Certain curvature restrictions on a quasi Einstein manifold

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Abstract. Quasi Einstein manifold is a simple and natural generalization of Einstein manifold. We prove that a quasi-conformally flat quasi Einstein manifold is of quasi-constant curvature, and that a conformally flat pseudo symmetric manifold is a quasi Einstein manifold. Also conditions are found for a quasi Einstein manifold to be quasi conformally conservative.

Introduction

The notion of quasi Einstein manifold was introduced by M. C. CHAKI and R. K. MAITY [1]. A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is defined to be a quasi Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[
S(X, Y) = a g(X, Y) + b A(X) A(Y) \tag{1}
\]

where \(a, b\) are scalars of which \(b \neq 0\) and \(A\) is a non-zero 1-form such that

\[
g(X, U) = A(X) \tag{2}
\]

for all vector fields \(X; U\) being a unit vector field. In such a case \(a, b\) are called associated scalars. \(A\) is called the associated 1-form and \(U\) is called

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the generator of the manifold. An \( n \)-dimensional manifold of this kind is denoted by the symbol \((QE)_n\). If either the 1-form \( A \) or the associated scalar \( b \), or both of them are zero, then the manifold reduces to an Einstein manifold.

A Riemannian manifold of quasi-constant curvature was given by B. Y. Chen and K. Yano [2] as a conformally flat manifold with the curvature tensor \( R^{'} \) of type \((0, 4)\) which satisfies the condition

\[
R^{'}(X, Y, Z, W) = p[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + q[g(X, W)T(Y)T(Z) - g(X, Z)T(Y)T(W)] + g(Y, Z)T(X)T(W) - g(Y, W)T(X)T(Z)\]

where \( R^{'}(X, Y, Z, W) = g(R(X, Y)Z, W) \), \( R \) is the curvature tensor of type \((1, 3)\), \( p, q \) are scalar functions, \( T \) is a non-zero 1-form defined by

\[
g(X, \tilde{\rho}) = T(X),
\]

and \( \tilde{\rho} \) is a unit vector field.

It can be easily seen that if the curvature tensor \( R^{'} \) is of the form (3), then the manifold is conformally flat. On the other hand, Gh. Vranceanu [3] defined the notion of almost constant curvature. Later A. L. Mocanu [4] pointed out that the manifold introduced by Chen and Yano and the manifold introduced by Gh. Vranceanu are the same. If \( q = 0 \), then it reduces to a manifold of constant curvature.

The notion of quasi-conformal curvature tensor

\[
C^{*}(X, Y)Z = a_1 R(X, Y)Z + b_1 [S(Y, Z)X - S(X, Z)Y]
+ g(Y, Z)QX - g(X, Z)QY
- \frac{r}{n} \left[ \frac{a_1}{n-1} + 2b_1 \right] [g(Y, Z)X - g(X, Z)Y]
\]

was given by Yano and Sawaki [5]. Here \( a_1 \) and \( b_1 \) are constants, \( R \) is the Riemannian curvature tensor of type \((1, 3)\), \( S \) is the Ricci tensor of type \((0, 2)\), \( Q \) is the Ricci operator and \( r \) is the scalar curvature of the manifold. If \( a_1 = 1 \) and \( b_1 = -\frac{1}{n-2} \), then (5) takes the form

\[
C^{*}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y]
\]
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\[ + g(Y, Z) QX - g(X, Z) QY \]

\[- \frac{r}{(n - 1)(n - 2)} [g(Y, Z) X - g(X, Z) Y] = C(X, Y) Z, \]

where \( C \) is the conformal curvature tensor [6]. Thus the conformal curvature tensor \( C \) is a particular case of the tensor \( C^* \). For this reason \( C^* \) is called the quasi-conformal curvature tensor.

A manifold \((M^n, g) (n > 3)\) shall be called quasi-conformally flat or quasi-conformally conservative according as \( C^* = 0 \) or \( \text{div} \, C^* = 0 \). It is known [7] that a quasi-conformally flat space is either conformally flat or Einstein. Since an Einstein manifold need not be conformally flat, a quasi-conformally flat manifold need not be conformally flat.

A non-flat Riemannian manifold \((M^n, g) (n \geq 2)\) is said to be a pseudo symmetric manifold [8] if its curvature tensor \( R \) satisfies the condition


where \( B \) is a non-zero 1-form,

\[ g(X, \tilde{U}) = B(X) \quad \forall X \]

and \( \nabla \) denotes the operator of covariant differentiation with respect to the metric tensor \( g \). Such a manifold is denoted by \((PS)_n (n \geq 2)\). It may be mentioned that Chaki’s pseudo symmetric manifold is different from that of R. Deszcz [9].

It is known [10, p. 93] that a conformally flat Einstein manifold is of constant curvature. In the present paper we have generalized this result to a quasi-conformally flat quasi Einstein manifold and we prove that a quasi-conformally flat \((QE)_n (n > 3)\) is a manifold of quasi-constant curvature. In Section 2 we look for a sufficient condition in order that a \((QE)_n (n > 3)\) may be quasi-conformally conservative. Next we study conformally flat pseudo symmetric manifolds and prove that such a manifold is a quasi Einstein manifold. Finally we obtain a sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold.
1. Quasi-conformally flat quasi Einstein manifold

From (5) we get
\[
\begin{align*}
\varphi \kappa (X, Y, Z, W) &= a_1 \varphi \kappa (X, Y, Z, W) + b_1 [S(Y, Z)g(X, W) \\
&- S(X, Z)g(Y, W) + S(X, W)g(Y, Z) - S(Y, W)g(X, Z)] \\
&- \frac{r}{n} \left( \frac{a_1}{n - 1} + 2b_1 \right) [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\end{align*}
\]

where \( \varphi \kappa (X, Y, Z, W) = g(C^*(X, Y)Z, W) \) and \( \varphi \kappa (X, Y, Z, W) = g(R(X, Y)Z, W) \). If the manifold is quasi-conformally flat, then we have
\[
\begin{align*}
\varphi \kappa (X, Y, Z, W) &= \frac{b_1}{a_1} [S(X, Z)g(Y, W) - S(Y, Z)g(X, W) \\
&+ S(Y, W)g(X, Z) - S(X, W)g(Y, Z)] \\
&- \frac{r}{n} \frac{a_1}{a_1 - 1} + 2b_1 \right] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)].
\end{align*}
\]

Using (1) in (1.2) we have
\[
\begin{align*}
\varphi \kappa (X, Y, Z, W) &= - \left[ 2b_1 a + \frac{r}{n} \left( \frac{a_1}{n - 1} + 2b_1 \right) \right] [g(Y, Z)g(X, W) \\
&- g(X, Z)g(Y, W)] - b_1 [g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z) \\
&+ g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)],
\end{align*}
\]

which implies that the manifold is a manifold of quasi-constant curvature. Hence we can state that

**Theorem 1.** A quasi-conformally flat quasi Einstein manifold \((QE)_n\) \((n > 3)\) is a manifold of quasi-constant curvature.

2. \((QE)_n\) \((n > 3)\) with divergence free quasi-conformal curvature tensor

In this section we look for a sufficient condition in order that a \((QE)_n\) \((n > 3)\) may be quasi-conformally conservative. Quasi-conformal curvature tensor is said to be conservative [11] if divergence of \(C^*\) vanishes, i.e., \(\text{div} C^* = 0\).
In a \((QE)_n\) if both \(a\) and \(b\) are constant, then contracting (1) we have
\[ r = an + b, \text{ i.e. } r = \text{constant}, \]
where \(r\) is the scalar curvature, i.e., \(dr = 0\). Using this from (5) we obtain
\[
(\nabla W C^*)(X, Y, Z) = a_1(\nabla W R)(X, Y)Z + b_1[(\nabla W S)(Y, Z)X
- (\nabla W S)(X, Z)Y + g(Y, Z)(\nabla W Q)X - g(X, Z)(\nabla W Q)Y]. \tag{2.1}
\]

We know that \((\text{div} R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\) [10], and from (1) we get
\[
(\nabla_X S)(Y, Z) = b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)],
\]
since both \(a\) and \(b\) are constant. Hence contracting (2.1) we obtain
\[
(\text{div} C^*)(X, Y, Z) = 2b(a_1 + b_1)[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)
- (\nabla_Y A)(X)A(Z) - (\nabla_Y A)(Z)A(X)] + bb_1[(\nabla_U A)(X)
+ A(X) \text{ div } U]g(Y, Z) - 2bb_1g(X, Z)B(U)A(Y). \tag{2.2}
\]

Imposing the condition that the generator \(U\) of the manifold is a recurrent vector field [12] with associated 1-form \(A\) not being the 1-form of recurrence, gives \(\nabla_X U = B(X)U\), where \(B\) is the 1-form of recurrence. Hence
\[
g(\nabla_X U, Y) = g(B(X)U, Y), \text{ that is,}

(\nabla_X A)(Y) = B(X)A(Y). \tag{2.3}
\]

In view of (2.3), (2.2) is expressed as follows
\[
(\text{div} C^*)(X, Y, Z) = 2b(a_1 + b_1)[B(X)A(Y)A(Z) - B(Z)A(X)A(Y)]
+ 2bb_1B(U)A(X)g(Y, Z) - 2bb_1g(X, Z)B(U)A(Y). \tag{2.4}
\]

Since \((\nabla_X A)(U) = 0\), it follows from (2.3) that \(B(X) = 0\). Hence from (2.4) it follows that \((\text{div} C^*)(X, Y, Z) = 0\). Thus we can state the following:

**Theorem 2.** If in a \((QE)_n\) \((n > 3)\) the associated scalars are constants and the generator \(U\) of the manifold is a recurrent vector field with the associated 1-form \(A\) not being the 1-form of recurrence, then the manifold is quasi-conformally conservative.
3. Conformally flat pseudo symmetric manifolds

It is known [8] that in a conformally flat \((PS)_n\) \((n \geq 3)\)

\[(n - 1)B(X)S(Y, Z) - (n - 1)B(Y)S(X, Z) - rB(X)g(Y, Z)
+ rB(Y)g(X, Z) + D(X)g(Y, Z) - D(Y)g(X, Z) = 0, \quad (3.1)\]

where \(D\) is a 1-form defined by

\[D(X) = B(QX), \quad (3.2)\]

\(Q\) denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor \(S\), i.e. \(g(QX, Y) = S(X, Y)\) for every vector fields \(X, Y\). Putting \(Z = \tilde{U}\) in (3.1), where \(g(X, \tilde{U}) = B(X)\) we get

\[B(X)D(Y) - B(Y)D(X) = 0. \quad (3.3)\]

Hence

\[D(X) = tB(X), \quad (3.4)\]

where \(t\) is a scalar. Using (3.4), it follows from (3.1) that

\[S(Y, Z) = \frac{r - t}{n - 1}g(Y, Z) + \frac{nt - r}{(n - 1)B(U)}B(Y)B(Z) \quad (3.5)\]

which implies that the manifold is a quasi Einstein manifold. Thus we state

**Theorem 3.** A conformally flat pseudo symmetric manifold \((PS)_n\) \((n \geq 3)\) is a quasi Einstein manifold.

4. Sufficient condition for a pseudo symmetric manifold to be a quasi Einstein manifold

Now contracting (7) we get

\[(\nabla_X S)(Y, Z) = 2B(X)S(Y, Z) + B(Y)S(X, Z) + B(Z)S(Y, X)
+ B(R(X, Y)Z) + B(R(X, Z)Y). \quad (4.1)\]
In a Riemannian manifold, a vector field $\rho$ defined by $g(X, \rho) = A(X)$ for any vector field $X$ is said to be a concircular vector field [12] if

$$(\nabla_X A)(Y) = \alpha g(X, Y) + \omega(X)A(Y),$$

(4.2)

where $\alpha$ is a non-zero scalar and $\omega$ is a closed 1-form. If $\rho$ is a unit vector, then the equation (4.2) can be written as

$$(\nabla_X A)(Y) = \alpha [g(X, Y) - A(X)A(Y)].$$

(4.3)

We suppose that a $(PS)_n$ admits a unit concircular vector field defined by (4.3), where $\alpha$ is a non-zero constant. Applying the Ricci idetity to (4.3) we obtain

$$A(R(X,Y)Z) = -\alpha^2 [g(X,Z)A(Y) - g(Y,Z)A(X)].$$

(4.4)

Putting $Y = Z = e_i$ in (4.4), and taking summation over $i$, $1 \leq i \leq n$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$A(QX) = (n-1)\alpha^2 A(X),$$

where $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$, which implies

$$S(X, \rho) = (n-1)\alpha^2 A(X).$$

(4.5)

From (4.5) we have

$$(\nabla_Y S)(X, \rho) = (n-1)\alpha^3 g(X,Y) - \alpha S(X, Y).$$

(4.6)

Using (4.4) we obtain

$$g(R(X,Y)Z, \rho) = -\alpha^2 g(X,Z)A(Y) - g(Y,Z)A(X)]$$

or,

$$g(R(Z,\rho)X,Y) = -\alpha^2 [g(X,Z)g(Y,\rho) - g(Z,Y)A(X)]$$

or,

$$R(Z,\rho)X = -\alpha^2 [g(X,Z)\rho - A(X)Z],$$

which implies

$$B(R(Z,\rho)X) = -\alpha^2 [g(X,Z)B(\rho) - A(X)B(Z)]$$

i.e.,

$$B(R(X,\rho)Y) = -\alpha^2 [g(X,Y)B(\rho) - A(Y)B(X)].$$

(4.7)
Similarly we have

\[ B(R(X,Y)\rho) = -\alpha^2[A(Y)B(X) - A(X)B(Y)]. \] (4.8)

In (4.1) putting \( Z = \rho \) and using (4.5),(4.6), (4.7) and (4.8) we have

\[ -(\alpha + B(\rho))S(X,Y) = -[\alpha^2B(\rho) + (n-1)\alpha^3]g(X,Y) \]
\[ + 2(n-1)\alpha^2B(X)A(Y) + n\alpha^2B(Y)A(X). \] (4.9)

Putting \( Y = \rho \) in (4.9) and using (4.5) we have

\[ B(\rho)A(X) + (n-1)A(X) = 0 \quad \forall X \]
\[ \text{i.e.} \quad B(X) = -\frac{B(\rho)}{n-1}A(X). \] (4.10)

Let us impose the condition

\[ \alpha + B(\rho) \neq 0. \] (4.11)

Putting (4.10) in (4.9) we obtain

\[ S(X,Y) = \frac{\alpha^2[B(\rho) + (n-1)\alpha]}{\alpha + B(\rho)}g(X,Y) + \frac{(3n-2)B(\alpha)}{(\alpha + B(\rho))(n-1)}A(X)A(Y) \]
\[ \text{i.e.} \quad S(X,Y) = ag(X,Y) + bA(X)A(Y), \] (4.12)

where \( a = \frac{\alpha^2[B(\rho) + (n-1)\alpha]}{\alpha + B(\rho)} \) and \( b = \frac{(3n-2)B(\alpha)}{(\alpha + B(\rho))(n-1)} \).

Thus we can state

**Theorem 4.** *If a pseudo symmetric manifold admits a unit concircular vector field whose associated scalar is a non-zero constant and satisfying the condition (4.11), then the manifold reduces to a quasi Einstein manifold.*

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References