Which $F$ loops are associative

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1. Introduction

It is known that Moufang loops of order $p$, $p^2$, $pq$, $p^3$ [3], $p^2q$ (odd) with $p < q$, $pqr$ (odd), $2p^2$ [12], $p^4(p \geq 5)$ [4] and $2pq$ ($p < q$, $p \nmid q - 1$) [6] are all groups. On the other hand, there exist nonassociative Moufang loops of order $2^4$ [3], $3^4$ [2], $p^5$ ($p \geq 5$) [15], $2^2q$ and $2pq$ ($p < q$, $p \mid q - 1$) [13].

Now we shall confine our study in a similar direction to a special class of Moufang loops called $F$ loops whose orders are $2^\alpha p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $0 \leq \alpha \leq 3$; $p_i$ are distinct odd primes such that $\alpha_i \leq 3$ if $p_i = 3$ and $\alpha_i \leq 4$ if otherwise. We shall prove that these $F$ loops are groups if

(i) $0 \leq \alpha \leq 2$ or
(ii) $r \leq 2$

2. Definition

1. A loop $(L, \cdot)$ is a Moufang loop if $xy \cdot zx = (x \cdot yz)x$ for all $x, y, z \in L$.
2. $La$, the associator subloop of $L$, is the subloop generated by all the associators $(x, y, z)$ where $xy \cdot z = (x \cdot yz)(x, y, z)$.
3. $N = N(L)$, nucleus of $L$, is the set of all $n \in L$ such that $(n, x, y) = (x, n, y) = (x, y, n) = 1$ for all $x, y \in L$.
4. $Z = Z(L)$, the centre of $L$, is the set of all $z \in N$ such that $(z, x) = 1$ where $zx = xz(z, x)$ for all $x \in L$.
5. An $F$ loop $L$ is a Moufang loop such that if $H$ is a subloop generated by any three elements $x, y, z$ of $L$, then $((x, y, z)) \subset Z(H)$, the centre of $H$.

Remark. It can be shown easily that $Ha = ((x, y, z))$ for any $F$ loop $H$ generated by $x, y$ and $z$. [6, p. 80, Lemma]
3. Results

From now on, $L$ is assumed to be a finite Moufang loop.

$R_1$ $L$ is diassociative, i.e. $\langle x, y \rangle$ is associative for all $x, y \in L$.
[1, p. 115, Lemma 3.1]

$R_2$ If $(x, y, z) = 1$, then $\langle x, y, z \rangle$ is a group for any $x, y, z \in L$.
[1, p. 117, Moufang’s Theorem]

$R_3$ $N$ and $Z$ are normal subloops of $L$. Clearly $N$ and $Z$ are associative.
[1, p. 114, Theorem 2.1]

$R_4$ There exist simple nonassociative Moufang loops $M(p^n)$ with $|M(p^n)| = p^{3n}(p^{4n} - 1)/d(p)$ where $d(2) = 1$ and $d(p) = 2$ if $p$ is an odd prime.
[11, p. 475, Theorem 4.5]

$R_5$ $L$ is simple if and only if $L$ is a simple group or $L$ is isomorphic with $M(p^n)$ for some prime $p$.
[10, p. 33, Theorem]

$R_6$ 120 is a divisor of $|M(p^n)|$. [14]

$R_7$ If $H$ is a subloop of $L$, $x \in L$, and $d$ is the smallest positive integer such that $x^d \in H$, then $|\langle H, x \rangle| \geq |H|d$.
[3, p. 31, Lemma 1]

$R_8$ $L_a \triangleleft L$ and $L_a \subseteq C_L(N) = \{x \mid x \in L, xn = nx \text{ for all } n \in N\}$
[5, p. 34, Corollary]

$R_9$ If $L$ is an $F$ loop, and $x, y, z \in L$,
(a) $(x, y, z) = (y, z, x) = (y, x, z^{-1})$
(b) $(x^n, y, z) = (x, y, z)^n$
[1, p. 125, Lemma 5.5]

$R_{10}$ If $L$ is an $F$ loop of order $2^{\alpha_1}3^{\alpha_2}p_1^{\beta_1} \cdots p_r^{\beta_r}$ where $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 3$, $0 \leq \beta_i \leq 4$ and $p_i$ are distinct primes $\geq 5$, then $L$ is a group
[6, p. 81, Corollaries 2 and 3].

4. $F$ loops of order $2^{\alpha_1}3^{\alpha_2}p_1^{\beta_1} \cdots p_r^{\beta_r}$

**Lemma 1.** Let $L$ be an $F$ loop of order $2^{\alpha_1}3^{\alpha_2}p_1^{\beta_1} \cdots p_r^{\beta_r}$; $p_i$ are distinct odd primes; $\alpha_i \leq 3$ if $p_i = 3$ and $\alpha_i \leq 4$ if $p_i \geq 5$. Suppose $\exists$ a maximal normal subloop $K$ of $L$ (In symbol, $K \triangleleft L$; it being understood that $K$ is neither trivial nor the entire loop) such that $K$ is associative. Then $L$ is a group.

**Proof.** Since $K$ is a maximal normal subloop of $L$, $L/K$ is simple. However, since 120 $\nmid |L|$, 120 $\nmid |L/K|$ and so by $R_5$ and $R_6$, $L/K$ is a group and $L_a \subseteq K$.
Let $|L/K| = 2^k m$, $(2, m) = 1$. 

Case 1: \(k = 2\).

Then \(|K|\) is odd. Let \(x\) be an element of \(L\) and \(|x| = 2^\alpha\). By \(R_9\), 
\[(x, y, z)^{2^\alpha} = (x^{2^\alpha}, y, z) = 1\]
for all \(y, z \in L\). But the order of \((x, y, z)\) is odd also since 
\((x, y, z) \in L_a \subset K\). Thus \((x, y, z) = 1\) and so all \(2\)-elements 
lie in \(N\). Then \(K \triangleleft KN \triangleleft L\). So \(L = KN\).
\(L_a = (KN, KN, KN) = (K, K, K) = 1\), as \(K\) is associative.

Case 2: \(k = 1\).

Then \(L/K\) is a group of order \(2m\). But a group of order \(2m\) has a 
normal subgroup \(L_0/K\) of order \(m\). Then \(K \triangleleft L_0 \triangleleft L\), a contradiction. 
Hence \(L\) is a group.

Case 3: \(k = 0\).

Then \(L/K\) is a simple group of odd order and hence isomorphic to \(C_p\) 
where \(p = p_j\) for some \(j\). Let \(x, y, z \in L\). Then \(x^p, y^p, z^p \in K\). Using \(R_9\) 
again, as \(K\) is a group,
\[(x^p, y, z) = (x, y, z)^{p^3} = (x^p, y^p, z^p) = 1\]
Thus \(x^p \in N\) and \(L/N\) is a \(p\)-loop. As \(|L/N| = p^\alpha \mid p_j^{\alpha_j}\), \(L/N\) is an 
Abelian group because of the restriction on \(\alpha_j\). So \(L_a \subset N\). By \(R_8\),
as \(L_a \subset C_L(N)\), \(L_a\) is an Abelian group. Also since \((x, y, z)^{p^3} = 1\) for 
all \(x, y, z \in L\), \(L_a\) is a \(p\)-group. Let \(P/L_a\) be a Sylow \(p\)-subgroup of 
\(L/L_a\). Then \(|P| = p^{\alpha_j}\). Since \(|L| = |N|p^\alpha\) and \(N \cap P\) is a \(p\)-group in \(N\),
\(|N \cap P| \leq p^{\alpha_j - \alpha}\). Now

\[|PN| = \frac{|P||N|}{|N \cap P|} \geq p_j^{\alpha_j} \frac{|L|}{p^\alpha p^{\alpha_j - \alpha}} = |L|\]

So \(L_a = (PN, PN, PN) = (P, P, P) = P_a = 1\) since a Moufang loop of this 
restricted order is a group. Hence \(L\) is a group.

Theorem 1. Let \(L\) be an \(F\) loop of order \(2^{2^p_1 \cdots p_r^{e_r}}\); \(p_i\) are distinct 
odd primes; \(\alpha_i \leq 3\) if \(p_i = 3\) and \(\alpha_i \leq 4\) if \(p_i \geq 5\). Then \(L\) is a group.

Proof. Suppose \(L\) is not associative. Since 120 is not a divisor of 
\(|L|\), \(L\) is not simple. Let \(L_1 \triangleleft L\). If \(L_1\) is a group, then \(L\) would be a group 
by Lemma 1, a contradiction. If \(L_1\) is not a group, since 120 \(\nmid |L_1|\), \(L_1\) 
is not simple. Let \(L_2 \triangleleft L_1\). In this manner, we have a series of subloops 
\(L_{j+1} \triangleleft L_j \triangleleft \cdots \triangleleft L_2 \triangleleft L_1 \triangleleft L\) where \(L_i\) is nonassociative for 
\(i \leq j\) and \(L_{j+1}\) is associative. (Note that 4 is a divisor of \(|L_i|\) for the nonassociative loop 
\(L_i\) by \(R_{10}\).) This series terminates as \(|L|\) is finite. Now by Lemma 1, \(L_j\) 
would be a group, a contradiction. So \(L\) must be a group.
5. F loops of order $2^3 p^a q^b$

**Lemma 2.** Let $L$ be an $F$ loop, $x$ a $p$-element and $x \in L - N$. Then \( \exists \) a nonassociative subloop $P$ of order $p^m$ in $L$ with $m \geq 4$ if $p = 2$ or $3$ and $m \geq 5$ if $p \geq 5$.

**Proof.** Since $x \notin N$, \( \exists y, z \in L \) such that $(x, y, z) \neq 1$. Using $R_9$, we can assume the order of $(x, y, z)$ is $p^r$ for some $r$. Let $H = \langle x, y, z \rangle$. Then $H_a = \langle (x, y, z) \rangle = C_{p^r} \subset Z(H) \subset N(H) \subset H$.

Let $f, g, h \in H$.

Then $(f, g, h) = (x, y, z)^j$ for some $j$.

$$(f, g, h)^{p^r} = (x, y, z)^{jp^r} = 1.$$ 

So $(f^{p^r}, g, h) = 1$ or $f^{p^r} \in N(H)$ for all $f \in H$. Therefore $H/N(H)$ is a group of exponent dividing $p^r$. Let $|H/N(H)| = p^\theta$ and $|N(H)| = m_0 p^\gamma$, where $m_0, p, \gamma \geq 0$.

Let $P/H_a$ be a Sylow $p$-subgroup of $H/H_a$. As $|H/H_a| = \frac{|H|}{|H_a|} = \frac{m_0 p^{\gamma + \theta}}{p^\theta} = m_0 p^{\gamma + \theta - r}, |P/H_a| = p^{\gamma + \theta - r}$ or $|P| = p^{\gamma + \theta}$. Since $P \cap N(H)$ is a $p$-subgroup of $N(H)$, $|P \cap N(H)| \leq p^\gamma$. Then $|PN(H)| = \frac{|P| |N(H)|}{|P \cap N(H)|} \geq p^{\theta + \gamma m_0 p^\gamma} = p^{\theta + \gamma m_0} = |H|$. Thus $PN(H) = H$.

$$H_a = (PN(H), P, N(H)) = (P, P, P) = P_a.$$ 

As $H_a \neq 1$, $P_a \neq 1$ and $P$ is not associative.

As $P$ is a nonassociative Moufang $p$-loop, $|P| = p^m$ with $m \geq 4$ if $p = 2$ or $3$ by [3] and with $m \geq 5$ if $p \geq 5$, by [4].

**Lemma 3.** If $H$ and $K$ are subloops of an $F$ loop $L$ with order $m$ and $n$ such that $(m, n) = 1$, then $|HK| = mn$.

**Proof.** Suppose $x_1 y_1 = x_2 y_2$ with $x_i \in H$, $y_i \in K$. \( \because \) $x_1^{-1}(x_1 y_1) = x_1^{-1}(x_2 y_2)$.

$$y_1 = (x_1^{-1} x_2 \cdot y_2)(x_1^{-1}, x_2, y_2).$$ 

Let $a = (x_1^{-1}, x_2, y_2)$. Then $a^m = a^n = 1$ using $R_9$. As $(m, n) = 1$, \( a = 1. \)

So $y_1 = x_1^{-1} x_2 \cdot y_2 \because y_1 y_2^{-1} = x_1^{-1} x_2 \in H \cap K$.

Since the order of an element divides the order of a diassociative loop [1, p. 92, Theorem 1.2], $H \cap K = \{1\}$.

So $y_1 = y_2$ and $x_1 = x_2$. 

**Lemma 4.** Let $L$ be an $F$ loop and $p$ a prime, $p | |L_a|$. Then

(a) $x$ is a $p$-element $\implies x \in N$

(b) $L/N$ is a group $\implies p | |L/N|$

**Proof.** Let $|x| = p^\alpha$ and $y, z \in L$. Then $(x, y, z)^{p^\alpha} = (x^{p^\alpha}, y, z) = (1, y, z) = 1$. But $(x, y, z)^{|L_a|} = 1$. Since $(p, |L_a|) = 1$, $(x, y, z) = 1$ and $x \in N$. Suppose $L/N$ is a group and $p | |L/N|$. Let $gN$ be an element of order $p$ in $L/N$. Then $g^p \in N$ and $g \notin N$. Let $|g| = p^\beta m$ with $(p, m) = 1$. Then $g^m$ is a $p$-element and hence $g^m \in N$. Since $(p, m) = 1$, $g \in N$. This is a contradiction. Hence $p | |L/N|$.

**Lemma 5.** Let $L$ be an $F$ loop of order $8p^\alpha$ with $\alpha \leq 3$ if $p = 3$ and $\alpha \leq 4$ if $p \geq 5$. Then $L$ is a group.

**Proof.** By [1, p. 92, Theorem 1.2], the order of each element of $L$ divides $8p^\alpha$. If each of the elements of $L$ has order a power of 2, then $|H|$ would be a power of 2 by [9, p. 415, Theorem]. On the other hand, if each of the elements of $L$ has order a power of $p$, then $|H|$ would be a power of $p$ by [8, p. 395, Theorem 1]. So, there exists 2-elements as well $p$-elements in $L$.

*Case 1:* Suppose $\exists$ a $p$-element $y$ such that $y \notin N$. By Lemma 2, $\exists$ a nonassociative subloop $P_p$ of order $p^m$ with $m \geq \alpha + 1$.

1.1: Suppose $\exists$ 2-element $x$ such that $x \in N$. By Lemma 2, $\exists$ a nonassociative subloop $P_2$ of order $2^k$ with $k \geq 4$. By Lemma 3, $|P_2P_p| = 2^kp^m \geq 2^4p^{\alpha+1} > 8p^\alpha = |L|$, a contradiction.

1.2: Suppose all 2-elements lie in $N$. Suppose $2 | |L/N|$. As $L/N$ is group by Theorem 1, there exists $g \in L - N$ such that $g^2 \in N$. Let $|g| = 2^ap^b$. Then $g^{p^b}$ is a 2-element. Thus $g^{p^b} \in N$. Since also $g^2 \in N$, we have $g \in N$, a contradiction. So $2 \nmid |L/N|$.

Then $2^3 | |N|$. Let $P_2$ be a Sylow 2-subgroup of the group $N$. Then $|P_2| = 8$. By Lemma 3, $|P_2P_p| \geq 8p^{\alpha+1} > |L|$ a contradiction.

*Case 2:* Suppose all $p$-elements of $L$ lie in $N$. We can similarly show that $p \nmid |L/N|$. Then $p^\alpha \mid |N|$. So, letting $P_p$ be a Sylow $p$-subgroup of $N$, we have $|P_p| = p^\alpha$.

2.1: Suppose $\exists$ a 2-element $x$ such that $x \notin N$.

By Lemma 2, $\exists$ a nonassociative subloop $P_2$ of order $2^k$ with $k \geq 4$.

By Lemma 3, $|P_2P_p| = 2^kp^\alpha \geq 16p^\alpha > 8p^\alpha = |L|$, a contradiction.

2.2: Suppose all 2-elements of $L$ lie in $N$.

Then clearly all elements of $L$ lie in $N$. Hence $L = N$ is a group.

**Lemma 6.** Let $L$ be an $F$ loop of order $2^3 \cdot 3 \cdot 5$. Then $L$ is a group.

**Proof.** *Case 1:* Suppose $L$ has an element $w$ of order 5.

1.1: Suppose $w \notin N$. By Lemma 2, $\exists$ a subloop $P_5$ with $|P_5| = 5^\alpha$, $\alpha \geq 5$. So $|P_5| \geq 5^5 > 2^3 \cdot 3 \cdot 5 = |L|$, a contradiction.
1.2: Suppose \( w \in N \). Then \( L/N \) is a group by Lemma 5 and Theorem 1. So \( L_a \subset N \).

(a) If \( 2 \nmid |L_a| \), then by Lemma 4, \( |L/N| = 3 \). So \( L/N = \langle \bar{x} \rangle \) or \( L = N \langle x \rangle \) for some \( x \in L \). So \( L \) is a group by diassociativity.

(b) If \( 3 \nmid |L_a| \), then by Lemma 4, any 3-element, if such exists, lies in \( N \). Suppose \( 3 \mid |L/N| \). Let \( g \) be an element of order 3 in \( L/N \). Then \( g^3 \in N \) but \( g \notin N \). Let \( |g| = 3^\alpha m \), \( (3,m)=1 \). Then \( |g^m| = 3^\alpha \). So \( g^m \in N \). This implies \( g \in N \), a contradiction. So \( 3 \nmid |L/N| \) and \( |L/N| = 2^3 \). If \( |L/N| = 2^3 \), then \( 2 \nmid |N| \). As \( L_a \subset N \), \( 2 \nmid |L_a| \). By Lemma 4, \( 2 \nmid |L/N| \), a contradiction. Hence \( |L/N| \leq 2^2 \). So \( L/N = \langle \bar{x}, \bar{y} \rangle \) or \( L = N \langle x, y \rangle \) for some \( x, y \in L \). Thus \( L \) is a group by diassociativity.

(c) We can assume \( 6 \nmid |L_a| \). So \( 30 \nmid |N| \). \( |L/N| = 2^2 \). Again \( L \) is a group by diassociativity.

Case 2: Suppose \( L \) has no element of order 5. Clearly \( 5 \nmid |N| \).

\( L \) must have an element \( u \) of order 3. Otherwise, \( L \) would be a 2-loop of order a power of two.

2.1: Suppose \( u \notin N \). By Lemma 2, \( \exists \) a subloop \( P_3 \) such that \( |P_3| = 3^m \), \( m \geq 4 \). Let \( v \in L - P_3 \). Then by \( R_7 \), \( |\langle v, P_3 \rangle| \geq 2 \cdot 3^m \geq 2 \cdot 3^4 > |L| \), a contradiction.

2.2: Suppose \( u \in N \). So \( |L/N| = 2^\alpha 5 \), \( \alpha \leq 3 \). By Lemma 5, Theorem 1 and \( R_{10} \), \( L/N \) is a group. Let \( \bar{x} \) be an element of order 5 in \( L/N \), i.e. \( x \in L - N \) and \( x^5 \in N \). Then \( x^5|N| = 1 \) or \( (x|N|)^5 = 1 \). As \( L \) has no element of order 5, \( x|N| = 1 \). As \( (5, |N|) = 1 \), \( x \in N \), a contradiction.

Lemma 7. Let \( L \) be an \( F \) loop of order \( 2^3 \cdot 3^3 \cdot 5 \). Then \( L \) is a group.

Proof. Case 1: Suppose \( L \) has an element \( x \) of order 5.

1.1: Suppose \( x \notin N \). By Lemma 2, \( \exists \) a subloop \( P_5 \) with \( |P_5| > 2^3 \cdot 3^3 \cdot 5 = |L| \), a contradiction.

1.2: Suppose \( x \in N \). So \( |L/N| \mid 2^3 \cdot 3^3 \). By Lemma 5, \( L/N \) is a group. Thus \( L_a \subset N \).

1.2(a) If there exist both 2-elements and 3-elements in \( L - N \), then by Lemma 2, \( \exists \) subloops \( P_2 \) and \( P_3 \) with \( |P_2| > 2^4 \) and \( |P_3| > 3^4 \). By Lemma 3, \( |P_2 P_3| \geq 2^4 \cdot 3^4 > 2^33^35 = |L| \), a contradiction.

1.2(b) If all the 2-elements lie in \( N \), then \( |L/N| \mid 3^3 \). If \( |L/N| \mid 3^2 \), then \( L \) is a group by disassociativity. So we assume \( |L/N| = 3^3 \). Then \( 3 \nmid |N| \). In other words, \( 3 \nmid |L_a| \). As in the case 1.2(b), we can use Lemma 4 to show that \( 3 \nmid |L/N| \). This is a contradiction.

1.2(c) If all the 3-elements lie in \( N \), we obtain a contradiction by a similar argument.

Case 2: Suppose \( L \) has no element of order 5. Clearly \( L \) has \( p \)-elements for \( p = 2 \) and \( p = 3 \).
2.1: Suppose it has a 2-element as well as a 3-element lying in $L - N$. By Lemma 2, $\exists$ subloops $P_2$ and $P_3$ with orders $2^\alpha$ and $3^\beta$ respectively, $\alpha, \beta \geq 4$. By Lemma 3, $|P_2P_3| = 2^\alpha 3^\beta \geq 2^4 3^4 > |L|$, a contradiction.

2.2: Suppose all the 2-elements of $L$ lie in $N$. It can be seen easily that $2 \not{|} |L/N|$. Clearly $5 \not{|} |N|$. So $|L/N| = 3^\gamma 5$, $\gamma \leq 3$.

Applying $R_{10}$, $L/N$ is a group. Let $\bar{x}$ be an element of order 5. Then $x \in L - N$ and $x^5 \in N$. So $x^{5|N|} = 1$ or $(x^{|N|})^5 = 1$. Since $L$ has no element of order 5, $x^{|N|} = 1$. But $(5, |N|) = 1$. So $x \in N$, a contradiction.

2.3: If all the 3-elements of $L$ lie in $N$, a contradiction arises in a similar way by applying Lemma 5.

**Lemma 8.** Let $L$ be a nonassociative $F$ loop of order $2^3p^\alpha q^\beta$; $p$ and $q$ distinct primes with $p < q$; $\alpha \leq 3$ if $p = 3$ and $\alpha \leq 4$ if $p \geq 5$; $\beta \leq 4$. Then $L$ is non-simple.

**Proof.** Suppose $L$ is simple. By $R_5$, $L$ is isomorphic to one of the $M(r^n)$. But $|M(r^n)| = 2^3p^\alpha q^\beta$ with $p, q, \alpha, \beta$ as specified if and only if $n = 1$, $r = 2$ or 3, but $|M(2)| = 2^3 \cdot 3 \cdot 5$ and $|M(3)| = 2^3 \cdot 3^3 \cdot 5$. By Lemma 6 and Lemma 7, we have a contradiction.

**Lemma 9.** Let $L$ be an $F$ loop of order $2^3p^\alpha q^\beta$ defined as above. Then $L$ has $p$-elements (as well as $q$-elements).

**Proof.** If $L$ is associative, then the result follows by Sylow theory. Suppose $L$ is not associative. By Lemma 8, $L$ is not simple. Let $L_1 \triangleleft \cdot L$. Suppose $2 \not{|} |L_1|$. Then $L/L_1$ is a group by Theorem 1 and $R_{10}$. Suppose $2 \not{|} |L_1|$. If $L/L_1$ is nonassociative, then $L/L_1$ is nonsimple by Lemma 8. But this contradicts the maximality of $L_1$. In any case, $L/L_1$ is a group. In fact, it is a simple group. Moreover if $L_1$ is nonassociative, then $|L_1| = 2^3p^\alpha q^\beta$ by Theorem 1 and $R_{10}$. So $|L/L_1| = p^\alpha q^\beta$. But a simple group of this odd order is isomorphic to $C_p$ or $C_q$.

Now suppose $L_1$ is nonassociative. By a similar argument, we have $L_2 \triangleleft \cdot L_1$ with $L_1/L_2$ a simple group. Continuing, we have a series of subgroups

$$L_{m+1} \triangleleft \cdot L_m \triangleleft \cdots \triangleleft L_2 \triangleleft \cdot L_1 \triangleleft \cdot L_0 = L$$

such that

(a) $L_i/L_{i+1}$ is a simple cyclic group for $0 \leq i \leq m$

(b) $L_i$ is nonassociative for $0 \leq i \leq m$

(c) $L_{m+1}$ is a group.

If $p \not{|} |L_{m+1}|$, we are through by (c).

Otherwise, let $j$ be the smallest integer such that $p \not{|} |L_{j+1}|$ but $p \not{|} |L_j|$.

Then $|L_j/L_{j+1}| = p$ by (a).

Let $x \in L_j - L_{j+1}$. Then $x^p \in L_{j+1}$. Write $|L_{j+1}| = \ell$. $(x^p)^\ell = 1$ i.e. $(x^\ell)^p = 1$. If $x^\ell \neq 1$, then we are through. If $x^\ell = 1$, as $(p, \ell) = 1$, $x \in L_{j+1}$, a contradiction.
Lemma 10. Let $L$ be an $F$ loop or order $2^3 p^\alpha q^\beta$ defined as above. Suppose $L_a \subset N$ with $|L_a| = 2^k m$, $k \geq 1$ and $m$ odd. Then $L$ is a group.

Proof. Case 1: $m = 1$
As $pq \nmid |L_a|$, $pq \nmid |L/N|$ by Lemma 4. So $|L/N| \nmid 2^2$. Thus $L/N = \langle \tilde{x}, \tilde{y} \rangle$ or $L = N\langle x, y \rangle$ for some $x, y \in N$. So $L$ is a group by diassociativity.

Case 2: $m > 1$
By $R_8$, $L_a \subset C_L(N)$. So $L_a \subset Z(N)$, the centre of $N$. Let $K$ be a subloop of order $2^k$ in $L_a$. As $L_a$ is an abelian group and $L_a < L$, $K < L$. $L/K$ is a group by Theorem 1 and $R_{10}$. Thus $L_a \subset K$. Hence $|L_a| = |K| = 2^k$, a contradiction.

Theorem 2. Let $L$ be an $F$ loop of order $2^3 p^\alpha q^\beta$ defined as above. Then $L$ is a group.

Proof. By Lemma 9, $\exists$ both $p$-elements and $q$-elements in $L$.

Case 1: Suppose $\exists$ both $p$-elements and $q$-elements in $L \subset N$.
By Lemma 2, $\exists$ nonassociative subloops $P$ and $Q$ such that $|P| \geq p^{\alpha+1}$ and $|Q| \geq q^{\beta+1}$.

By Lemma 3, $|PQ| \geq p^{\alpha+1} q^{\beta+1} 2^3 p^\alpha q^\beta = |L|$, a contradiction.

Case 2: Suppose all $p$-elements lie in $N$ and some $q$-element lies in $L - N$. Then $p^{\alpha} \mid |N|$. Also, by Lemma 2, $\exists$ a $q$-subloop $Q$ of $L$, such that $|Q| \geq q^{\beta+1}$. By Lemma 5 and $R_{10}$, $L/N$ is a group and $L_a \subset N$.
By Lemma 10, we may assume that $|L_a|$ is odd. As $2 \nmid |L_a|$, $2^3 \mid |N|$ by Lemma 4. So $|N| = 2^3 p^\alpha$. Then, since $N \cap Q = \{1\}$, $|NQ| = \frac{|N||Q|}{|N\cap Q|} \geq 2^3 p^\alpha q^{\beta+1} > |L|$, a contradiction.

Case 3: Suppose all $q$-elements lie in $N$ and some $p$-element lies in $L - N$. A contradiction arises just as in case 2.

Case 4: Suppose all $p$-elements and $q$-elements lie in $N$.
Then $|L/N| \mid 2^3$.
If $|L/N| \leq 2^2$, then $L/N = \langle \tilde{x}, \tilde{y} \rangle$ or $L = N\langle x, y \rangle$ for some $x, y \in L$ and $L$ is a group by diassociativity. If $|L/N| = 2^3$, then $2 \nmid |N|$. Therefore $2 \nmid |L_a|$ because $L_a \subset N$ and by Lemma 4, $2 \nmid |L/N|$, a contradiction.

References

Which $F$ loops are associative


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