Consecutive binomial coefficients satisfying
a quadratic relation
By FLORIAN LUCA (Morelia) and LÁSZLÓ SZALAY (Sopron)

Abstract. In this note, we study the diophantine equation $A\binom{n}{k}^2 + B\binom{n}{k+1}^2 + C\binom{n}{k+2}^2 = 0$ in positive integers $(n, k)$, where $A$, $B$ and $C$ are fixed integers.

1. Introduction

D. Singmaster (see [7]) found infinitely many positive integer solutions $(n, k)$ to the diophantine equation

$$\binom{n}{k} = \binom{n-1}{k+1}. \quad (1)$$

All such solutions arise in a natural way from the sequence of Fibonacci numbers $(F_m)_{m\geq 0}$ given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for $m \geq 0$. Goetgheluck (see [2]) extended the above result and found infinitely many positive integer solutions $(n, k)$ for the diophantine equation

$$2\binom{n}{k} = \binom{n-1}{k+1}. $$

These solutions arise in a natural way from the positive integer solutions

---

Mathematics Subject Classification: 11D09.

Key words and phrases: quadratic diophantine equations.
of the Pell equation \(x^2 - 3y^2 = -2\). The general linear diophantine equation

\[A\binom{n}{k} + B\binom{n}{k+1} + C\binom{n}{k+2} = 0 \tag{2}\]

was treated in [5].

All the consecutive binomial coefficients satisfying the Pythagorean relation

\[\binom{n}{k}^2 + \binom{n}{k+1}^2 = \binom{n}{k+2}^2 \tag{3}\]

were determined in [4]. It turns out, that in searching for the integer solutions \((n, k)\) with \(1 \leq k < k+2 \leq n-1\) of equation (3), one is naturally led to Fibonacci numbers which are a square or twice a square. The similar looking diophantine equations

\[a\binom{n}{k}^2 + b\binom{n}{k+1}^2 = \binom{n}{k+2}^2, \tag{4}\]

for \((a, b) = (1, 2), (2, 1)\), as well as the diophantine equation

\[\binom{n}{k}^2 + \binom{n+1}{k}^2 = \binom{n+2}{k}^2,\]

were considered in [9]. Other diophantine equations involving binomial coefficients appear in [8].

In this note, we fix three integers \(A, B, C\), not all zero, and look at the positive integer solutions \((n, k)\) of the equation \(A\binom{n}{k}^2 + B\binom{n}{k+1}^2 + C\binom{n}{k+2}^2 = 0\). To avoid degenerate cases, we shall assume that \(1 \leq k < k+2 \leq n-1\). We assume that \(\gcd(A, B, C) = 1\). We shall also assume that \(AC \neq 0\). Indeed, say if \(A = 0\), then the above equation simplifies to \(B(k+2)^2 + C(n-k-1)^2 = 0\), which implies, up to changing signs, that we may assume \(B = B_0^2\), \(C = -C_0^2\), where \(B_0\) and \(C_0\) are coprime positive integers. Since both \(n\) and \(k\) are positive, we get that the given equation implies that \(C_0n = (C_0 + B_0)k + C_0 + 2B_0\), and it is clear that this last equation has infinitely many solutions, which are all effectively computable.

We shall also assume that \(B \neq 0\). Indeed, if \(B = 0\), then the only case when equation (5) might have any integer solutions \((n, k)\) with \(1 \leq
Consecutive binomial coefficients satisfying a quadratic relation

$k < k + 2 ≤ n - 1$ is when $A = A_0^2$ and $C = -C_0^2$ hold with some positive integers $A_0$ and $C_0$. Equation (5) now leads to

$$A_0 \binom{n}{k} - C_0 \binom{n}{k+2} = 0,$$

which is a particular case of the more general equation of the form (2).

2. Main result

It is clear that we may assume that gcd($A, B, C$) = 1 and that $A > 0$. Our main result is the following.

**Theorem 1.** Assume that $A, B, and C$ are nonzero integers. Then the diophantine equation

$$A \binom{n}{k}^2 + B \binom{n}{k+1}^2 + C \binom{n}{k+2}^2 = 0$$

has at most finitely many effectively computable integer solutions $(n, k)$ with $1 ≤ k < k + 2 ≤ n - 1$.

3. Preliminary results

Before proceeding to the proof of Theorem 1, we recall a criterion due to Legendre for the existence of a nonzero integer solution $(x, y, z)$ to the diophantine equation

$$ax^2 + by^2 + cz^2 = 0,$$

where $a$, $b$ and $c$ are nonzero integers. We may certainly assume, up to relabelling the coefficients $a, b, c$ and the variables $x, y, z$, that $a > 0, b < 0$ and $c < 0$. Furthermore, we may also assume that gcd($a, b, c$) = 1 and that $a, b, c$ are squarefree (if $a = d^2a_1$ and $(x, y, z)$ is a solution of equation (6), then $(dx, y, z)$ is a solution of equation (6) with $a$ replaced by $a_1$). We now show that we may even assume that gcd($a, b$) = gcd($b, c$) = gcd($a, c$) = 1.

Indeed assume that $d_{ab} = \text{gcd}(a, b)$, $d_{bc} = \text{gcd}(b, c)$ and $d_{ac} = \text{gcd}(a, c)$.
Then \( a = d_{ab}d_{ac}a_1, b = d_{ab}d_{bc}b_1, c = d_{ac}d_{bc}c_1 \) and equation (6) becomes
\[
d_{ab}d_{ac}a_1 x^2 + d_{ab}d_{bc}b_1 y^2 + d_{ac}d_{bc}c_1 z^2 = 0.
\]
The above equation shows that \( d_{ab} | z, d_{bc} | x \) and \( d_{ac} | y \), and writing \( x = d_{bc}x_1, y = d_{ac}y_1, z = d_{bc}z_1 \), we get the equation
\[
a_1 d_{bc}x_1^2 + b_1 d_{ac}y_1^2 + c_1 d_{ab}z_1^2 = 0,
\]
which is an equation of the same type as (6) with the coefficients \( a, b, c \) replaced by \( a_1 d_{bc}, b_1 d_{ac}, c_1 d_{ab} \), which are pairwise coprime because of the definitions of \( d_{ab}, d_{ac}, d_{bc} \) and the fact that all three numbers \( a, b \) and \( c \) are squarefree.

Legendre’s Theorem asserts the following. (See, for example, [1], p. 62 and p. 73.)

**Lemma 2.** Let \( a, b, c \) be three squarefree integers, \( a > 0, b < 0, c < 0 \) which are pairwise coprime. Then there exists a nonzero integer solution \((x, y, z)\) to the diophantine equation (6) if and only if all three congruences
\[
t^2 \equiv -ab \pmod{c} \quad t^2 \equiv -ac \pmod{b} \quad t^2 \equiv -bc \pmod{a}
\]
are solvable.

Knowing, via Lemma 2, that a certain equation of the form (6) has infinitely many nonzero integer solutions \((x, y, z)\), it is of interest to us to know how to compute all of them. This is achieved in the following lemma. (J. Kelemen [3] described the solutions of (6), but because of relative unaccessibility of this paper and for the convenience of the reader we give the proof.)

**Lemma 3.** Assume that \((x_0, y_0, z_0)\) is an integer solution of equation (6) with \( z_0 \neq 0 \). Then, all integer solutions \((x, y, z)\) with \( z \neq 0 \) of equation (6) are of the form
\[
x = \pm \frac{D}{d} \left( -a x_0 s^2 - 2 b y_0 r s + b x_0 r^2 \right),
\]
\[
y = \pm \frac{D}{d} \left( a y_0 s^2 - 2 a x_0 r s - b y_0 r^2 \right),
\]
\[
z = \pm \frac{D}{d} \left( a z_0 s^2 + b z_0 r^2 \right),
\]
where \( r \) and \( s > 0 \) are coprime integers, \( D \) is a nonzero integer, and \( d \) is a bounded positive integer.
Consecutive binomial coefficients satisfying a quadratic relation 189

Proof. Let \((x, y, z)\) be a nonzero integer solution of equation (6) with \(z \neq 0\). Note that since we are assuming that \(\gcd(a, b) = \gcd(b, c) = \gcd(a, c) = \gcd(x, z) = \gcd(y, z)\). We write \(D\) for this number. Write \(X = \frac{x}{z},\ Y = \frac{y}{z},\ X_0 = \frac{x_0}{z_0},\ Y_0 = \frac{y_0}{z_0}\) and let \(t\) be such that \(Y - Y_0 = t(X - X_0)\).

Clearly, \(t\) is a rational number if \((X, Y) \neq (X_0, Y_0)\). Let \(t = \frac{r}{s}\) with \(s > 0\) and \(\gcd(r, s) = 1\). Equation (6) implies that

\[
aX^2 + bY^2 = -c, \quad (7)
\]

and

\[
aX_0^2 + bY_0^2 = -c. \quad (8)
\]

Replacing \(X\) by \(X_0 + (X - X_0)\) and \(Y\) by \(Y_0 + t(X - X_0)\) in equation (7) and using equation (8), we get

\[
-c = a(X_0 + (X - X_0))^2 + b(Y_0 + t(X - X_0))^2
\]

\[
= (aX_0^2 + bY_0^2) + (X - X_0)(2aX_0 + 2bY_0 t) + (X - X_0)^2(a + bt^2),
\]

which leads to

\[
0 = (X - X_0)(2aX_0 + 2bY_0 t + (X - X_0)(a + bt^2)).
\]

If \(X \neq X_0\), we get

\[
X = X_0 + \frac{-2aX_0 - 2bY_0 t}{a + bt^2} = \frac{-aX_0 - bY_0 t + bX_0 t^2}{a + bt^2},
\]

so

\[
Y = Y_0 + t(X - X_0) = Y_0 + t\frac{-2aX_0 - 2bY_0 t}{a + bt^2} = \frac{aY_0 - 2aX_0 t - bY_0 t^2}{a + bt^2},
\]

and replacing \(t\) by \(r/s\) we get

\[
x = \frac{z}{z} = X = \frac{-ax_0 s^2 - 2by_0 rs + bx_0 r^2}{az_0 s^2 + bz_0 r^2},
\]

\[
y = \frac{z}{z} = Y = \frac{ay_0 s^2 - 2ax_0 rs - by_0 r^2}{az_0 s^2 + bz_0 r^2}.
\]

Since \(D = \gcd(x, z) = \gcd(y, z)\), it follows that the two fractions appearing on the right hand sides of the two formulae above have the same denominator when written in simplified form. Let \(z_0(az^2 + br^2)/d\) be this denominator. Then, \(d|az_0 s^2 + bz_0 r^2\). Let \(d_0 = \gcd(d, z_0)\). Thus, \(d_0 \leq z_0\). Now let
\[ d_1 = d/d_0. \] Then \( br^2 \equiv -as^2 \pmod{d_1}. \) Since also \( d|ay_0s^2 - 2ax_0rs - by_0r^2, \)
it follows that \( d_1|ay_0s^2 - 2ax_0rs + ay_0s^2, \) so \( d_1|2as(y_0s - x_0r). \) Let \( d_2 = \gcd(d_1, 2a), \) \( d_3 = \gcd(d_1, s) \) and \( d_4 = \gcd(d_1, y_0s - x_0r). \) Clearly, \( d_2 \leq 2a. \) Now \( d_3|s \) and \( d_3|as^2 + br^2, \) therefore \( d_3|br^2, \) and since \( \gcd(r, s) = 1, \) we get that \( d_3|b. \) Thus, \( d_3 \leq b. \) Finally, \( d_4|y_0s - x_0r, \) therefore \( y_0^2s^2 \equiv x_0^2r^2 \pmod{d_4}. \) Since \( as^2 \equiv -br^2 \pmod{d_4}, \) we also get that \( r^2(ax_0^2 + by_0^2) \equiv 0 \pmod{d_4}. \) Let \( d_5 = \gcd(d_4, r^2) \) and \( d_6 = \gcd(d_4, cz_0^2). \) Since \( d_5|r^2 \) and \( d_5|as^2 + br^2, \) we get that \( d_5|as^2, \) and since \( r \) and \( s \) are coprime, we get that \( d_5|a. \) Thus, \( d_5 \leq a. \) Finally, \( d_6|cz_0^2, \) therefore \( d_6 \leq cz_0^2. \) We now get that \( d \leq d_0d_2d_3d_5d_6 \leq 2a^2bcz_0^2, \) which completes the proof of Lemma 3. \( \square \)

**Remark.** The above Lemma 3 addresses only those solutions \((x, y, z)\) with \( z \neq 0. \) However, if \((x, y, z)\) is a nonzero solution with \( z = 0, \) then \( x/y = \pm \sqrt{-b/a}. \) On the other hand, this is impossible except for the case \( x/y = 1, \) because \( \gcd(a, b) = 1, \) and \( a, b \) are squarefree.

### 4. The proof of the theorem

We shall assume that \( A > 0 \) and \( B < 0, C < 0, \) for the remaining cases can be dealt with in a similar way. Equation (5) can be rewritten as

\[ (\alpha + 1)^2(A\alpha^2 + B\beta^2) = -C\beta^2(\beta - 1)^2, \tag{9} \]

where \( \alpha = k + 1 \) and \( \beta = n - k \) are positive integers. The above equation shows that \( A\alpha^2 + B\beta^2 = -C\delta^2 \) holds with the rational number \( \delta = \beta(\beta - 1)/(\alpha + 1). \) Thus, there exists a positive integer \( C_1 \) such that \( C_1^2|C, C_1\delta \) is an integer, and \( A\alpha^2 + B\beta^2 = (C/C_1^2)(C_1\delta)^2. \) Let \( \gamma = C_1\delta. \) By arguments similar to the ones employed before Lemma 2, there exist integers \( a, b, c, u, v, w, \) which are easily obtained from \( A, B \) and \( C, \) where the three integers \( a, b \) and \( c \) satisfy \( a > 0, b < 0, c < 0, \) are squarefree and coprime any two, and \( u, v \) and \( w \) are positive, such that every integer solution \((\alpha, \beta, \gamma)\) of the equation \( A\alpha^2 + B\beta^2 = (-C/C_1^2)\gamma^2 \) has the property that \((x, y, z) = (ua, v\beta, w\gamma)\) is a solution of

\[ ax^2 + by^2 = -cz^2. \]

In the coordinates \((x, y)\), equation (9) can be rewritten as

\[ ax^2 + by^2 = -c \left( \frac{C_1w\beta(\beta - 1)}{(\alpha + 1)} \right)^2 = -c \left( \frac{C_1uw(y - v)}{v^2(x + u)} \right)^2. \]
Thus,
\[ z = \frac{C_1uwy(y-v)}{v^2(x+u)}, \]
or, equivalently,
\[ v^2(x+u)z = C_1uwy(y-v). \]  

We now use Lemma 3, where we write \( x_1 := x_1(r,s) = |-ax_0s^2-2by_0rs+bx_0r^2|/d, \ y_1 := y_1(r,s) = |ay_0s^2-2ax_0rs-by_0r^2|/d \) and \( z_1 := z_1(r,s) = |az_0s^2+bz_0r^2|/d. \) With these notations, we have that \( x_1, y_1 \) and \( z_1 \) positive integers which are coprime any two. Moreover, by Lemma 3, we also have that \((x,y,z) = (Dx_1,Dy_1,Dz_1)\). Here, we neglect the signs because our unknowns \( x, y \) and \( z \) are positive. Equation (10) can be rewritten as
\[ v^2(Dx_1+u)z_1 = C_1uwy_1(Dy_1-v), \]
and since \( z_1 \) and \( y_1 \) are coprime, we get that \( z_1|C_1uw(Dy_1-v) \). Thus,
\[ \frac{v^2(Dx_1+u)}{y_1} = \frac{C_1uw(Dy_1-v)}{z_1} = E, \]
where \( E \) is a positive integer. The above equation leads to the linear system of two equations in the unknowns \( D \) and \( E \), namely
\[(v^2x_1)D-Ey_1 = -uw^2, \quad (C_1uwy_1)D-Ez_1 = C_1uwv.\]

Let \( \Delta = \Delta(r,s) = (v^2x_1)(-z_1) - (C_1uwy_1)(-y_1) = -(v^2x_1z_1 - C_1uwy_1^2) \) be the discriminant of the above system. We first note that \( \Delta \) is an homogeneous form of degree 4 in the variables \( r \) and \( s \). Moreover, since both \( D \) and \( E \) are integers, we get, by Cramer’s rule, that \( \Delta \) divides both \((-uw^2)(-z_1) - (C_1uwv)(-y_1) = uv(vz_1 + C_1wy_1) \) and \((v^2x_1)(C_1uwv) - (C_1uwy_1)(-uw^2) = C_1uw^2w(vx_1 + uy_1) \). It now follows easily that \( \Delta \neq 0 \). Indeed, if \( \Delta = 0 \), then we must have \( uvw(vx_1 + vy_1) = 0 \), which is impossible because all of \( u, v, w, x_1 \) and \( y_1 \) are positive integers. We now let \( \Delta_1 = \gcd(\Delta, uv), \ \Delta_2 = \gcd(\Delta, C_1uw^2w) \) and \( \Delta_3 = \Delta / \text{lcm} [\Delta_1, \Delta_2] \). Then \( \Delta_1 \leq uv, \ \Delta_2 \leq C_1uw^2w \) and \( \Delta_3 \) divides both \( F(r,s) = vz_1 + C_1wy_1 \) and \( G(r,s) = vx_1 + uy_1 \). Note that both \( F(r,s) \) and \( G(r,s) \) are homogeneous forms of degree 2. From now on, we proceed as follows. We first prove that the two homogeneous forms \( F(r,s) \) and \( G(r,s) \) have at most one linear form in common with multiplicity 1 (or none). This will show that
either $|\Delta_3|$ is bounded, or that $\Delta_3$ divides a linear form in $r$ and $s$. In the first case, $|\Delta|$ is bounded. In the second case, $\Delta_3$ is a linear form and $(\Delta/\Delta_3)(r,s)$ is a homogeneous form of degree 3, which then must be bounded in absolute value. This argument therefore shows that there exists a constant $K$ (obviously, effectively computable), and an homogeneous factor of $\Delta$, let’s call it $\Delta'$, of degree either 3 or 4, such that $|\Delta'(r,s)| < K$.

We shall then show that $\Delta$ has no multiple roots. In particular, each one of the above inequalities will then be a Thue inequality, and it is well-known that such inequalities have at most finitely many integers solutions $(r, s)$, which furthermore are effectively computable by using the theory of linear forms in logarithms (see [6]). This will conclude the proof of Theorem 1.

The polynomials $F$ and $G$. Suppose that $F$ and $G$ have more than one root in common. Since they are quadratic, it follows that they differ by a scalar multiple. Thus, we may assume that $\lambda F + \mu G = 0$ holds with some coefficients $\lambda$ and $\mu$, not both zero. Note now that $F(r, s) = C_1vw((z_1/C_1w) + (y_1/v))$, and $G(r, s) = uv((y_1/v) + (x_1/u))$, and it is now easy to see $(X(r, s), Y(r, s), Z(r, s)) = (x_1/u, y_1/v, z_1/(C_1w))$ is simply a parametrization of all (but finitely many) nonzero rational points on the quadratic curve

$$AX^2 + BY^2 = -CZ^2.$$ (11)

Since $\lambda F + \mu G = 0$, we get, with $\lambda_1 = C_1vw\lambda$ and $\mu_1 = uv\mu$, that $\lambda_1(Z + Y) + \mu_1(Y + X) = 0$, or $\mu_1 X + (\lambda_1 + \mu_1) Y + \lambda_1 Z = 0$. Since $\lambda$ and $\mu$ are not both zero, the above relation is nontrivial. We thus get that all rational points $(X, Y, Z)$ on the curve (11) (except for finitely many of them) lie on a line, which is certainly impossible. Thus, $F$ and $G$ can have at most one root in common.

The roots of $\Delta$. Here, we show that all the roots of $\Delta$ are simple. With the previous notations, we recognize that

$$\Delta = -C_1uv^2 w \left( \left( \frac{X_1}{w} \right) \left( \frac{Z_1}{C_1w} \right) - \left( \frac{Y_1}{v} \right)^2 \right) = -C_1uv^2 w (XZ - Y^2).$$

Assume that $XZ - Y^2$ has a double root. We now set up $s = 1$ and let $U_1 = U_1(r) = X(r, 1)/Z(r, 1)$ and $V_1 = V_1(r) = Y(r, 1)/Z(r, 1)$ be rational functions. If $XZ - Y^2$ has a double root, it follows that the rational function $U_1 - V_1^2 = (XZ - Y^2)/Z^2$ also has a double root (note that $Z$
Consecutive binomial coefficients satisfying a quadratic relation (11) shows that

\[ AU_1^2 + BV_1^2 = -C. \]

Taking derivatives in the above relation (with respect to \( r \)), we get

\[ 2AU_1U_1' + 2BV_1V_1' = 0. \]  \hspace{1cm} (12)

Now let \( r_0 \) be the double root of \( U_1 - V_1^2 \). We then have that \( U_1(r_0) = V_1(r_0)^2 \) and by taking derivatives we also have \( U_1(r_0)' = 2V_1(r_0)V_1(r_0)' \).

Evaluating the above relation (12) in \( r_0 \), we get

\[ 2AU_1(r_0)U_1(r_0)' = -2BV_1(r_0)V_1(r_0)' = -2BU_1(r_0)', \]

two equations for \( U_1(r_0)'(AU_1(r_0) + B) = 0. \)

If \( U_1(r_0)' = 0 \), then, since \( U_1(r_0) = V_1(r_0)^2 \) and by taking derivatives we also have \( U_1(r_0)' = 2V_1(r_0)V_1(r_0)' \), we either get \( V_1(r_0)^2 = 0 \) (which is impossible because \( (U_1(r), V_1(r)) \) is nonsingular), or \( V_1(r_0) = 0 \). In this later case, since \( U_1(r_0) = V_1(r_0)^2 \), we get that \( U_1(r_0) = 0 \), therefore \( X(r_0, 1) = Y(r_0, 1) = 0 \), which is again impossible. Thus, \( U_1(r_0)' \neq 0 \) and we are therefore left with the situation \( U_1(r_0) = -B/A \). Since \( U_1(r_0) = V_1(r_0)^2 \), we get that \( V_1(r_0)^2 = -B/A \). Thus,

\[ -C = AU_1(r_0)^2 + BV_1(r_0)^2 = A \left( -\frac{B}{A} \right)^2 + B \left( -\frac{B}{A} \right) = 0, \]

which is again impossible. Thus, \( \Delta \) has only simple roots. Of course, this argument is valid only if \( r_0 \) is not at infinity. In this last case, we interchange the roles of \( r \) and \( s \) (i.e., we set \( U_1 = U_1(s) = X(1,s)/Z(1,s) \) and \( V_1 = V_1(s) = Y(1,s)/Z(1,s) \)) and we apply the same argument.

This completes the proof of Theorem 1.

Acknowledgements. This paper was written during a very enjoyable visit by the first author to University of West Hungary in Sopron; he wishes to express his thanks to that institution for the hospitality and support. The first author’s research was also partly supported by grants SEP-CONACYT 37259-E and 37260-E.
References


FLORIAN LUCA
INSTITUTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
C.P. 58180, MORELIA, MICHOACÁN
MÉXICO
E-mail: fluca@matmor.unam.mx

LÁSZLÓ SZALAY
INSTITUTE OF MATHEMATICS AND STATISTICS
UNIVERSITY OF WEST HUNGARY
9400, SOPRON, ERSZÉBET UTCA 9.
HUNGARY
E-mail: laszalay@ktk.nyme.hu

(Received November 15, 2004, revised May 2, 2005)