On the diophantine equation $x^2 + 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta} = y^n$

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Abstract. Let $S = \{p_1, \ldots, p_s\}$ be a set of distinct primes and denote by $S$ the set of non-zero integers composed only of primes from $S$. Further, denote by $Q$ the product of the primes from $S$. Let $f \in \mathbb{Z}[X]$ be a monic quadratic polynomial with negative discriminant $D_f$ contained in $S$. Consider equation $f(x) = y^n$ (2) in integer unknowns $x$, $y$, $n$ with $n \geq 3$ prime and $y > 1$. It follows from a general result of [13] that in (2) $n$ can be bounded from above by an effectively computable constant depending only on $Q$. This bound is, however, large and is not given explicitly. Using some results of Bugeaud and Shorey [8] we derive, apart from certain exceptions, a good and completely explicit upper bound for $n$ in (2) (see Theorems 1 and 2). Further, combining our Theorem 2 with some deep results of Cohn [12] and de Weger [25] we give all non-exceptional (see Section 1) solutions of equation $x^2 + 2^{\alpha}3^{\beta}5^{\gamma}7^{\delta} = y^n$ (6), where $x$, $y$, $n$, $\alpha$, $\beta$, $\gamma$, $\delta$ are unknown non-negative integers with $x \geq 1$, $\gcd(x, y) = 1$ and $n \geq 3$ (cf. Theorem 3). When, in (6), $\alpha \geq 1$ is also assumed then our Theorem 3 is a generalization of a result of Luca [19]. In this case all the solutions of equation (6) are listed.

1. Introduction

There are many results concerning the generalized Ramanujan–Nagell equation

$$x^2 + D = \mu y^n, \quad (1)$$

where $D > 0$ is a given integer, $\mu \in \{1, 4\}$ and $x$, $y$, $n$ are positive integer unknowns with $n \geq 3$ and $\gcd(x, y) = 1$. First consider the case $\mu = 1$. Then the first result was due to V. A. Lebesque [15] who proved that there are no

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solutions for $D = 1$. Ljunggren [16] solved (1) for $D = 2$, and Nagell [22], [23] solved it for $D = 3, 4$ and 5. In his elegant paper [11], Cohn gave a fine summary of work on equation (1). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (1) was solved by Mignotte and de Weger [20]. By using the theory of Galois representations and modular forms Bennett and Skinner [5] solved (1) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner’s method and with several additional ideas, Bugeaud, Mignotte and Siksek [7] gave all the solutions of (1) for the remaining 19 values of $D \leq 100$. Bugeaud and Shorey [8] used a beautiful result of Bilu, Hanrot and Voutier [6] to solve completely several equations of type (1) both for $\mu = 1$ and for $\mu = 4$ when $D$ is an odd positive square-free integer, $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-D})$ and $D \not\equiv 7 \pmod{8}$ if $\mu = 1$ (see Corollaries 3, 5 and 7 of [8]).

Let $S = \{p_1, \ldots, p_s\}$ denote a set of distinct primes and $\mathbf{S}$ the set of non-zero integers composed only of primes from $S$. Denote by $P$ and $Q$ the greatest and the product of the primes of $S$, respectively. In recent years, equation (1) has been considered also in the more general case when $D$ is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [24] that in (1) $n$ can be bounded from above by an effectively computable constant depending only on $f', P$ and $s$. In [13] an effective upper bound was derived for $n$ which depends only on $Q$. By using the powerful method of Bilu, Hanrot and Voutier [6] equation (1) can be completely solved for $\mu = 1$ and some special sets of primes $S$. Namely, if in (1) $D \in \mathbf{S}$ with $S = \{2\}$ then all solutions of (1) were given by Cohn [10] and Arif and Muriefa [1] and [3]. For $S = \{3\}$, equation (1) was solved completely by Arif and Muriefa [2] and Luca [18]. When $S = \{q\}$, where $q \geq 5$ is an odd prime with $q \not\equiv 7 \pmod{8}$, Arif and Muriefa [4] determined all solutions of the equation $x^2 + q^{2k+1} = y^n$, where $\text{gcd}(n, 3h_0) = 1$ and $n \geq 3$. Here $h_0$ denotes the class number of the field $\mathbb{Q}(\sqrt{-q})$. For $S = \{2, 3\}$, Luca [19] gave the complete solution of (1).

To formulate our results we introduce some notation. Let $f(x) = x^2 + Ax + B$ where $A, B \in \mathbb{Z}$ and denote by $D_f$ the discriminant of $f$. Set

$$\Delta = \begin{cases} -\frac{D_f}{4} & \text{if } D_f \text{ is even}, \\ -D_f & \text{if } D_f \text{ is odd}. \end{cases}$$

Suppose that $\Delta \in \mathbf{S}$ and $\Delta > 0$. Let $c$ and $d$ be non-zero integers such that $\Delta = dc^2$ and $d > 0$ denotes the square-free part of $\Delta$. Further, for any $k \in \mathbb{Z}$ and rational prime $p$ denote by $\text{ord}_p(k)$ the greatest power of $p$ to which $p$ divides $k$. 

\begin{align*}
S & = \{p_1, \ldots, p_s\} \\
\Delta & = \begin{cases} -\frac{D_f}{4} & \text{if } D_f \text{ is even}, \\ -D_f & \text{if } D_f \text{ is odd}. \end{cases}
\end{align*}
On the diophantine equation $x^2 + 2^a3^b5^c7^d = y^n$ \(\text{151}\)

Consider the equation

$$f(x) = y^n$$  \(\text{(2)}\)

in integer unknowns $x, y, n$ with $n \geq 3$ prime and $y > 1$. We say that a solution $(x, y, n)$ of (2) is exceptional if

$$\text{ord}_2(D_f) = 2, \ y \text{ is even and } d \equiv 7 \pmod{8}.$$  

Write $h$ for the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Further, denote by $h(-4\Delta)$ the number of classes of positive binary quadratic forms with discriminant $-4\Delta$ (for the definition see Section 2).

**Theorem 1.** If $(x, y, n)$ is a non-exceptional solution of (2) with $x \neq -\frac{A}{2}$ and $\gcd(y, \Delta) = 1$ then, except for the infinite families of equations

$$x^2 + Ax + B = y^n$$

where $(A, B, x, y, \Delta, n) \in \{(A, (A^2 + 7)/4, (11 - A)/2, 2, 7, 5), (A, (A^2 + 7)/4, (181 - A)/2, 2, 7, 13), (A, (A^2 + 11)/4, (31 - A)/2, 3, 11, 5), (A, (A^2 + 19)/4, (559 - A)/2, 5, 19, 7)\}$, where $A$ is odd and $(A, B, x, y, \Delta, n) \in \{(A, (A^2 + 76)/4, (44868 - A)/2, 55, 19, 5), (A, (A^2 + 1364)/4, (5519292 - A)/2, 377, 341, 5)\}$, where $A$ is even, we have

$$n = 3 \quad \text{or} \quad n \mid h(-4\Delta).$$

Further, in the latter case

$$n \leq \max\{3, P\} \quad \text{if} \quad n \nmid h$$

and

$$n < \frac{4}{\pi} \sqrt{Q} \log(2e\sqrt{Q}) \quad \text{if} \quad n \mid h.$$

We note that the assumption $x \neq -\frac{A}{2}$ is necessary. Otherwise using (2) and supposing that $D_f$ is even we get $y^n = \Delta$, whence by $\Delta \in S$ we see that $n$ cannot be bounded.

Equation (2) can be reduced to an equation of the type

$$X^2 + \Delta = \mu Y^n,$$  \(\text{(3)}\)

where $\mu \in \{1, 4\}$,

$$\gcd(X, Y) = \gcd(Y, \Delta) = 1$$  \(\text{(4)}\)
We shall deduce Theorem 1 from the following Theorem 2. We say that a solution $(X, Y, n)$ of (3) is \textit{exceptional} if

$$
\mu = 1, \ \text{ord}_2(D_f) = 2, \ Y \text{ is even and } d \equiv 7 \pmod{8}.
$$

\textbf{Theorem 2.} If $(X, Y, n)$ is a non-exceptional solution of equation (3) satisfying (4) and (5) then, except for $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 5), (4, 2, 7, 13), (4, 3, 11, 5), (4, 5, 19, 7), (1, 55, 19, 5), (1, 377, 341, 5)\}$, we have

$$
n = 3 \text{ or } n \mid h(-4\Delta).
$$

Further, in the latter case

$$
n \leq \max\{3, P\} \text{ if } n \nmid h
$$

and

$$
n < \frac{4}{\pi} \sqrt{Q} \log(2e\sqrt{Q}) \text{ if } n \mid h.
$$

This should be compared with Corollaries 5 and 7 of Bugeaud and Shorey \cite{8}, where equations of type (3) were considered with square-free $\Delta > 0$. In Corollary 5 they showed that the equation $x^2 + 4\Delta = y^n$ has no solution with $n \geq 5$. Here $\Delta$ is square-free and $n$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Further, in Corollary 7 of \cite{8} the authors considered the equation (3), where $\mu \in \{1, 4\}$, $\Delta$ is an odd positive square-free integer and $n \geq 3$ is an odd prime not dividing the class number of the field $\mathbb{Q}(\sqrt{-\Delta})$. Under these assumptions they solved completely equation (3) in the case when

$$
\mu = 1, \ \Delta \equiv 1 \pmod{4}, \ n \geq 3 \text{ or } \mu = 4, \ \Delta \equiv 7 \pmod{8}, \ n \geq 3 \text{ or } \mu = 4, \ \Delta \equiv 3 \pmod{8}, \ n \geq 5.
$$

In contrast with \cite{8}, in our Theorem 2 it is not assumed that $\Delta$ is square-free. Using the approach of \cite{8} we give completely explicit upper bounds for $n$ in (3) depending only on $P$ and $Q$. This allows us to solve completely equation (3) in the case when $S = \{2, 3, 5, 7\}$ and $\mu = 1$. Combining Theorem 2 with some results of Cohn \cite{12} and de Weger \cite{25}, we give all non-exceptional solutions of the equation

$$
x^2 + 2^\alpha3^\beta5^\gamma7^\delta = y^n
$$

(6)
On the diophantine equation $x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n$

where $x, y, n, \alpha, \beta, \gamma, \delta$ are unknown non-negative integers with $x \geq 1$, $y \geq 2$, $\gcd(x, y) = 1$ and $n \geq 3$. We recall that in this special case a solution is called \textit{exceptional} if $\alpha = 0$, $y$ is even and $3^\beta 5^\gamma 7^\delta$ is either of the form $7c^2$ or of the form $15c^2$. We note that if our equation (6) is of the form $x^2 + 7c^2 = y^n$ or $x^2 + 15c^2 = y^n$ and $(x, y, n)$ is an exceptional solution of (6), then we cannot use the parametrization for $(x, y)$ provided by Lemma 2 (see e.g. [11]). Hence we consider only the non-exceptional solutions of (6). We note that using another approach Bugeaud, Mignotte and Siksek [7] solved the equations $x^2 + 7c^2 = y^n$ and $x^2 + 15c^2 = y^n$ when $1 \leq 7c^2 < 15c^2 \leq 100$.

\textbf{Theorem 3.} All non-exceptional solutions of equation (6) are listed in the table occurring in Section 4.

If in (6) $\alpha \geq 1$ is assumed then, by $\gcd(x, y) = 1$, $y$ is odd. Hence the solutions $(x, y, n)$ of (6) are always non-exceptional. Thus in this case we can list all the solutions of equation (6).

\textbf{Corollary.} All solutions of (6) with $\alpha \geq 1$ are listed in the table in Section 4.

We note that the solutions of equation (6) with $\alpha \geq 1$ are those which are not marked with an asterisk in the table. Further, in this case our Theorem 3 is a generalization of a result of Luca [19] mentioned above.

\section{2. Auxiliary results}

We keep the notations of the preceding section. For a non-zero integer $m$ denote by $\omega(m)$ the number of distinct prime factors of $m$. By definition, for $a, b, c \in \mathbb{Z}$, the discriminant of the binary quadratic form $aX^2 + 2bXY + cY^2$ is $4b^2 - 4ac$, thus $-4\Delta$ is the discriminant of the form $X^2 + \Delta Y^2$. We say that a binary quadratic form is positive if $a > 0$. The set of positive binary quadratic forms of discriminant $-4\Delta$ is partitioned into a finite number of equivalence classes which we denote by $h(-4\Delta)$.

The next lemma is a special case of Lemma 1 of [8] (see also LE [14]).

\textbf{Lemma 1.} Consider equation

$$X_1^2 + \Delta Y_1^2 = \mu Y Z_1$$

(7)

in integer unknowns $X_1, Y_1, Z_1$ with $Z_1 > 0$ and $\gcd(X_1, Y_1) = 1$. Then the solutions of the above equation can be put into at most $2^\omega(Y) - 1$ classes. Further, in each class there is a unique solution $(X_1, Y_1, Z_1)$ such that $X_1 > 0, Y_1 > 0$ and
$Z_1$ is minimal among the solutions of the class. This minimal solution satisfies $Z_1 \mid h(-4\Delta)$, where $h(-4\Delta)$ is the number of classes of positive binary forms of discriminant $-4\Delta$.

**Proof.** See [8].

**Lemma 2.** Suppose that equation (3) has a solution under the assumptions (4) and (5) with $\mu = 1$. Denote by $d > 0$ the square-free part of $\Delta = dc^2$. If $d \not\equiv 7 \pmod{8}$ and $Y$ is odd then one of the following cases holds:

(a) there exist $a_1, b_1 \in \mathbb{Z}$ with $b_1 \mid c$, $b_1 \neq \pm c$ such that $Y = a_1^2 + b_1^2d$ and $\pm X + c\sqrt{-d} = (a_1 + b_1\sqrt{-d})^n$;

(b) $n \mid h$, where $h$ denotes the class number of the field $\mathbb{Q}(\sqrt{-d})$;

(c) $d \equiv 3 \pmod{8}$, $n = 3$ and there exist odd integers $A_1, B_1$ with $B_1 \mid c$ such that $Y = \frac{1}{4}(A_1^2 + B_1^2d)$, $\pm X + c\sqrt{-d} = \frac{1}{8}(A_1 + B_1\sqrt{-d})^3$;

(d) $(n, \Delta, X) = (3, 3u^2 \pm 8, u^3 \pm 3u)$ or $(n, \Delta, X) = (3, 3u^2 \pm 1, 8u^3 \pm 3u)$, where $u \in \mathbb{Z}$;

(e) $(n, \Delta, X) = (5, 19, 22434)$ or $(n, \Delta, X) = (5, 341, 2759646)$.

**Proof.** If $d \equiv 7 \pmod{8}$ then the lemma is a reformulation of a theorem of Cohn [12]. So, it remains the case when in (3) $d \equiv 7 \pmod{8}$ and $Y$ is odd. In this case we may apply a result of Ljunggren [17] (pp. 593–594) to conclude that if in equation (3) $n \nmid h$ then there exist $a_1, b_1 \in \mathbb{Z}$ such that

$$\pm X + c\sqrt{-d} = \left(\frac{a_1 + b_1\sqrt{-d}}{2}\right)^n, \quad a_1 \equiv b_1 \pmod{2}. \quad (8)$$

If in (8) $a_1$ and $b_1$ are both odd then since $d \equiv 7 \pmod{8}$, we get

$$a_1^2 + db_1^2 \equiv 0 \pmod{8},$$

whence, by

$$Y = \frac{a_1^2 + db_1^2}{4},$$

it follows that $Y$ is even, a contradiction. So $a_1$ and $b_1$ are both even and the lemma is proved.

The next lemma provides an upper bound for the class number of an imaginary quadratic field.

**Lemma 3.** Let $D > 0$ be a square-free integer, and denote by $h$ the class number of the field $K = \mathbb{Q}(\sqrt{-D})$. Then

$$h < \frac{4}{\pi}\sqrt{D}(\log 2\sqrt{D}).$$
On the diophantine equation \( x^2 + 2^a 3^b 5^c 7^d = y^n \)

PROOF. Denote by \( h(-4D) \) the class number of the unique quadratic order in \( K \) with discriminant \(-4D\). Then \( h(-4D) \) is the number of classes of positive quadratic forms of discriminant \(-4D\) (see e.g. COHEN [9], Definition 5.2.7). Further, we have

\[
h(-4D) < \frac{4}{\pi} \sqrt{D} \log 2e \sqrt{D}
\]

(cf. e.g. Proposition 1 of [8]). Since \( h | h(-4D) \) (see e.g. [21]), the assertion follows. \( \square \)

Lemma 4. Denote by \( h(-4\Delta) \) the number of classes of positive binary forms of discriminant \(-4\Delta\). Then, for \( d \equiv 3 \pmod{4} \),

\[
h(-4\Delta) = h(-4c^2d) = h2c \times \prod_{p|2c} \left( 1 - \frac{-d/p}{p} \right) \frac{1}{u},
\]

where \( u = 3 \), if \( d = 3 \) and \( u = 1 \) otherwise; for \( d \equiv 1, 2 \pmod{4} \),

\[
h(-4\Delta) = h(-4c^2d) = hc \times \prod_{p|c} \left( 1 - \frac{-4d/p}{p} \right) \frac{1}{u},
\]

where \( u = 2 \), if \( d = 1 \) and \( u = 1 \) otherwise. Here \( (\cdot)_p \) denotes the Kronecker symbol.

PROOF. See MOLLIN [21]. \( \square \)

The next lemma is a deep result of DE WEGE [25]. It will be utilized in the proof of Theorem 3.

Lemma 5. Let \( S = \{2, 3, 5, 7\} \). Consider the equation \( U + V = W^2 \) in unknowns \( U, V, W \), where \( U, V \) or \(-V \in S \cap \mathbb{Z}_{>0}, W \in \mathbb{Z}_{>0} \). Suppose that \( U \geq V \) and that \( \gcd(U, V) \) is square-free. Then the above equation has exactly 388 solutions which are given explicitly in [25].

PROOF. This is Theorem 7.2 of [25]. \( \square \)

3. Proofs of theorems

PROOF OF THEOREM 2. Consider equation (3) satisfying (4) and (5). We follow the approach of [8] and we introduce two infinite sets. Denote by \( F_k \) the Fibonacci sequence defined by \( F_0 = 0, F_1 = 1 \) and satisfying \( F_k = F_{k-1} + F_{k-2} \)

for all $k \geq 2$ and by $L_k$ the Lucas sequence defined by by $L_0 = 2$, $L_1 = 1$ and satisfying $L_k = L_{k-1} + L_{k-2}$ for all $k \geq 2$. Then

$$F := \{(F_{k+\varepsilon}, L_{k-\varepsilon}, F_k) \mid k \geq 2, \varepsilon \in \{\pm 1\}\},$$

and

$$H := \{(1, \Delta, Y) \mid \text{there exist } r, s \in \mathbb{Z}_{>0} \text{ such that}$$

$$s^2 + \Delta = \mu Y^r \text{ and } 3s^2 - \Delta = \mp \mu\}.$$ 

If $(X, Y, n)$ is a non-exceptional solution of (3) then it corresponds to a solution $(X_1, Y_1, Z_1) = (X, 1, n)$ of (7). Since by Lemma 1 the solutions of (7) can be put into at most $2^{\omega(Y) - 1}$ classes we have to distinguish two cases. Firstly, if $(X, 1, n)$ is the minimal solution in the class then by Lemma 1 we have $n \mid h(-4\Delta)$. Secondly, if $(X, 1, n)$ is not the minimal solution then there exist at least two solutions of (7) in the class. By using Theorem 2 of [8] and noting that $n$ is an odd prime we see that in this case either $(\mu, Y, \Delta, n) \in \{(4, 2, 7, 3), (4, 7, 3, 3), (4, 2, 7, 5), (4, 2, 7, 13), (4, 3, 11, 5), (4, 5, 19, 7), (1, 55, 19, 5), (1, 377, 341, 5)\}$ or we have

$$n \in \{1, 5\} \quad \text{and} \quad (1, \Delta, Y) \in F$$

or

$$n \in \{r, 3r\} \quad \text{and} \quad (1, \Delta, Y) \in H, \quad \text{with } r \in \mathbb{Z}_{>0}.$$ 

Since $n$ is an odd prime we obtain that

$$n = 5 \quad \text{and} \quad (1, \Delta, Y) \in F \quad \text{or} \quad n = 3 \quad \text{and} \quad (1, \Delta, Y) \in H.$$ 

If $n = 5$ and $(1, \Delta, Y) \in F$ then by the definition of the set $F$ we get

$$F_{k-2} = 1, \quad L_{k+1} = \Delta, \quad F_k = Y$$

or

$$F_{k+2} = 1, \quad L_{k-1} = \Delta, \quad F_k = Y.$$ 

We see that $F_{k+2} = 1$ cannot hold since in this case it follows that $k + 2 \in \{1, 2\}$ and hence $k = 0$. Thus $F_0 = Y = 0$ follows which contradicts the assumption $Y > 1$. If $F_{k-2} = 1$ we get $k - 2 \in \{1, 2\}$, whence $k \in \{3, 4\}$ which implies by $F_k = Y$ that

$$(Y, \Delta) \in \{(2, 7), (3, 11)\}.$$ 

Hence using (3) we get

$$X^2 + 7 = \mu \cdot 2^i \quad \text{and} \quad X^2 + 11 = \mu \cdot 3^i.$$
On the diophantine equation \( x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n \)

We see that if \( \mu = 4 \) the above equations have solutions which are already listed (i.e. \( (\mu, Y, \Delta, n) \in \{(4, 2, 7, 5), (4, 3, 11, 5)\} \)). If \( \mu = 1 \) then \( X^2 + 11 = 3^5 \) is impossible, while the equation \( X^2 + 7 = 2^5 \) leads to an exceptional solution of (3), which contradicts the assumption that \( (X, Y, n) \) is non-exceptional. Hence we obtain that

\[
 n \mid h(-4\Delta) \quad \text{or} \quad n = 3,
\]

according as \( (X, 1, n) \) is the minimal solution in the class or not. We recall that \( n \) is an odd prime and \( \Delta = dc^2 \in S \). Thus if \( n \mid h(-4\Delta) \) but \( n \nmid h \) then by Lemma 4 we obtain that \( n \) cannot exceed the greatest prime lying in \( S = \{p_1, \ldots, p_s\} \). Hence

\[
 n \leq \max\{3, P\}.
\]

If \( n \mid h \) then since \( d \) is the square-free part of \( \Delta \) we have

\[
 d \leq Q = p_1 \cdots p_s.
\]

Hence using Lemma 3 the assertion follows.

Proof of Theorem 1. Put \( f(x) = x^2 + Ax + B \), where \( A, B \in \mathbb{Z} \). One can easily see that equation (2) leads to the equation of type (3)

\[
 X^2 + \Delta = \mu Y^n,
\]

where

\[
 (X, \Delta, \mu, Y) = \begin{cases} 
 (x + \frac{A}{2}, \frac{-D_f}{4}, 1, y) & \text{if } D_f \text{ is even,} \\
 (2x + A, -D_f, 4, y) & \text{if } D_f \text{ is odd,}
\end{cases}
\]

According to the definition of \( \Delta \) and the assumption \( x \neq \frac{-A}{2} \), we may suppose that in equation (9)

\[
 \Delta \in S, \; \Delta > 0, \; X \geq 1, \; n \geq 3 \text{ prime.}
\]

Since, by assumption, \( \gcd(Y, \Delta) = 1 \) we can apply Theorem 2 to equation (9) and we get Theorem 1.

Proof of Theorem 3. There is no loss of generality by supposing that in (6) \( n = 4 \) or \( n \) is an odd prime. Keeping the notations of the preceding sections we have \( dc^2 = \Delta = 2^a 3^b 5^c 7^d \), where \( d \in H \) with \( H = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\} \). Assume that \( n \) is an odd prime. Since \( (x, y, n) \) is a non-exceptional solution of (6) and for every \( d \in H \) the class number \( h \) of the imaginary
quadratic field $\mathbb{Q}(\sqrt{-d})$ is 1 or a power of 2 by Theorem 2 we get $n \leq 7$. Hence (6) can have a solution only if $n \in \{3, 4, 5, 7\}$.

The case $n \in \{5, 7\}$. We recall that $dc^2 = \Delta = 2^a 3^\beta 5^\gamma 7^\delta$, where $d \in \mathcal{H}$ with $\mathcal{H} = \{1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105, 210\}$.

Consider equation (6) with $n = 5$. Assume first that $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 6$, $\delta \geq 0$.

By Lemma 2 we get
\[ \pm x + c\sqrt{-d} = (a + b\sqrt{-d})^5 \tag{11} \]
where $a, b \in \mathbb{Z}$, $b \mid c$, $y = a^2 + db^2$. Hence, by comparing the imaginary parts of (11) we obtain
\[ 5a^4b - 10a^2b^3d + b^5d^2 = c. \tag{12} \]
Since $\gamma \geq 6$ we have $\text{ord}_5(c) \geq 3$.

**Case 1.** $1 \leq \text{ord}_5(b) \leq \text{ord}_5(c) - 2$

Since $b \mid c$ and $\text{ord}_5(b) \leq \text{ord}_5(c) - 2$, we see that $\frac{c}{5b} \in \mathbb{Z}$ and $\text{ord}_5\left(\frac{c}{5b}\right) \geq 1$. Using (12) we get
\[ a^4 - 2a^2b^2d + \frac{b^4}{5}d^2 = \frac{c}{5b}. \tag{13} \]
Since $\text{ord}_5\left(\frac{c}{5b}\right) \geq 1$ and $\text{ord}_5(b) \geq 1$, we obtain by (13) that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \geq 6$, we obtain that $5 \mid y$ which is impossible since $x$ and $y$ are relatively prime.

**Case 2.** $\text{ord}_5(b) = \text{ord}_5(c)$.

In this case we have $\text{ord}_5(b) \geq 3$ since $\text{ord}_5(c) \geq 3$. By (12) it follows that
\[ 5a^4 - 10a^2b^2d + b^5d^2 = \frac{c}{b}. \tag{14} \]
Hence the left-hand side of (14) is divisible by 5 but the right-hand side is not, a contradiction.

**Case 3.** $\text{ord}_5(b) = 0$

By $\text{ord}_5(b) = 0$ we have $\text{ord}_5(c) = \text{ord}_5\left(\frac{c}{5b}\right) \geq 3$. Thus using (14) we see that $5 \mid b^2d^2$ follows, whence by $\text{ord}_5(b) = 0$ we get $5 \mid d$. Hence from (14) we infer that
\[ a^4 - 2a^2b^2d + \frac{b^4}{5}d^2 = \frac{c}{5b}. \tag{15} \]
Clearly $\frac{a^4}{5}$ and $\frac{c}{5b}$ are integers and $\text{ord}_5\left(\frac{c}{5b}\right) \geq 2$. Thus by (15) it follows that $5 \mid a^4$, whence by $a \mid x$ we get $5 \mid x$. Thus using equation (6) and the assumption $\gamma \geq 6$, we obtain that $5 \mid y$ which contradicts $\gcd(x, y) = 1$.

**Case 4.** $\text{ord}_5(b) = \text{ord}_5(c) - 1$
On the diophantine equation \( x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n \)

By assumption we see that in (15) \( \frac{b_4d_2}{5} \) and \( \frac{c}{55} \) are integers and \( \text{ord}_5\left(\frac{c}{55}\right) = 0 \). If now \( \text{ord}_2\left(\frac{c}{55}\right) \geq 1 \) then clearly \( \alpha \geq 1 \) and by (15) we get

\[ a^4 \equiv \frac{b_4d^2}{5} \pmod{2}. \] (16)

We may suppose that \( a \) is odd since otherwise we obtain by (6), \( a \mid x \) and \( \alpha \geq 1 \) that \( 2 \leq \gcd(x, y) \) contradicting the assumption \( \gcd(x, y) = 1 \). Thus by (16) we see that \( b \) and \( d \) are odd integers, whence it follows that \( 2 \mid y = a^2 + db^2 \). Using equation (6) and \( \alpha \geq 1 \) we get \( 2 \mid x \) which cannot hold since \( x \) and \( y \) are relatively prime integers.

Suppose now that \( \text{ord}_2\left(\frac{c}{55}\right) = 0 \) and \( \text{ord}_3\left(\frac{c}{55}\right) \geq 1 \). Then obviously \( \beta \geq 1 \). We may assume that \( 3 \nmid d \) and \( 3 \nmid b \) since otherwise we get by (15) that \( 3 \mid a^4 \) whence \( 3 \mid x \). Thus by (6) and \( \beta \geq 1 \) we see that \( 3 \mid y \) which leads to a contradiction.

By \( \Delta = dc^2, y = a^2 + db^2 \) and (6) we have

\[ x^2 + dc^2 = (a^2 + db^2)^5 \] (17)

Clearly \( 3 \nmid x \) since otherwise we obtain a contradiction by (6), \( \beta \geq 1 \) and \( \gcd(x, y) = 1 \). Thus by \( 3 \mid x \) and \( \beta \geq 1 \) we have

\[ x^2 + dc^2 \equiv 1 \pmod{3}. \] (18)

If \( 3 \mid a \), then by \( a \mid x, \beta \geq 1 \) and (6) we obtain a contradiction. Hence

\[ (a^2 + db^2)^5 \equiv (1 + d)^5 \pmod{3}. \] (19)

Since for every \( d \in H \) with \( 3 \nmid d \) we have \( 1 + d \equiv -1, 0 \pmod{3} \) we see by (19) that

\[ (a^2 + db^2)^5 \equiv -1, 0 \pmod{3}. \] (20)

Combining (17), (18) and (20) we get a contradiction.

If \( \text{ord}_2\left(\frac{c}{55}\right) = 0 \) and \( \text{ord}_3\left(\frac{c}{55}\right) = 0 \) then we have by (15) that

\[ a^4 - 2a^2b^2d + \frac{b^4d^2}{5} = \pm 7^{\delta'}, \] (21)

for some non-negative integer \( \delta' \). If \( \delta' \geq 1 \) and \( 7 \mid d \) then by (21) we infer that \( 7 \mid a^4 \), whence by \( a \mid x \) we have \( 7 \mid x \) and \( 7 \mid y = a^2 + db^2 \). This cannot hold by \( \gcd(x, y) = 1 \).
If \( \delta' \geq 1 \) and \( 7 \nmid d \) then (21) is impossible mod 7 for every \( d \in \mathcal{H} \).

If \( \delta' = 0 \) then (21) is a Thue equation. By solving (21) for every \( d \in \mathcal{H} \) we obtain the solution \((\Delta, y, n) = (2 \cdot 5^3, 11, 5)\). It remains the case when in (6) \( n = 5 \) and \( \gamma \in \{0, 1, 2, 3, 4, 5\} \). If \( \gamma \in \{0, 3, 4, 5\} \) then we may apply the argument used in the Case 4 to the equation

\[
5a^4 - 10a^2b^2d + b^4d^2 = \frac{c}{b}, \quad \text{and} \quad a^4 - 2a^2b^2d + \frac{b^4d^2}{5} = \frac{c}{5b},
\]

respectively. Thus in this case there are no other solutions with \( n = 5 \).

If \( \gamma \in \{1, 2\} \) we see that \( \text{ord}_5(c) = 0 \) and \( \text{ord}_5(d) = 1 \), or \( \text{ord}_5(c) = 1 \) and \( \text{ord}_5(d) = 0 \). This leads to a contradiction in view of (12).

In the case when in (6) \( n = 7 \) we can work as above and we conclude that there are no solutions to (6) with \( n = 7 \).

The case \( n \in \{3, 4\} \).

First consider equation (6) with \( n = 4 \). Then factorizing in (6) we get

\[
(y^2 + x)(y^2 - x) = \Delta. \tag{22}
\]

Hence \( y^2 + x \in S \) and \( y^2 - x \in S \), where \( S = \{2, 3, 5, 7\} \). Thus we have the following equations

\[
\begin{align*}
\frac{y^2 + x}{2} + \frac{y^2 - x}{2} &= y^2, & \text{if } x \equiv y \pmod{2}, \\
2(y^2 + x) + 2(y^2 - x) &= (2y)^2, & \text{if } x \not\equiv y \pmod{2}.
\end{align*} \tag{23}
\]

Since \( x \) and \( y \) are relatively prime we have \( \gcd \left( \frac{y^2 + x}{2}, \frac{y^2 - x}{2} \right) = 1 \) and \( \gcd(2(y^2 + x), 2(y^2 - x)) = 2 \). Further, since \( x > 0 \) we get that \( \frac{y^2 + x}{2} \geq \frac{y^2 - x}{2} \) and \( 2(y^2 + x) \geq 2(y^2 - x) \) always holds. Thus we see that equations (23) satisfies the conditions of Lemma 5 with

\[
(U, V, W) = \left( \frac{y^2 + x}{2}, \frac{y^2 - x}{2}, y \right) \quad \text{and} \quad (U, V, W) = (2(y^2 + x), 2(y^2 - x), 2y).
\]

By applying Lemma 5 to (23) we obtain all non-exceptional solutions of (6) with \( n = 4 \) (see the table).

Next suppose that in (6) \( n = 3 \). By Lemma 2 we see that (6) can have a solution with \( n = 3 \) only in the following cases:
On the diophantine equation \( x^2 + 2^\alpha 3^\beta 5^\gamma 7^\delta = y^n \)

(I) \( 3a_1^2 b_1 - b_1^3 d = c \), where \( a_1, b_1 \in \mathbb{Z} \) and \( b_1 \mid c \),

(II) \( 3A_1^2 B_1 - B_1^3 d = 8c \), where \( A_1, B_1 \) are odd integers and \( B_1 \mid c \),

(III) \( \Delta = 3u^2 \pm 8 \) and \( x = u^3 \pm 3u \), where \( u \in \mathbb{Z} \),

(IV) \( \Delta = 3u^2 \pm 1 \) and \( x = 8u^2 \pm 3u \), where \( u \in \mathbb{Z} \).

Each of the above cases leads to an equation of the form

\[ U + V = W^2, \quad U, V \in S \]

with the following choices of the triple \((U, V, W)\).

**Case (I)** In this case we have the equation

\[ 3a_1^2 b_1 - b_1^3 d = c, \]

where \( a_1, b_1 \in \mathbb{Z} \) and \( b_1 \mid c \). We distinguish three subcases according to \( 3 \mid d \) or \( 3 \nmid d \).

If \( 3 \mid d \) then obviously \( 3 \mid c \) and hence we get from the above equation

\[ a_1^2 = c \frac{d}{3b_1} + \frac{d^2}{3b_1^2}. \]

We see that \( \text{gcd}\left( \frac{d}{3b_1}, \frac{d^2}{3b_1^2} \right) \) is square-free since \( \frac{d}{3b_1} \) and \( \frac{d^2}{3b_1^2} \) are relatively prime and \( \frac{d}{3} \) is square-free. The last two subcases can be reduced in a similar way. Thus we have to solve for every \( d \in H \) the following equations:

\[
(U, V, W) = \begin{cases}
\left( \frac{d}{3b_1}, \frac{d}{3b_1^2}, a_1 \right), & \text{if } 3 \mid d \text{ and } \frac{d}{3b_1} \geq \frac{c}{3b_1}, \\
\left( \frac{c}{3b_1}, \frac{d}{3b_1^2}, a_1 \right), & \text{if } 3 \mid d \text{ and } \frac{c}{3b_1} > \frac{d}{3b_1^2}, \\
\left( 3db_1^2, \frac{3c}{b_1}, 3a_1 \right), & \text{if } 3 \nmid d, 3 \mid b_1 \text{ and } 3db_1^2 \geq \frac{3c}{b_1}, \\
\left( \frac{3c}{b_1}, 3db_1^2, 3a_1 \right), & \text{if } 3 \nmid d, 3 \mid b_1 \text{ and } \frac{3c}{b_1} > 3db_1^2, \\
\left( 3db_1^2, \frac{c}{3b_1}, a_1 \right), & \text{if } 3 \nmid d, 3 \mid b_1 \text{ and } 3db_1^2 \geq \frac{c}{3b_1}, \\
\left( \frac{c}{3b_1}, 3db_1^2, a_1 \right), & \text{if } 3 \nmid d, 3 \mid b_1 \text{ and } \frac{c}{3b_1} > 3db_1^2.
\end{cases}
\]
Case (II) In this case we deal only with those values of \( d \in \mathcal{H} \) for which \( d \equiv 3 \pmod{8} \). Thus \( d \in \{3, 35\} \) and we get the following equations:

\[
(U, V, W) = \begin{cases} 
(B_1^2, \frac{8c}{3B_1^1}, A_1), & \text{if } d = 3 \text{ and } B_1^2 \geq \frac{8c}{3B_1^1}, \\
\left(\frac{8c}{3B_1^1}, B_1^2, A_1\right), & \text{if } d = 3 \text{ and } \frac{8c}{3B_1^1} > B_1^2, \\
(105B_1^2, \frac{24c}{9B_1^1}, 3A_1), & \text{if } d = 35, \ 3 \mid B_1 \text{ and } 105B_1^2 \geq \frac{24c}{9B_1^1}, \\
\left(\frac{24c}{9B_1^1}, 105B_1^2, 3A_1\right), & \text{if } d = 35, \ 3 \mid B_1 \text{ and } \frac{24c}{9B_1^1} > B_1^2, \\
(105B_1^2, \frac{8c}{9B_1^1}, A_1), & \\
B_1 = 3B_1', & \text{if } d = 35, \ 3 \mid b_1, \text{ and } 105B_1^2 \geq \frac{8c}{9B_1^1}, \\
\left(\frac{8c}{9B_1^1}, 105B_1^2, A_1\right), & \\
B_1 = 3B_1', & \text{if } d = 35, \ 3 \mid b_1, \text{ and } \frac{8c}{9B_1^1} > 105B_1^2. 
\end{cases}
\]

(25)

Case (III)

\[
(U, V, W) = \begin{cases} 
(3\Delta, \pm 24, 3u), & \text{if } \alpha \in \{0, 1\}, \\
\left(\frac{3\Delta}{4}, \pm 6, \frac{3u}{2}\right), & \text{if } \alpha \geq 2.
\end{cases}
\]

(26)

Case (IV)

\[
(U, V, W) = (3\Delta, \pm 3, 3u).
\]

(27)

One can easily see that each of the above equations satisfies the conditions of Lemma 5. By applying Lemma 5 to equations (24)–(27) we get all non-exceptional solutions to (6) with \( n = 3 \) (see the table).

\\

4. Non-exceptional solutions of equation (6)

The solutions of (6) with \( \alpha = 0 \) are marked with an asterisk.

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Acknowledgements. The author is grateful to KÁLMÁN GYÖRY, ÁKOS PINTÉR, LÁJOS HAJDU and the referee for their help and numerous valuable remarks.

References

1. Pink : On the diophantine equation $x^2 + 2^a3^b5^c7^d = y^n$


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