On the maximal and minimal exponent of the prime power divisors of integers

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Abstract. For some integer \( n \) and prime \( p \) let \( \nu_p(n) \) be the largest non-negative integer for which \( p^{\nu_p(n)} \) is a divisor of \( n \). Let \( h(n) = \min_{p|n} \nu_p(n) \), \( H(n) = \max_{p|n} \nu_p(n) \). The mean value of \( h, H \) over some subsets of integers is investigated.

1. Let \( \mathcal{P} \) be the set of primes, and for a prime divisor \( p \) of \( n \) let \( \nu_p(n) \) be defined as \( p^{\nu_p(n)} || n \). Then

\[
n = \prod_p p^{\nu_p(n)}.
\]

As usual let \( \pi(x, k, l) \) be the number of primes \( p \leq x \) in the arithmetical progression \( \equiv l \pmod{k} \). Let

\[
H(n) := \max_{p|n} \nu_p(n), \quad h(n) = \min_{p|n} \nu_p(n).
\]

NIVEN proved in [1] that

\[
\sum_{n \leq x} h(n) = x + \frac{\zeta(3/2)}{\zeta(3)} \sqrt{x} + o(\sqrt{x}) \quad (x \to \infty), \quad (1.1)
\]

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and W. Schwarz and J. Spilker in [2], that

\[ \sum_{n \leq x} H(n) = \mathcal{M}(H)x + O\left(x^{3/4} \exp \left(-\gamma \sqrt{\log x}\right)\right), \quad (1.2) \]

\[ \sum_{n \leq x} \frac{1}{H(n)} = \mathcal{M}\left(\frac{1}{H}\right)x + O\left(x^{3/4} \exp \left(-\gamma \sqrt{\log x}\right)\right), \quad (1.3) \]

where \( \gamma > 0 \) is a suitable constant.

D. Suryanarayana and Sita Ramachandra Rao [8] proved that

\[ \sum_{n \leq x} H(n) = \mathcal{M}(H)x + O\left(\sqrt{x} \exp \left(-\gamma (\log x)^{3/5}(\log \log x)^{-1/5}\right)\right). \quad (1.4) \]

Furthermore they proved that

\[ \sum_{i \leq x} \frac{1}{H(i)} = cx + O\left(\sqrt{x} \exp \left(-\gamma (\log x)^{3/5}(\log \log x)^{-1/5}\right)\right), \quad (1.5) \]

\[ \sum_{i \leq x} h(i) = c_1 x + c_2 x^{1/2} + c_3 x^{1/3} + c_4 x^{1/4} + c_5 x^{1/5} + O\left(x^{1/6}\right) \]

\[ \sum_{i \leq x} \frac{1}{h(i)} = d_1 x + d_2 x^{1/2} + d_3 x^{1/3} + d_4 x^{1/4} + d_5 x^{1/5} + O\left(x^{1/6}\right), \quad (1.6) \]

hold with suitable constants \( c, c_j, d_j \).

**Remark.** Gu Tongxing and Cao Huizhong in their paper entitled “On sums of exponents of factoring integers” published in Journal of Mathematical Research and Exposition 13(2), 1993 page 166 announced that they can improve the error form in (1.4) to \( O(\sqrt{x} \exp(-c(\log x)^{3/5}(\log \log x)^{1/5}) \) by using some results of A. Ivić and P. Shiu in Illinois J. Math., 26 (1982), 576–590.
2. Let $\zeta(s) = \sum 1/n^s$ be the Riemann zeta function, and
$$\eta(s) := \frac{1}{\zeta(s)} - 1.$$ 

The following assertion (which is quoted now as Lemma 1) is an unpublished result due to Michael Filaseta who communicated to Jean-Marie De Koninck and with the allowance of Dr. Filaseta his proof has been written in the paper [9].

**Lemma 1.** Let $k \geq 2$ be an integer. Let $g(x)$ be a function satisfying $1 \leq g(x) \leq \log x$ for $x$ sufficiently large, and set
$$h = x^{\frac{1}{t(k+1)^2}} g(x)^3.$$ 

Then the number of $k$-free integers in the interval $(x, x+h]$ is
$$\frac{h}{\zeta(k)} + O\left(\frac{h \cdot \log x}{g(x)^3}\right) + O\left(\frac{h}{g(x)}\right).$$

As a direct consequence we formulate the following

**Corollary.** Let $Y = x^{\frac{1}{t(k) + 1}} \log x$. Using the abbreviation $\eta(s)$ defined at the beginning of this section, we have
$$\#\{n \in [x, x+Y] \mid H(n) = r\} = Y(\eta(r + 1) - \eta(r)) + O\left(\frac{Y}{\log x}\right).$$

3. An asymptotic formula for the number of primes $p$ with a fixed value $H(p+1) = r$ is given in

**Theorem 1.** Let $\varepsilon > 0$ be fixed, $Y = x^{\frac{1}{t(k) + \varepsilon}}$. Let $r \geq 1$. Then
$$\#\{p \in \mathcal{P}, \ p \in [x, x+Y] \mid H(p+1) = r\} = e(r) \frac{Y}{\log x} + O\left(\frac{Y}{(\log x)^2}\right),$$

where
$$e(1) = \prod_p \left(1 - \frac{1}{p(p-1)}\right), \quad \text{and for } r \geq 2,$$
$$e(r) = \prod_p \left(1 - \frac{1}{(p-1)p^r}\right) - \prod_p \left(1 - \frac{1}{(p-1)p^r-1}\right).$$
PROOF. We shall estimate

\[ E_r(x, Y) := \# \{ p \in [x, x + Y], \ p \in \mathcal{P}, \ H(p + 1) \leq r \}. \]

Let \( z = (\log x)^3 \). Let \( F \) be the number of those primes \( p \) in \([x, x + Y]\), for which \( qr^{r+1} | p + 1 \) holds for some prime \( q > z \). We deduce that

\[ F = O \left( \frac{H}{\log x} \right). \]  \hspace{1cm} (3.1)

We have \( F \leq F_1 + F_2 + F_3 \), where

\[ F_1 := \sum_{\substack{z < q < Y/t \\frac{1}{r+1} < q \in \mathcal{P}}} \left( \pi \left( x + Y, q^{r+1}, -1 \right) - \pi \left( x, q^{r+1}, -1 \right) \right), \]

\[ F_2 := \sum_{Y^{1/2r+2} < q < Y^{1/(r+1)}} \left( \pi \left( x + Y, q^{r+1}, -1 \right) - \pi \left( x, q^{r+1}, -1 \right) \right) \]

and

\[ F_3 = \sum_{Y^{1/(r+1)} \leq q} \left( \pi \left( x + Y, q^{r+1}, -1 \right) - \pi \left( x, q^{r+1}, -1 \right) \right). \]

By using the Brun–Titchmarsh inequality (see Halberstam Richert [5], Theorem 3.7, page 107) we can obtain that

\[ F_1 < \frac{cY}{\log x} \sum_{q > z} 1/q^{r+1} \ll \frac{Y}{(\log x)^3}, \]  \hspace{1cm} \text{say.}

Furthermore

\[ F_2 \ll \sum_{q^{r+1} > \sqrt{Y}} \left( \left[ \frac{Y}{q^{r+1}} \right] \right) \ll \frac{Y}{(\log x)^3}. \]

Let us estimate \( F_3 \). If \( qr^{r+1} > Y \), \( qr^{r+1} | p + 1 \), then \( p + 1 = qr^{r+1} \nu \), \( \nu < \frac{2}{qr} \). For a fixed \( \nu < \frac{2}{qr} \) the number of those primes \( q \) for which
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\[ x \leq q^r \nu \leq x + Y \text{ is less than } c \left( \frac{x}{y} \right)^{1/r+1} \cdot \frac{Y}{x}, \text{ and so} \]

\[ F_3 \ll \sum_{\nu < \frac{Y}{x}} \left( \frac{x}{y} \right)^{\frac{1}{r+1}} \cdot \frac{Y}{x} \]

whence we can obtain that \( F_3 \ll Y^{\frac{1}{r+1}} \).

Let us observe that

\[
H(p+1) \leq r \text{ if and only if } \sum_{d | p+1} \mu(d) = 1.
\]

From the argument used earlier we obtain that

\[ E_r(x, Y) = \sum_{p \in [x, x+Y]} \sum_{d | p+1 \text{ with } P(d) < z} \mu(d) + O \left( \frac{Y}{(\log x)^2} \right). \]

Here \( P(n) \) denotes the largest prime divisor of \( n \).

If \( p + 1 \) is such a number for which there is a \( d > z^2 \), such that \( d^{r+1} | p+1 \), and \( P(d) < z \), then there is a divisor \( \delta \) of \( d \) such that \( \delta \in [z, z^2] \), and so \( \delta^{r+1} | p+1 \).

Since

\[
\sum_{\delta \in [z, z^2]} \left( \pi(x + Y \delta^{r+1}, -1) - \pi(x, \delta^{r+1}, -1) \right) \lesssim \frac{Y}{\log x} \sum_{z \leq \delta \leq z^2} \frac{1}{\varphi(\delta^{r+1})} \lesssim \frac{Y}{(\log x)^2},
\]

therefore

\[ E_r(x, Y) = \sum_{d < z^2} \mu(d) \left( \pi(x + Y d^{r+1}, -1) - \pi(x, d^{r+1}, -1) \right) \]

\[ + O \left( \frac{Y}{(\log x)^2} \right), \]

and so by the prime number theorems for short intervals due to Huxley [7] we obtain that

\[ E_r(x, Y) = \frac{Y}{\log x} \sum_{d < z^2} \mu(d) \frac{1}{\varphi(d^{r+1})} + O \left( \frac{Y}{(\log x)^2} \right). \]
One can observe furthermore that
\[ \sum_{d < z} \frac{\mu(d)}{\varphi(d^{r+1})} = \prod_p \left( 1 - \frac{1}{(p-1)^r} \right) + O \left( \frac{1}{\log x} \right) \]
say.

If \( r \geq 2 \), then \( H(n) = r \) holds if and only if \( H(n) \leq r \), and \( H(n) \leq r-1 \) does not hold, therefore
\[ \# \{ p \in [x, x+Y], H(p+1) = r \} = E_r(x,Y) - E_{r-1}(x,Y). \]

Furthermore
\[ \# \{ p \in [x, x+Y], H(p+1) = 1 \} = E_1(x,Y), \]
thus our theorem immediately follows. \( \square \)

4. Let \( 1 \leq Y \leq \sqrt{x} \). The number of those \( n \leq x \) which have a divisor \( p^2 \) such that \( p > Y \), is less than \( O(\sqrt{x}) \). Hence, we can deduce easily that
\[ x^{-1} \# \{ n \leq x \mid H(n+j) = r_j, j = 0, \ldots, s \} = c(r_0, r_1, \ldots, r_s) + O \left( (\log x)^{-2} \right), \] (4.1)
say, with a constant \( c(r_0, \ldots, r_s) \).

Similarly, one can get that
\[ \frac{1}{\log x} \# \{ p \leq x \mid H(p+l) = r_l, l = 1, \ldots, s \} = d(r_1, \ldots, r_s) + O \left( (\log x)^{-2} \right). \] (4.2)

The relations (4.1), (4.2) hold for every choice of \( r_0, r_1, \ldots, r_s \in \mathbb{N} \).

(4.1) readily follows from the relation
\[ x^{-1} \# \{ n \leq x \mid H(n+j) \leq r_j, j = 0, \ldots, s \} = d(r_0, r_1, \ldots, r_s) + O \left( (\log x)^{-2} \right) \] (4.3)
which is almost a direct consequence of the relation
\[ \sum_{d \mid m} \mu(d) = \begin{cases} 1 & \text{if } H(m) < t, \\ 0 & \text{otherwise.} \end{cases} \] (4.4)
Hence

\[ \# \{ n \leq x \mid H(n + j) \leq r_j, \ j = 0, \ldots, s \} = \sum_{d_0, d_1, \ldots, d_s} \mu(d_0) \cdots \mu(d_s) \# \{ n \leq x \mid n + j \equiv 0 \pmod{d_j^{r_j}}, \ j = 0, \ldots, s \} \]

The right hand side can be evaluated simply, since we can drop all those \( d_0, \ldots, d_s \) for which \( \max d_j \geq (\log x)^2 \), their contribution is \( O(x/(\log x)^2) \).

We can argue similarly, by the proof (4.2). Here we should use the Siegel–Walfisz theorem also, which asserts that

\[ \pi(x, k, l) = \text{li} x \frac{\varphi(k)}{\varphi(l)} \left( 1 + O(e^{-c\sqrt{\log x}}) \right) \]

uniformly as \( (l, k) = 1, k \leq (\log x)^B \), where \( c > 0 \) is a suitable, and \( B > 0 \) is an arbitrary large constant (see [10], Theorem 8.3 Chapter IV).

**Theorem 2.** Let \( g \in \mathbb{Z}[x] \), irreducible over \( \mathbb{Q} \), \( r = \deg g \), \( r \geq 3 \). Assume that \( g(n) \in \mathbb{N} \) for \( n > 0 \).

Then

\[ \frac{1}{x} \# \{ n \leq x \mid H(g(n)) \leq s \} = c(g, s) + O((\log \log x)^{-1}) \]

if \( s \geq r - 2 \), \( c(g, s) \) is a suitable constant, and

\[ \frac{1}{\text{li} x} \# \{ p \leq x \mid H(g(p)) \leq s \} = d(g, s) + O((\log \log x)^{-1}) \]

if \( s \geq r - 1 \), \( d(g, s) \) is a suitable constant

**Remark.** The proof is based upon an important theorem of C. Hooley [4] which is quoted now as

**Lemma 2.** Let \( g \) be a polynomial satisfying the conditions of Theorem 2. Let \( N''(x) \) be the number of those \( n \leq x \) for which \( q^{r-1} \) divides \( g(n) \) for at least one \( q \geq \frac{1}{B} \log x \). Then

\[ N''(x) \ll x \cdot (\log x)^{r-1}. \]

Let \( S(x) \) be the number of those primes \( p \leq x \) for which there exists a prime \( q > \log x \) such that \( q^r | g(p) \), then

\[ S(x) = O \left( \frac{\text{li} x}{\log \log x} \right). \]
Proof of Theorem 2. By using Lemma 2

\[
\sum_{n \leq x \atop H(g(n)) \leq s} 1 = \sum_{n \leq x} \sum_{d \mid g(n)} \mu(d) + O(N''(x)),
\]

where \(d\) runs over those square free integers the largest prime factor of which is smaller than \(\frac{x}{\log x}\). Hence (4.5) can be deduced immediately. (See [4], Chapter 4, Theorem 3) (4.6) can be proved similarly, by using Lemma 2, the Siegel–Walfisz and the Bombieri–Vinogradov theorem.

\[ \square \]

Theorem 3. Let \(\varepsilon > 0\) be a constant, \(Y = x^{\frac{2}{3} + \varepsilon}\). Then, for every \(s \in \mathbb{N}\), and every fixed \(A > 0\),

\[
\frac{1}{Y} \sum_{n \in [x,x+Y] \atop H(n^2+1) = s} 1 = a(s) + O((\log x)^{-A}), \quad (4.7)
\]

\[
\frac{1}{Y} \sum_{p \in [x,x+Y] \atop H(p^2+1) = s} 1 = b(s) + O((\log x)^{-A}). \quad (4.8)
\]

Remark. In [6] we proved that the number of those \(n \in [x,x+Y]\) for which \(q^2|n^2+1\) holds for at least one \(q \geq \sqrt{Y}\) is \(O\left(x^{2/3} \log x\right)\). By using this, and standard techniques we obtain (4.7). (4.8) follows similarly, by using the Hoheisel–Tatuzawa theorem, the short interval version of the Siegel–Walfisz theorem ([10], Theorem 3.2, Chapter IX).

5. Theorem 4. We have

\[
\left( \sum : \right) = \# \{ n \leq x \mid H(n) = 1, \; h(n + 1) \geq 2 \} \equiv C\sqrt{x} + O\left( \frac{\sqrt{x}}{\log x} \right). \quad (5.1)
\]

Proof of Theorem 4. If \(H(n) = 1, \; h(n + 1) \geq 2\), then \(n + 1\) can be uniquely written as \(am^2\), where \(a\) is cube-full, \(m\) is square-free and \(am^2 - 1\) is square free.
Let
\[ \sum_a := \sum_{m^2 \leq \frac{x}{a}} |\mu(m)| |\mu(am^2 - 1)|. \]  
(5.2)

Then
\[ \sum = \sum_{a < x} \sum_{a \leq Y} \sum_{a \geq Y} \sum a = \sum^{(1)} + \sum^{(2)}. \]

Since \( \sum_a \leq \sqrt{x/a} \), and summing over cube-full \( \sum_{a \geq Y} \frac{1}{a} \ll \frac{1}{Y^{1/3}} \), we obtain that \( \sum^{(2)} \ll \frac{x}{\log x^2} \), if \( Y = (\log x)^6 \).

To evaluate (5.2) for a fixed \( a \), first we overestimate the number of those \( m \) for which \( p > Z \) and either \( p^2 \mid m \), or \( p^2 \mid am^2 - 1 \). \( Z = \frac{1}{6} \log x \). We have
\[ \sum_{m < \sqrt{x/a}} \sum_{p > Z} 1 \leq 2 \sqrt{x/a} \sum_{p > Z} \frac{1}{p^2} \ll 2 \left( \sqrt{x/a} \right) \frac{1}{Z}, \]

furthermore
\[ \sum_{m < \sqrt{x/a}} \sum_{p < \sqrt{x}} \frac{1}{p} \leq 3 \sqrt{x/a} \frac{1}{Z}, \]
\[ \sum_{m < \sqrt{x/a}} \sum_{\sqrt{x} \leq p < (\sqrt{x})(\log x)^{c}} \frac{1}{p} \leq 2 \sqrt{x \log \log x} \frac{1}{\sqrt{x/a} \log x}. \]

For fixed integers \( a, b > 0 \) the number of solutions \( m, n, am^2 \leq x \) of the equation \( am^2 - bn^2 = 1 \) is no more than \( O(\log x) \). This follows from the identity
\[ \left( \sqrt{\frac{a}{b}} - \frac{n}{m} \right) \left( \sqrt{\frac{a}{b}} + \frac{n}{m} \right) = \frac{1}{bm^2}, \]
whence one get that if \( n, m \) is a solution, then \( n/m \) is an approximant from the continued fraction of \( \sqrt{\frac{a}{b}} \). As it is known, no more than \( O(\log \sqrt{\frac{a}{b}}) = O(\log x) \) such \( m, n \) pairs exist.
Collecting our inequalities, we get that
\[
\sum_a = \sum_{m<\sqrt{\frac{x}{a}}} \left( \sum_{m \mid \delta_1}^* \mu(\delta_1) \right) \left( \sum_{\delta_2 \mid am^2-1}^* \mu(\delta_2) \right) \\
+ O \left( \frac{1}{x^{1/6}} \sqrt[3]{a} \right) + O \left( \frac{x \log \log x}{a \log x} \right) \\
+ O \left( \sqrt{\frac{x}{a} (\log x)^{-c}} \right),
\]
where \(c\) is an arbitrary fixed positive constant. The asterisk means that we sum over those \(\delta_1, \delta_2\) the largest prime factor of which is no larger than \(\frac{1}{6} \log x\). In this case \(\delta_1 \leq x^{\frac{1}{6}+\varepsilon}, \delta_2 \leq x^{\frac{1}{6}+\varepsilon}\). For fixed \(\delta_1, \delta_2, (\delta_1, \delta_2) = 1\) we have to sum over those \(\nu \leq \frac{1}{11} \sqrt{\frac{x}{a}}\) for which \(a \delta_1^4 \nu^2 - 1 \equiv 0 \pmod{\delta_2^2}\) which is equivalent to \(\nu^2 \equiv a \pmod{\delta_2^2}\) if \((a, \delta_2) = 1\).

Let \(\rho_a(D)\) be the number of solutions of \(\nu \pmod{D}\), for which \(\nu^2 \equiv a \pmod{D}\), if \(D\) is square free. It is clear that \(\rho_a\) is multiplicative in \(D\), \(\rho_a(p) = 1 + \left(\frac{a}{p}\right)\) for \(p\) prime, \(p \nmid D\), and so
\[
\# \left\{ \nu \leq \frac{1}{\delta_1^4} \sqrt{\frac{x}{a}}, \nu^2 \equiv a \pmod{\delta_2^2} \right\} = \rho(\delta_2^2) \left( \frac{1}{\delta_1^4 \delta_2^2} \sqrt[3]{\frac{x}{a}} + O(1) \right).
\]
Thus
\[
\sum_a = \sqrt{\frac{x}{a}} \sum_{(\delta_1, \delta_2) = 1}^* \frac{\mu(\delta_1) \mu(\delta_2)}{\delta_1^4 \delta_2^2} \rho(\delta_2^2) + O \left( \sum_{\delta_1}^* \sum_{\delta_2}^* 1 \right).
\]
The error term is \(O \left( x^{1/3+2\varepsilon} \right)\). For the fixed \(\delta_2\),
\[
\sum_{(\delta_1, \delta_2) = 1}^* \frac{\mu(\delta_1)}{\delta_1^4} = \prod_{\substack{p \leq (\log x)^{1/6} \\ (p, \delta_2) = 1}} \left( 1 - \frac{1}{p^2} \right) + O \left( \frac{1}{x^{1/2}} \right) \\
= \frac{1}{\zeta(2)} \prod_{p \mid \delta_2} \frac{1}{1 - 1/p^2} + O \left( \frac{1}{(\log x)^C} \right).
\]
Thus

\[
\sum_{a} = \sqrt{x} \frac{1}{\zeta(2)} \sum_{(\delta_2,a)=1} \frac{\mu(\delta_2)\rho(\delta_2^2)}{\delta_2} \prod_{p|\delta_2} \frac{1}{1 - 1/p^2} + O \left( \sqrt{x} \frac{1}{(\log x)^3} \right),
\]

the sum on the right hand side

\[
\sum_{(\delta_2,a)=1} \frac{\mu(\delta_2)\rho(\delta_2^2)}{\delta_2} \prod_{p|\delta_2} \frac{1}{1 - 1/p^2} = \prod_{p|a} \left( 1 - \rho(p^2) \frac{1}{p^2} \right) = \prod_{p|a} \left( 1 - \rho(p^2) \frac{1}{p^2 - 1} \right).
\]

Summing \(\sum_{a}\) over the cube-full \(a\), we obtain (5.1) easily. \[\square\]

One can prove similarly

**Theorem 5.** Let \(r, s\) be arbitrary positive integers. Then

\[
\#\{n \leq x \mid H(n) \leq r, \ h(n + 1) \geq s\} = c(r, s)x^{1/s} + O \left( x^{1/s}/\log x \right).
\]

The following two conjectures seem to be quite plausible.

**Conjecture 1.** We have

\[
\#\{n \leq x \mid h(n) \geq 2, \ h(n + 1) \geq 2\} \geq \frac{cx^{1/4}}{\log x}.
\]

**Conjecture 2.** We have

\[
\#\{p \leq x \mid h(p + 1) \geq 2\} \to \infty \quad (x \to \infty).
\]

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