u)} < 1\) for any \(a \in (0,1)\). So \(\alpha(M) = \beta(p) = 1\) and \(\varepsilon_0(L^M) = 0\) since \(M \in \Delta_2\).

**Example 3.** Let \(a \in (0,1)\). Define Orlicz function \(M\) is even and for \(u \geq 0\),

\[
M(u) = \begin{cases} 
\frac{u^2}{2}, & \text{if } u \in [0,1] \\
\frac{1}{a} - \frac{1}{2}, & \text{if } u \in \left(1, \frac{1}{a}\right) \\
\frac{u^2}{2} - \frac{1 - a}{2a^2} u + \frac{1 - a^2}{2a^2}, & \text{if } u \in \left(\frac{1}{a}, \infty\right).
\end{cases}
\]
Then
\[
p(u) = \begin{cases} 
  u, & \text{if } u \in [0, 1] \\
  1, & \text{if } u \in \left(1, \frac{1}{a}\right) \\
  u - \frac{1 - a}{a}, & \text{if } u \in \left(\frac{1}{a}, \infty\right),
\end{cases}
\]
for \( u \geq 0 \). Since \( \lim_{u \to 0^+} \frac{p(\varepsilon u)}{p(u)} = \varepsilon = \lim_{u \to \infty} \frac{p(\varepsilon u)}{p(u)} < 1 \) for any \( \varepsilon \in (0, 1) \) and \( \inf_k \frac{2k}{u_k} = a \), so \( \alpha(M) = \beta(p) = a \) and \( \varepsilon_0(L_M) = \frac{2(1-a)}{1+a} \) since \( M \in \Delta_2 \).

**Example 4.** Given any number \( a \in (0, 1) \), define the function \( p \) by \( p(0) = 0 \) and \( p(t) = a^{-i} \) for \( t \in \left[\frac{1}{a^i}, \frac{1}{a^{i+1}}\right) \) \( (i = 0, \pm 1, \pm 2, \ldots) \). Then \( p \) is a nondecreasing and right continuous function on \( \mathbb{R}_+ \), that is, \( M(u) = \int_0^u p(t)dt \) is an Orlicz function. Moreover, \( \beta(p) = a \) and \( M \) satisfies the \( \Delta_2 \)-condition on the whole \( \mathbb{R} \). Consequently, \( \varepsilon_0(L_M) = \frac{2(1-a)}{1+a} \).

**Proof.** It is evident that \( p(at) = ap(t) \) for any \( t \in [0, \infty) \). Moreover, for any \( b > a \) there is \( u > 0 \) such that \( p(bu) \geq p(u) \), whence \( \beta(p) = a \). Let \( k \in \mathbb{N} \) be chosen in such a way that \( 2 \leq a^{-k} \). Since the equality \( p(at) = ap(t) \) for any \( t \in [0, \infty) \) can be written as \( p(a^{-i}t) = a^{-1}p(t) \) for any \( t \in [0, \infty) \), we have for any \( u \geq 0 \),
\[
M(2u) = \int_0^{2u} p(t)dt \leq 2up(2u) \leq 2up(a^{-k}u) = 2ua^{-k}p(u) \\
= 2a^{-k} \frac{1}{a(1-a)} (1-a)up(au) \leq \frac{2}{a^{k+1}(1-a)} \int_0^u p(t)dt \\
\leq \frac{2}{a^{k+1}(1-a)} \int_0^u p(t)dt = \frac{2}{a^{k+1}(1-a)} M(u),
\]
which means that \( M \in \Delta_2 \). In consequence, \( \varepsilon_0(L_M) = \frac{2(1-a)}{1+a} \). \( \square \)

**Remark 1.** We conclude from Examples 3 and 4 that for any number \( b \in (0, 2) \) there is an Orlicz function (not being a power function) such that \( \varepsilon_0(L_M) = b \). It is enough to get Orlicz functions from that examples corresponding to the number \( a = \frac{2-b}{2+b} \). For any Orlicz function \( M \) not satisfying the \( \Delta_2 \)-condition on \( \mathbb{R} \), we have \( \varepsilon_0(L_M) = 2 \). For Orlicz functions \( M \) being uniformly convex (which means that \( \alpha(M) = \beta(p) = 1 \)) and satisfying the \( \Delta_2 \)-condition on \( \mathbb{R} \), we have \( \varepsilon_0(L_M) = 0 \).
References


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