A problem of Galambos on Oppenheim series expansions

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Abstract. In this paper, we investigate the Hausdorff dimension of exceptional sets in the metric properties of digits of Oppenheim series expansions and answer a question posed by Galambos.

1. Introduction

For any $x \in (0, 1]$, the algorithm

\[ x = x_1, \quad d_n = \lceil 1/x_n \rceil + 1, \quad x_n = 1/d_n + a_n/b_n \cdot x_{n+1}, \]

where $a_n = a_n(d_1, \ldots, d_n)$ and $b_n = b_n(d_1, \ldots, d_n)$ are positive integer valued functions and $\lceil y \rceil$ denotes the integer part of $y$, leads to the Oppenheim expansion [12]

\[ x \sim \frac{1}{d_1} + \frac{a_1}{b_1} \frac{1}{d_2} + \cdots + \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n} \frac{1}{d_{n+1}} + \cdots \]

By (1),

\[ \frac{1}{d_n} < x_n \leq \frac{1}{d_n - 1}, \]

and hence by the last equality in (1),

\[ d_{n+1} > \frac{a_n}{b_n} d_n (d_n - 1). \]

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The expansion defined by (1) and (2) is convergent and its sum is equal to $x$.

A sufficient condition for a series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is (see [12])

$$d_{n+1} \geq \frac{a_n d_n}{b_n} (d_n - 1) + 1 \text{ for all } n \geq 1.$$  

Definition 1.1. We call the expansion (2) (obtained by the algorithm (1)) restricted Oppenheim expansion of $x$ if $a_n$ and $b_n$ depend on the last denominator $d_n$ only and if the function

$$h_n(j) = \frac{a_n(j)}{b_n(j)} j(j - 1)$$

is integer-valued, for all $n \geq 1$ and $j \geq 2$.

In the present paper, we deal with restricted Oppenheim expansions only. In this case, (4) and (5) are equivalent.

The representation (2) under (1) was first studied by Oppenheim [12], including Lüroth ([11]), Engel, Sylvester expansions ([2]) and Cantor infinite product ([13]) as special cases. Oppenheim established the arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory of such expansions were laid down by Galambos [5], [6], [7], [9], see also the monographs of Galambos [8], Schweiger [14], Verbaat [15], Dajani and Kraaikamp [1]. From [8], Chapter 6, it can be seen that the integer approximations $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_n-1(x))$ defined by

$$T_n(x) < \frac{d_n(x)}{h_{n-1}(d_n-1(x))} \leq T_n(x) + 1, \quad n \geq 1,$$

where $h_0(x) \equiv 1$, plays an important role in the metric theory of Oppenheim expansions, see Galambos [8] Chapter VI. Moreover, they are stochastically independent and are distributed as the denominators in the Lüroth expansion. Galambos, see [8] Page 132, posed the question to calculate the Hausdorff dimension of the set

$$B_m = \{x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}, \quad m \geq 2,$$

and compare this with the Lüroth case. In [16], the second author concerned this problem under the condition $h_n(j)$ is of order $t$ ($t \geq 1$), see [16] for the definition. In this paper, we continue to consider this problem. Under more natural conditions, we obtain the Hausdorff dimension of $B_m$ and thus answer
the question of Galambos. To obtain the lower bound of the Hausdorff dimension of a fractal set, a mass distribution is needed, which is a necessary (and sufficient) tool for this. The mass distribution constructed here is quite technical and subtle.

We use $|\cdot|$ to denote the diameter of a subset of $(0, 1]$, $\dim_H$ to denote the Hausdorff dimension and ‘cl’ the closure of a subset of $(0, 1]$ respectively.

2. Hausdorff dimension of $B_m$

For any $m \geq 2$, let

$B_m = \{ x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1 \}.

By (7), it is easy to check that

$B_m = \left\{ x \in (0, 1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m + 1 \text{ for all } n \geq 1 \right\}, \quad (8)$

where $h_0(n) \equiv 1$. Thus in order to calculate the Hausdorff dimensions of $B_m$, $m \geq 2$, it is sufficient to consider the following sets

$C_m = \left\{ x \in (0, 1] : 1 < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq m \text{ for all } n \geq 1 \right\}, \quad m \geq 3.$

From now on, we fix $m \geq 3$ be a positive integer.

Lemma 2.1. For any integer $a \geq 1$, let $S(a)$ be determined by the following equation

$\sum_{a \leq b \leq ma} \left( \frac{a}{b(b-1)} \right)^{S(a)} = 1. \quad (9)$

Then

$\lim_{a \to +\infty} S(a) = 1.$

Proof. Since

$\sum_{a < b \leq ma} \left( \frac{a}{b(b-1)} \right) = 1 - \frac{1}{m} < 1,$

we have $S(a) \leq 1$ for all $a \geq 1$.

On the other hand, for any $1/2 < s < 1$,

$\sum_{a < b \leq ma} \left( \frac{a}{b(b-1)} \right)^s \geq \sum_{a \leq b \leq ma} \left( \frac{a}{b(b-1)} \right)^s - \left( \frac{1}{a-1} \right)^s$
\[ \int_{a}^{ma} \frac{a^s}{x^{2s}} \, dx = \left( \frac{1}{a-1} \right)^s \]

\[ = \frac{1}{1-2s}((ma)^{1-2s} - a^{1-2s}) \cdot a^s - \left( \frac{1}{a-1} \right)^s \]

\[ = \frac{(1-m^{1-2s}) \cdot a^{1-s}}{2s-1} - \left( \frac{1}{a-1} \right)^s > 1, \ a \text{ is large enough.} \]

Thus when \( a \) is large enough, \( S(a) > s \). The proof of Lemma 2.1 is finished. \( \square \)

We now state the mass distribution principle, see [4] Proposition 2.3, that will be used later.

**Lemma 2.2.** Let \( E \subset (0,1] \) be a Borel set and \( \mu \) be a measure with \( \mu(E) > 0 \). If for any \( x \in E \),

\[ \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s, \]

where \( B(x,r) \) denotes the open ball with center at \( x \) and radius \( r \), then \( \dim_H E \geq s \).

Now we are in the position to prove the main result of this paper.

**Theorem 2.3.** Suppose \( h_j(d) \geq d - 1 \) for all \( j \geq 1 \) and \( d \geq 2 \), then for each \( m \geq 3 \),

\[ \dim_H C_m = 1. \]

**Proof.** For any \( j \geq 1 \) and \( d \geq 2 \), define

\[ G_j(d) = m \cdot h_j(d); \]

\[ M_j(m) = G_{j-1} \circ G_{j-2} \circ \cdots \circ G_1(m), \ M_1(m) := m. \]

From the assumption on \( h_j(d) \), it is easy to check that

\[ M_j(m) \geq m^j - m^{j-1} - \cdots - m^2 - m \quad \text{for each } j \geq 1, \]

thus

\[ \lim_{j \to \infty} M_j(m) = +\infty. \quad (10) \]

For any \( 0 < s < 1 \), from Lemma 2.1, since \( \lim_{a \to \infty} S(a) = 1 \), there exists \( a_0 \in \mathbb{N} \) such that for any \( a \geq a_0 \), \( S(a) > s \). By (10), there exists \( k_0 \geq 1 \) such that for any \( k \geq k_0 \),

\[ M_k(m) \geq a_0 + 1. \quad (11) \]
Define

\[ E_m = \left\{ x \in (0, 1] : d_j(x) = M_j(m) \text{ for all } 1 \leq j \leq k_0, \right. \]
\[ \left. \quad \text{and } 1 < \frac{d_{j+1}(x)}{h_j(d_j(x))} \leq m \text{ for all } j \geq k_0 \right\}. \]

It is clear that \( E_m \subset C_m \). Now we estimate the Hausdorff dimension of \( E_m \).

For any \( x \in E_m \), since \( h_j(d) \geq d - 1 \) for all \( j \geq 1 \) and \( d \geq 2 \), by (5), we have,

\[ d_k(x) \geq h_{k-1}(d_{k-1}(x)) + 1 \geq d_{k-1}(x) \geq \cdots \geq d_{k_0+1}(x) \]
\[ \geq h_{k_0}(d_{k_0}(x)) + 1 \geq d_{k_0}(x) = M_{k_0}(m) \geq a_0 + 1, \quad (12) \]

and

\[ h_k(d_k(x)) \geq d_k(x) - 1 \geq a_0. \quad (13) \]

Now we introduce a symbolic space defined as follows:

For any \( k \geq k_0 \), let

\[ D_k = \left\{ \sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{N}^k : \sigma_j = M_j(m) \text{ for all } 1 \leq j \leq k_0, \right. \]
\[ \left. \quad \text{and } 1 < \frac{\sigma_{j+1}}{h_j(\sigma_j)} \leq m \text{ for all } k_0 \leq j \leq k-1 \right\}, \]

and define

\[ D = \bigcup_{k=k_0}^{\infty} D_k. \]

For any \( k \geq k_0 \) and \( \sigma = (\sigma_1, \ldots, \sigma_k) \in D_k \), let \( J_\sigma \) and \( I_\sigma \) denote the following closed subintervals of \( (0, 1] \):

\[ J_\sigma = \bigcup_{h_k(\sigma_k) < d \leq mh_k(\sigma_k)} \text{cl}\{x \in (0, 1] : d_1(x) = \sigma_1, d_2(x) = \sigma_2, \ldots, d_k(x) = \sigma_k, d_{k+1}(x) = d\}, \]
\[ I_\sigma = \text{cl}\{x \in (0, 1] : d_1(x) = \sigma_1, \ d_2(x) = \sigma_2, \ldots, d_k(x) = \sigma_k\}, \]

and each \( J_\sigma \) is called an interval of \( k \)th-order. Finally, define

\[ E = \bigcap_{k=k_0}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma. \]
It is obvious that
\[ E = E_m. \]
From the proof of Theorem 6.1 in [8], we have, for any \( k \geq k_0 \) and \( \sigma \in D_k \),
\[
|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdot \ldots \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k},
\]
(14)
thus by (6), we have
\[
|J_\sigma| = \sum_{h_k(\sigma_k) < d \leq mh_k(\sigma_k)} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \ldots \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{(d - 1)d}
\]
\[
= \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \ldots \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left( \frac{1}{h_k(\sigma_k)} - \frac{1}{mh_k(\sigma_k)} \right)
\]
\[
= \left( 1 - \frac{1}{m} \right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \ldots \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{h_k(\sigma_k)}
\]
\[
= \left( 1 - \frac{1}{m} \right) \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \ldots \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k},
\]
(15)
For any \( k \geq k_0, \sigma \in D_k \), define
\[
\mu(J_\sigma) = \prod_{i=k_0}^{k-1} \left( \frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))}, \quad \text{if } k \geq k_0 + 1,
\]
(16)
and
\[ 5\mu(J_\sigma) = 1, \quad \text{if } \sigma \in D_{k_0}. \]
\( \mu \) is a probability mass distribution supported on \( E_m \), because
\[
\sum_{\sigma_{k+1} = h_k(\sigma_k) + 1}^{mh_k(\sigma_k)} \mu(J_{\sigma_1, \sigma_2, \ldots, \sigma_{k+1}})
\]
\[
= \sum_{\sigma_{k+1} = h_k(\sigma_k) + 1}^{mh_k(\sigma_k)} \prod_{i=k_0}^{k} \left( \frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \right)^{S(h_i(\sigma_i))} = \mu(J_{\sigma_1, \sigma_2, \ldots, \sigma_k}),
\]
and
\[
\sum_{\sigma_{k_0+1} = h_{k_0}(\sigma_{k_0}) + 1}^{mh_{k_0}(\sigma_{k_0})} \mu(J_{\sigma_1, \sigma_2, \ldots, \sigma_{k_0+1}})
\]
A problem of Galambos on Oppenheim series expansions

\[ \lim_{r \to 0} \inf \frac{\log \mu(B(x, r))}{\log r} \geq s. \]  

(17)

If (17) is proved, by Lemma 2.2, we have \( \dim_H E_m \geq s \). Since \( 0 < s < 1 \) is arbitrary, this implies \( \dim_H C_m = 1 \).

Now we prove (17).

For any \( x \in E_m \), there exists \( \sigma = (\sigma_1, \sigma_2, \ldots) \) such that for any \( k \geq k_0 \), \( \sigma(k) := (\sigma_1, \sigma_2, \ldots, \sigma_k) \in D_k \) and \( d_j(x) = \sigma_j \) for each \( j \geq 1 \). Thus

\[ x \in J_{\sigma_1, \sigma_2, \ldots, \sigma_k} \]

for all \( k \geq k_0 \).

From the proof of Theorem 6.1 in [8], we have, for any \( k \geq k_0 \), the right endpoint of the interval \( J_{\sigma_1, \sigma_2, \ldots, \sigma_k} \), i.e., \( \max\{y \in (0, 1] : y \in J_{\sigma_1, \sigma_2, \ldots, \sigma_k}\} \), is

\[
\frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{k}(\sigma_{k})}{b_{k}(\sigma_{k})} \cdot \frac{1}{\mu_{k}(\sigma_{k})}.
\]

(18)

The left endpoint of the interval \( J_{\sigma_1, \sigma_2, \ldots, \sigma_k} \), i.e., \( \min\{y \in (0, 1] : y \in J_{\sigma_1, \sigma_2, \ldots, \sigma_k}\} \), is

\[
\frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{k}(\sigma_{k})}{b_{k}(\sigma_{k})} \cdot \frac{1}{\mu_{k}(\sigma_{k})}.
\]

\[
\frac{1}{\sigma_1} + \sum_{j=2}^{k} \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{j-1}(\sigma_{j-1})}{b_{j-1}(\sigma_{j-1})} \cdot \frac{1}{\sigma_j} + \frac{a_{1}(\sigma_{1})}{b_{1}(\sigma_{1})} \cdot \frac{a_{k}(\sigma_{k})}{b_{k}(\sigma_{k})} \cdot \frac{1}{\mu_{k}(\sigma_{k})}.
\]
Thus by (14), (20) and (21), that is,\
\[
\frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{1}{\sigma_2} + \ldots + \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \left( \frac{1}{\sigma_k} + \frac{1}{m\sigma_k(\sigma_k - 1)} \right).
\] (19)

If \(\sigma_k - 1 > h_{k-1}(\sigma_{k-1})\), from (18), (19), we know the gap between \(J_{\sigma_1,\ldots,\sigma_k}\) and \(J_{\sigma_1,\ldots,\sigma_k-1}\) is

\[
\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m(\sigma_k - 1)(\sigma_k - 2)}.
\] (20)

In the same way, if \(\sigma_k + 1 \leq mh_{k-1}(\sigma_{k-1})\), from (18), (19), we know the gap between \(J_{\sigma_1,\ldots,\sigma_k}\) and \(J_{\sigma_1,\ldots,\sigma_k-1,\sigma_k+1}\) is

\[
\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{m\sigma_k(\sigma_k - 1)}.
\] (21)

For any \(0 < r < \frac{1}{m}|I_{\sigma_0}| \leq \frac{1}{m}(M_1(m), M_2(m), \ldots, M_{k_0}(m))\), since

\[
(\sigma|k_0) = (M_1(m), M_2(m), \ldots, M_{k_0}(m))
\]

and \(|I_{(\sigma|k)}| \to 0\) as \(k \to \infty\), there exists \(k\) (depends on \(x\)) such that

\[
\frac{1}{m}|I_{(\sigma|k+1)}| < r \leq \frac{1}{m}|I_{(\sigma|k)}|,
\]

that is,

\[
\frac{1}{m} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{\sigma_k + 1(\sigma_k + 1 - 1)} < r \leq \frac{1}{m} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{\sigma_k(\sigma_k - 1)}.
\] (22)

By (14), (20) and (21), \(B(x, r)\) can intersect only one \(k\)th-order interval \(J_{\sigma_1,\ldots,\sigma_k}\).

On the other hand, for every \(h_k(\sigma_k) < j \leq mh_k(\sigma_k)\), from (14), we have

\[
|I_{\sigma_1,\ldots,\sigma_j}| \geq \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{mh_k(\sigma_k)(mh_k(\sigma_k) - 1)}.
\]

Thus \(B(x, r)\) can intersect at most

\[
\frac{4r(mh_k(\sigma_k))^2}{\frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_k(\sigma_k)}{b_k(\sigma_k)}} := l.
\]
Combining (6), (16) and (23), we have thus

\[ \mu(B(x,r)) \leq \min \left\{ \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_k}), \sum_i \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_i}) \right\}, \]

where the sum is over all \( i \) such that \( \max\{\sigma_{k+1} - l, h_k(\sigma_k) + 1\} \leq i \leq \sigma_{k+1} + l \).

By (16), we have

\[ \mu(B(x,r)) \leq \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_k}) \min \left\{ 1, \sum_i \frac{h_k(\sigma_k)}{i(i-1)} S(h_k(\sigma_k)) \right\} \]

\[ \leq \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_k}) \min \left\{ 1, 2l \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right\} \]

\[ = \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_k}) \min \left\{ 1, 8r(mh_k(\sigma_k))^2 \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right\} \]

\[ \leq \mu(J_{\sigma_1,\sigma_2,\ldots,\sigma_k}) \cdot 1^{1-s} \left( 8r(mh_k(\sigma_k))^2 \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right)^s. \]

From (13), we have, for any \( n \geq k_0 \),

\[ h_n(\sigma_n) \geq a_0, \]

thus

\[ S(h_n(\sigma_n)) \geq s \quad \text{for all } n \geq k_0. \quad (23) \]

Combining (6), (16) and (23), we have

\[ \mu(B(x,r)) \leq \prod_{i=k_0}^{k-1} \frac{h_i(\sigma_i)}{\sigma_{i+1}(\sigma_{i+1} - 1)} \left( 8r(mh_k(\sigma_k))^2 \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \right)^s \]

\[ = \left( h_{k_0}(M_{k_0}(m)) \frac{a_{k_0+1}(\sigma_{k_0+1})}{b_{k_0+1}(\sigma_{k_0+1})} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \frac{1}{h_k(\sigma_k)} \right)^s \]

\[ \cdot \left( 8r(mh_k(\sigma_k))^2 \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \right)^s \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \]

\[ = \left( h_{k_0}(M_{k_0}(m)) \cdot \frac{b_1(M_1(m))}{a_1(M_1(m))} \cdots \frac{b_{k_0}(M_{k_0}(m))}{a_{k_0}(M_{k_0}(m))} \right)^s. \]
\[
8r(mh_k(\sigma_k))^2 \left( \frac{1}{h_k(\sigma_k)} \right)^s \leq c_1 \left( r \cdot h_k(\sigma_k) \left( \frac{1}{h_k(\sigma_k)} \right)^s \right),
\]

where \( c_1 \) is a positive constant which does not depend on \( x \) and \( r \).

From the definition of \( S(a) \), we have

\[
1 = \sum_{a \leq b \leq ma} \left( \frac{a}{b(b-1)} \right)^{S(a)} \geq (m-1)a \left( \frac{1}{ma \cdot ma} \right)^{S(a)} = (m-1)a \left( \frac{1}{m^2a} \right)^{S(a)},
\]

thus

\[
\frac{a}{a^{S(a)}} \leq \frac{m^{2S(a)}}{m - 1} \leq \frac{m^2}{m - 1},
\]

and this implies

\[
h_k(\sigma_k) \left( \frac{1}{h_k(\sigma_k)} \right)^{S(h_k(\sigma_k))} \leq \frac{m^2}{m - 1}.
\]

Therefore

\[
\mu(B(x, r)) \leq c_2^2 \cdot r^s, \quad (24)
\]

where \( c_2 \) is a positive constant which does not depend on \( x \) and \( r \).

From (24), we know (17) holds. This completes the proof of Theorem 2.3. \( \Box \)

From (8) and Theorem 2.3, we have

**Corollary 2.4.** Suppose \( h_j(d) \geq d - 1 \) for all \( j \geq 1 \) and \( d \geq 2 \), then for each \( m \geq 2 \), we have \( \dim_H B_m = 1 \).

**Remark 2.5.** Let \( a_n(d_1, \ldots, d_n) = 1, b_n(d_1, \ldots, d_n) = d_n(d_n - 1), \)
\((n = 1, 2, \ldots)\). Then the algorithm (1) leads to the Lüroth expansion of \( x \),

\[
x = \frac{1}{d_1(x)} + \ldots + \frac{1}{d_1(x)(d_1(x) - 1) \ldots d_{n-1}(x)(d_{n-1}(x) - 1)d_n(x)} + \ldots \quad (25)
\]
Here \( h_n(j) = 1 \) and \( T_n(x) = d_n(x) - 1 \). For the L"uroth series, with the help of the theory of self similar set, see [3], Chapter 9, the Hausdorff dimension \( s \) of the \( B_m \) is determined by the following equation

\[
\sum_{2 \leq b \leq m+1} \left( \frac{1}{b(b-1)} \right)^s = 1.
\]

To some extent, L"uroth series expansion stands as a special case to say that the assumption on \( h_j \) in the main theorem is not superfluous. Moreover, we can obtain: if \( l \leq h_j(d_j(x)) \leq L \), for all \( x \in C_m = B_{m-1} \) and \( j \) larger than some fixed integer \( k_0 \), then one can has

\[
0 < \inf_{l \leq a \leq L} S(a) \leq \dim H \{ x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1 \} \leq \sup_{l \leq a \leq L} S(a) < 1.
\]

We now list some special cases which satisfy the assumption in Theorem 2.3.

**Example 1.** Engel expansion. Let \( a_n(d_1, \ldots, d_n) = 1 \), \( b_n(d_1, \ldots, d_n) = d_n \), \( n = 1, 2, \ldots \). Then (2), together with the algorithm (1), become Engel expansion of \( x \),

\[
x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x) \cdots d_n(x)} + \cdots. \tag{26}
\]

In this case, \( h_n(j) = j - 1 \) and \( T_n(x) = \frac{d_n(x)}{d_{n-1}(x)^2} - 1 \) if \( \frac{d_n(x)}{d_{n-1}(x)^2} \) is an integer and \( \left[ \frac{d_n(x)}{d_{n-1}(x)^2} \right] \) otherwise. By Corollary 2.4, we have for each \( m \geq 2 \),

\[
\dim H \{ x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1 \} = 1.
\]

**Example 2.** Sylvester expansion. Choose \( a_n(d_1, \ldots, d_n) = 1 \), \( b_n(d_1, \ldots, d_n) = 1 \), \( n = 1, 2, \ldots \). We get the Sylvester expansion of \( x \),

\[
x = \frac{1}{d_1(x)} + \frac{1}{d_2(x)} + \cdots + \frac{1}{d_n(x)} + \cdots. \tag{27}
\]

Here \( h_n(j) = j(j-1) \) and \( T_n(x) = \frac{d_n(x)}{d_{n-1}(x) d_{n-1}(x) - 1} - 1 \) if \( \frac{d_n(x)}{d_{n-1}(x) d_{n-1}(x) - 1} \) is an integer and \( \left[ \frac{d_n(x)}{d_{n-1}(x) d_{n-1}(x) - 1} \right] \) otherwise. By Corollary 2.4, we have for each \( m \geq 2 \),

\[
\dim H \{ x \in (0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1 \} = 1.
\]
Example 3. Cantor product. Take $a_n(d_1, \ldots, d_n) = d_n + 1$, $b_n(d_1, \ldots, d_n) = d_n$, ($n = 1, 2, \ldots$), the expansion (2) yields the Cantor product,

$$1 + x = \left(1 + \frac{1}{d_1(x)}\right) \left(1 + \frac{1}{d_2(x)}\right) \ldots \left(1 + \frac{1}{d_n(x)}\right) \ldots \quad (28)$$

Here $h_n(j) = j^2 - 1$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H \{x \in [0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 4. Modified Engel expansion. Let $a_n(d_1, \ldots, d_n) = 1$, $b_n(d_1, \ldots, d_n) = d_n - 1$, ($n = 1, 2, \ldots$). We get the modified Engel expansion of $x$,

$$x = \frac{1}{d_1(x)} + \frac{1}{(d_1(x)-1)(d_2(x)-1)} \ldots \frac{1}{(d_{n-1}(x)-1)d_n(x)} \ldots \quad (29)$$

Thus $h_n(j) = j$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H \{x \in [0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Example 5. Daróczy–Kátai-Birthday expansion. Choose $a_n(d_1, \ldots, d_n) = d_n$, $b_n(d_1, \ldots, d_n) = 1$, ($n = 1, 2, \ldots$), the resulting series expansion of $x$ takes the form,

$$x = \frac{1}{d_1(x)} + \frac{d_1(x)}{d_2(x)} \ldots \frac{d_1(x)d_2(x) \ldots d_{n-1}(x)}{d_n(x)} \ldots \quad (30)$$

The Daróczy–Kátai-Birthday expansion was introduced for the first time in Galambos [9]. Here $h_n(j) = j^2(j - 1)$ and $T_n(x) = \frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)} - 1$ if $\frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)}$ is an integer and $\left[\frac{d_n(x)}{d_{n-1}(x)(d_{n-1}(x)-1)}\right]$ otherwise. By Corollary 2.4, we have for each $m \geq 2$,

$$\dim_H \{x \in [0, 1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\} = 1.$$

Remark 2.6. A modification of (1) and (3) to the algorithm $0 < x \leq 1$, $x = x_1$, and

$$\frac{1}{D_n+1} < x_n \leq \frac{1}{D_n}, \quad \frac{1}{D_n} = x_n = \frac{a_n}{b_n} \cdot x_{n+1} \quad (31)$$
A problem of Galambos on Oppenheim series expansions generates an alternating series representation
\[ x \sim \frac{1}{D_1} - \frac{a_1}{b_1} \frac{1}{D_2} + \cdots + (-1)^n \frac{a_1 a_2 \ldots a_n}{b_1 b_2 \ldots b_n} \frac{1}{D_{n+1}} + \ldots, \tag{32} \]
called alternating Oppenheim expansion. The metric theory for the alternating Oppenheim expansion was studied recently in [10]. Using the same method, we can get the corresponding results of Theorem 2.3 and Corollary 2.4 for this expansion.

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