Abstract. We prove a result concerning additive $n$-commuting maps on prime rings and then apply it to $n$-commuting linear generalized differential polynomials.

1. Results

Throughout, unless specially stated, $R$ always denotes a prime ring with center $Z$. We let $U$ be the maximal ring of right quotients of $R$ and let $Q$ stand for the symmetric Martindale quotient ring of $R$. The center $C$ of $U$ (and $Q$) is called the extended centroid of $R$. See [3] for its details. An additive map $d : R \to R$ is called a derivation if $(xy)^d = x^d y + xy^d$ for all $x, y \in R$. A map $f : R \to U$ is called $n$-commuting on a subset $S$ of $R$, where $n$ is a positive integer, if $[f(x), x^n] = 0$ for all $x \in S$. The map $f$ is merely called commuting if it is 1-commuting. The study of these mappings was initiated by Posner’s Theorem: The existence of a nonzero derivation commuting on $R$ implies the commutativity of $R$ [21, Theorem 2]. More related results have been obtained in [17]–[19], [4], [5], [13]–[6]. Also, see [11], [1], [2] for $n$-commuting maps. Applying [2, Theorem 1.1] and [1, Theorem 4.4] we have the result: Let $R$ be a prime
ring such that either \( \text{char } R = 0 \) or a prime \( p > n \), or \( \deg(R) > n \). Then every additive \( n \)-commuting map of \( R \) into \( U \) is commuting. The goal of this paper is to prove a theorem related to the result above and then apply it to some applications on \( n \)-commuting linear differential polynomials. We now state the main result:

**Theorem 1.1.** Let \( R \) be a prime ring with center \( Z \), its maximal ring of right quotients \( U \) and \( n \) a fixed positive integer. Suppose that \( f : R \to U \) is an additive \( n \)-commuting map such that \( f \) is \( Z \)-linear if \( Z \neq 0 \). Then there exist \( \lambda \in C \) and a map \( \mu : R \to C \) such that \( f(x) = \lambda x + \mu(x) \) for all \( x \in R \), unless \( R \cong M_2(\text{GF}(2)) \).

Here, \( \text{GF}(2) \) denotes the Galois field of two elements. The following gives a counterexample for the case \( R = M_2(\text{GF}(2)) \).

**Example 1.2.** Let \( R = M_2(\text{GF}(2)) \) and let \( f : R \to R \) be defined by

\[
f \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha + \gamma & 0 \\ 0 & \beta + \delta \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in R.
\]

Then \( f \) is a \( \text{GF}(2) \)-linear map. A direct computation proves that \( [f(x), x^6] = 0 \) for all \( x \in R \). However, \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \).

Hence, \( f \) is a 6-commuting linear map but it is not commuting.

We now apply Theorem 1.1 to \( n \)-commuting linear generalized differential polynomials. To state these results precisely, let us recall some notation. We denote by \( \text{Der}(U) \) the set of all derivations of \( U \). For \( d \in \text{Der}(U) \) and \( x \in U \), one usually writes \( x^d \) for \( d(x) \). Also, if \( \beta \in C \), define \( x^{d\beta} = x^d \beta \). It follows that \( \text{Der}(U) \) forms a right \( C \)-module. Let \( D \) be the \( C \)-submodule of \( \text{Der}(U) \) defined by

\[
D = \{ \delta \in \text{Der}(U) \mid I^\delta \subseteq R \text{ for some nonzero ideal } I, \text{ depending on } \delta, \text{ of } R \}.
\]

By a derivation word we mean an additive map \( \Delta \) from \( U \) into itself assuming the form \( \Delta = \delta_1 \delta_2 \ldots \delta_t \), where \( \delta_i \in D \). If \( \Delta \) is empty, we define \( x^\Delta = x \) for \( x \in U \). A linear generalized differential polynomial means a linear generalized polynomial with coefficients in \( U \) and with an indeterminate \( X \).
which are acted on by derivation words. Thus every linear generalized differential polynomial can be written in the form \( \sum_i \sum_j a_{ij}X^{\Delta_i}b_{ij} \), where \( a_{ij}, b_{ij} \in U \) and the \( \Delta_i \)'s are derivation words.

**Theorem 1.3.** Let \( R \) be a noncommutative prime ring, \( R \not\cong M_2(\mathbb{GF}(2)) \), and \( n \) a fixed positive integer. Suppose that

\[
[\psi(x), x^n] = 0
\]

for all \( x \in R \), where \( \psi(x) \) is a linear generalized differential polynomial. Then \( \psi(x) = \lambda x + \mu(x) \) for all \( x \in R \), where \( \lambda \in C \) and \( \mu : R \rightarrow C \).

**Proof.** Applying the identities (1)–(5) given in [8, p. 155], we can find finitely many distinct regular words \( \Delta_0, \Delta_1, \ldots, \Delta_t \) with \( \Delta_0 = \emptyset \) such that

\[
\psi(x) = \sum_{i=0}^t \sum_j a_{ij}x^{\Delta_i}b_{ij} \quad (1.1)
\]

for all \( x \in R \), where \( a_{ij}, b_{ij} \in U \). By assumption,

\[
\left[ \sum_{i=0}^t \sum_j a_{ij}x^{\Delta_i}b_{ij}, x^n \right] = 0 \quad (1.2)
\]

for all \( x \in R \). Applying Kharchenko’s Theorem [9, Theorem 2] to (1.2) yields

\[
\left[ \sum_{i=1}^t \sum_j a_{ij}y_ib_{ij} + \sum_j a_{0j}xb_{0j}, x^n \right] = 0 \quad (1.3)
\]

for all \( y_i, x \in R \). For \( i > 0 \) we see that \( \sum_j a_{ij}y_ib_{ij}, x^n = 0 \) for all \( x, y \in R \) and so for all \( x, y \in U \) (see [3, Theorem 6.4.1] or [7, Theorem 2]). In view of [12, Theorem], we have \( \sum_j a_{ij}y_ib_{ij}, x = 0 \) for all \( x, y \in R \). Thus \( \sum_j a_{ij}y_ib_{ij} \in C \) for all \( y \in U \). In particular, \( \sum_{i=1}^t \sum_j a_{ij}x^{\Delta_i}b_{ij} \in C \) for all \( x \in U \). Thus (1.3) is reduced to \( \sum_j a_{0j}xb_{0j}, x^n = 0 \) for all \( x \in U \). By Theorem 1.1, there exist \( \lambda \in C \) and \( \eta : R \rightarrow C \) such that \( \sum_j a_{0j}xb_{0j} = \lambda x + \eta(x) \) for all \( x \in U \). We are now done by setting \( \mu(x) = \eta(x) + \sum_{i=1}^t \sum_j a_{ij}x^{\Delta_i}b_{ij} \in C \) for all \( x \in R \). This proves the theorem. \( \square \)
A special case of Theorem 1.3 is the following

**Theorem 1.4.** Let \( R \) be a noncommutative prime ring, \( R \not\cong M_2(\text{GF}(2)) \), with a derivation \( \delta \), \( n \geq 1 \). Suppose that \( [\psi(x), x^n] = 0 \) for all \( x \in R \), where \( \psi(x) = \sum_{i=0}^{t} a_i x^{d_i} \) with \( a_i \in R \). Then \( a_0 \in \mathcal{Z} \) and \( \psi(x) = a_0 x \) for all \( x \in R \).

We remark that Park and Jung studied the case: a derivation \( d \) on an \( n! \)-torsion-free semiprime ring \( R \) such that \( d^2 \) is \( n \)-commuting on \( R \), where \( n \geq 2 \) [20, Theorem 3.1]. Applying the theory of orthogonal completion for semiprime rings (see [3]), [20, Theorem 3.1] can be reduced to the prime case and so can be solved as a special case of Theorem 1.4. To prove it we first quote Chang’s Theorem [6, Theorem 3.2]:

**Theorem 1.5** *(Chang [6]).* Let \( R \) be a noncommutative prime ring with a derivation \( d \). Suppose that \( \sum_{i=1}^{n} a_i x^{d_i} \in \mathcal{Z} \), where \( a_i \in R \). Then \( a_0 = 0 \) and \( \sum_{i=0}^{n} a_i x^{d_i} = 0 \) for all \( x \in U \).

Before giving the proof of Theorem 1.4 we need the following generalization of Theorem 1.5

**Theorem 1.6.** Let \( R \) be a noncommutative prime ring with a derivation \( d \). Suppose that \( \sum_{i=0}^{n} a_i x^{d_i} \in \mathcal{Z} \), where \( a_i \in R \). Then \( a_0 = 0 \) and \( \sum_{i=0}^{n} a_i x^{d_i} = 0 \) for all \( x \in R \).

**Proof.** In view of Theorem 1.5, it is enough to show that \( a_0 = 0 \). Obviously we can assume that \( d \neq 0 \). We set \( \phi(x) = \sum_{i=0}^{n} a_i x^{d_i} \) for \( x \in U \), and note that \( [\phi(x), y] = 0 \) for all \( x, y \in R \). According to [10, Theorem 2], \( [\phi(x), y] = 0 \) for all \( x, y \in U \) and so \( \phi(x) \in C \) for all \( x \in U \). In particular, \( a_0 = \phi(1) \in C \). Suppose that \( a_0 \neq 0 \). Replacing \( \phi(x) \) with \( a_0^{-1} \phi(x) \) we reduce the proof to the case when \( a_0 = 1 \). The aim is to derive a contradiction.

Given \( x, y \in U \), it follows directly from Leibniz’s rule that \( \phi(yx) = \sum_{i=1}^{n} b_i x^{d_i} + \phi(y)x \) for some \( b_i \in U \), depending on \( y \). Therefore

\[
\sum_{i=1}^{n} (b_i - \phi(y)a_i) x^{d_i} = \phi(yx) - \phi(y)\phi(x) \in C
\]

for all \( x, y \in U \). Theorem 1.5 now yields that \( \phi(yx) = \phi(y)\phi(x) \) for all \( x, y \in U \). Therefore \( \phi : U \to C \) is a ring homomorphism. Next,
\[ \sum_{i=0}^{n} a_i x^{d^{i+1}} = \phi(x^d) \in C \] and so Theorem 1.5 yields that \( x^d \in \ker(\phi) \).

Since \( d \neq 0 \), \( \ker(\phi) \neq 0 \) as well. We see that \( \ker(\phi) \) is a nonzero ideal of \( U \) and \( \phi \) is a generalized differential polynomial identity on \( \ker(\phi) \). Therefore [10, Theorem 2] implies that \( \phi(x) = 0 \) for all \( x \in U \). In particular, \( 1 = \phi(1) = 0 \), a contradiction. The proof is now complete. \( \square \)

Proof of Theorem 1.4. In view of Theorem 1.3, \( \psi(x) = \lambda x + \mu(x) \) for all \( x \in R \), where \( \lambda \in C \) and \( \mu : R \to C \). That is, \( \sum_{i=0}^{t} a_i x^{\delta_i} - \lambda x \in C \) for all \( x \in R \) and so for all \( x \in U \) [10, Theorem 2]. In view of Theorem 1.6, \( \sum_{i=0}^{t} a_i x^{\delta_i} - \lambda x = 0 \) for all \( x \in U \). In particular, we set \( x = 1 \) to get \( a_0 = \lambda \in Z \), and hence \( \sum_{i=1}^{t} a_i x^{\delta_i} = 0 \) for all \( x \in U \). Thus \( \psi(x) = a_0 x \) for all \( x \in R \). This proves the theorem. \( \square \)

2. Proof of Theorem 1.1

We begin with the following special case.

Lemma 2.1. Theorem 1.1 holds if \( R = M_m(C) \), the \( m \times m \) matrix ring over a field \( C \), unless \( m = 2 \) and \( C = GF(2) \).

Proof. For \( n = 1 \) we are done by Brešar’s Theorem [4, Theorem A]. Therefore, we always assume \( n > 1 \). Let \( \{e_{ij} \mid 1 \leq i \leq j \leq m\} \) be the set of usual matrix units of \( R \). The aim is to prove that there exists \( \lambda \in C \) such that \( f(e_{ij}) - \lambda e_{ij} \in C \) for all \( 1 \leq i, j \leq m \). Indeed, we then have \( f(x) - \lambda x \in C \) for all \( x \in R \) as \( f \) is \( C \)-linear. Hence, the lemma is proved by setting \( \mu(x) = f(x) - \lambda x \in C \) for \( x \in R \).

For \( m \geq 3 \) we claim that

\[ [f(u), e] = 0 \quad \text{if} \quad u^2 = eu = ue = 0 \quad \text{and} \quad e = e^2 \quad \text{for} \quad e, u \in R. \quad (2.1) \]

Indeed, \( (e + u)^n = e \) since \( n > 1 \). Thus, by assumption, \( 0 = [f(e + u), (e + u)^n] = [f(e) + f(u), e] = [f(u), e] \), as desired. We claim that there exist \( \lambda_{ij} \in C \) such that

\[ f(e_{ij}) - \lambda_{ij} e_{ij} \in C \quad \text{and so} \quad [f(e_{ij}), e_{ij}] = 0 \quad (2.2) \]

for \( i \neq j \). Let \( 1 \leq p \leq m \) be distinct from \( i, j \). Note that \( e_{pp}^2 = e_{pp} \) and \( e_{ij}^2 = 0 = e_{pp} e_{ij} = e_{ij} e_{pp} \). Thus, by (2.1), \( [f(e_{ij}), e_{pp}] = 0 \) follows. Write
Applying (2.3) and we obtain that (2.1) we have

\[ e \]

the other hand, the idempotent \( e \) of (2.4), we get

\[ \text{diagonal, that is, } e \]

Right-multiplying by \( e \), we have that \( e \) is an idempotent where

Since the idempotent \( e_{ip} + e_{pp} \) satisfies \( e_{ij}(e_{ip} + e_{pp}) = 0 = (e_{ip} + e_{pp})e_{ij} \), by (2.1) we have \([f(e_{ij}), e_{ip} + e_{pp}] = 0 \) and so \([f(e_{ij}), e_{ip}] = 0 \). Applying (2.3) we obtain that \( \alpha_{ii}e_{ip} + \alpha_{ji}e_{jp} = \alpha_{pp}e_{ip} \). Hence, \( \alpha_{ji} = 0 \) and \( \alpha_{ii} = \alpha_{pp} \). On the other hand, the idempotent \( e_{pj} + e_{pp} \) satisfies \( e_{ij}(e_{pj} + e_{pp}) = (e_{pj} + e_{pp})e_{ij} = 0 \). By (2.1) again, \([f(e_{ij}), e_{pj} + e_{pp}] = 0 \) and so \([f(e_{ij}), e_{pj}] = 0 \). Applying (2.3) and \( \alpha_{ji} = 0 \) we obtain that \( \alpha_{pp}e_{pj} = \alpha_{jj}e_{pj} \) and so \( \alpha_{pp} = \alpha_{jj} \). This implies that \( f(e_{ij}) = \alpha_{ij}e_{ij} \in C \). Set \( \lambda_{ij} = \alpha_{ij} \in C \). In particular, \([f(e_{ij}), e_{ij}] = 0 \). This proves (2.2).

Next, we write \( f(e_{ii}) = \sum_{s,t} \alpha_{st}e_{st} \), where \( \beta_{st} \in C \). By assumption, we have \([f(e_{ii}), e_{ii}] = 0 \). This implies \( f(e_{ii})e_{ii} = e_{ii}f(e_{ii}) \) and so \( e_{ii}f(e_{ii})e_{pp} = 0 \) for all \( p \neq i \). Hence \( \beta_{ipp} = 0 \). Using the fact that \( e_{ii} + e_{ij} \) is an idempotent where \( j \neq i \), we have that

\[ 0 = [f(e_{ii} + e_{ij}), e_{ii} + e_{ij}] = [f(e_{ii}), e_{ij}] + [f(e_{ij}), e_{ii}] \]

Note that \( f(e_{ij}) - \lambda_{ij}e_{ij} \in C \). This implies that

\[ [f(e_{ii}), e_{ij}] + [\lambda_{ij}e_{ij}, e_{ii}] = 0. \]

Right-multiplying by \( e_{pp} \) where \( p \neq j \), we see that \( \beta_{ipp} = 0 \) and so \( f(e_{ii}) \) is diagonal, that is, \( \beta_{ist} = 0 \) for \( s \neq t \) and so \( f(e_{ii}) = \sum_{t=1}^{m} \beta_{itt}e_{tt} \). Making use of (2.4), we get \( \sum_{t=1}^{m} \beta_{tt}e_{tt}, e_{ii}] + [\lambda_{ij}e_{ij}, e_{ii}] = 0 \) and so \( \beta_{iii} = \beta_{ijj} + \lambda_{ij} \). Let \( 1 \leq k \leq m \) be such that \( k \neq i,j \). By assumption,

\[ 0 = [f(e_{ii} + e_{kj} + e_{ji}), (e_{ii} + e_{kj} + e_{ji})^{n}] \]

\[ = \left[ \sum_{t=1}^{n} \beta_{tt}e_{tt} + \lambda_{kk}e_{kk} + \lambda_{jj}e_{jj}, e_{ii} + e_{ki} + e_{ji} \right] \]

\[ = (\beta_{ikk} - \beta_{iii} + \lambda_{kk})e_{ki} + (\beta_{ijj} - \beta_{iii} + \lambda_{ij})e_{ji}, \]

since \( n > 1 \). This implies that \( (\lambda_{kk} - \lambda_{ik})e_{ki} + (\lambda_{jj} - \lambda_{ij})e_{ji} = 0 \), since \( \beta_{iii} - \beta_{ikk} = \lambda_{ik} \). That is, \( \lambda_{ji} = \lambda_{ij} \) and \( \lambda_{kj} = \lambda_{jk} = \lambda_{ki} \). So \( \beta_{iii} = \)}
Let Posner’s Theorem for prime PI-rings, suppose that

\[ 0 = \lambda_{ij} + \lambda_{ij} = \beta_{kk} + \lambda_{ik}. \]

But \( \lambda_{ik} = \lambda_{jk} = \lambda_{kj} = \lambda_{ij} \), this implies that

\[ \beta_{ij} + \lambda_{ij} = \beta_{kk} \]

and so

\[ f(e_{ii}) - \lambda_{ij}e_{ii} = \sum_{s=1}^{m} \beta_{ss}e_{ss} - \lambda_{ij}e_{ii} = \beta_{jj} \sum_{s=1}^{m} e_{ss} \in C. \]

We let \( \lambda = \lambda_{ij} \in C \). Then \( f(e_{st}) - \lambda e_{st} \in C \) for \( 1 \leq s, t \leq m \).

We assume next that \( m = 2 \). By assumption, we have \([f(e_{11}), e_{11}] = 0\), implying that \( f(e_{11}) = \alpha e_{11} + \beta e_{22} \) for some \( \alpha, \beta \in C \). Setting \( \lambda_{11} = \alpha - \beta \) we have \( f(e_{11}) - \lambda_{11}e_{11} \in C \). Analogously, \( f(e_{22}) - \lambda_{22}e_{22} = 0 \) for some \( \lambda_{22} \in C \). As \( |C| > 2 \), there exists \( \alpha \in C \) with \( \alpha \neq 0, 1 \). Note that \( e_{11} + e_{12} \) and \( e_{11} + \alpha e_{12} \) are two idempotents. Thus \([f(e_{11} + e_{12}), e_{11} + e_{12}] = 0\) and \([f(e_{11} + \alpha e_{12}), e_{11} + \alpha e_{12}] = 0\). Since \( f \) is \( C \)-linear and \( \alpha \neq 0, 1 \), this implies \([f(e_{12}), e_{12}] = 0\). So \( f(e_{12}) - \lambda_{12}e_{12} = 0 \) for some \( \lambda_{12} \in C \). Analogously, \( f(e_{21}) - \lambda_{21}e_{21} = 0 \) for some \( \lambda_{21} \in C \). On the other hand, 

\[ 0 = [f(e_{11} + e_{12}), e_{11} + e_{12}] = [f(e_{11}), e_{11}] + [f(e_{12}), e_{11}] = [\lambda_{11}e_{11}, e_{11}] + [\lambda_{12}e_{12}, e_{11}] = (\lambda_{11} - \lambda_{12})e_{12}, \]

implying that \( \lambda_{11} = \lambda_{12} \). It follows from an analogous argument that \( \lambda_{12} = \lambda_{22} \) and \( \lambda_{11} = \lambda_{21} \). Set \( \lambda = \lambda_{11} \). We see that \( f(e_{ij}) - \lambda e_{ij} \in C \) for \( i, j = 1, 2 \). This proves the lemma.

\( \square \)

**Lemma 2.2.** Let \( R \) be a prime PI-ring with center \( Z \). Then every \( Z \)-linear map from \( R \) into \( RC \) is defined by a linear generalized polynomial with coefficients in \( RC \).

**Proof.** By Posner’s Theorem for prime PI-rings, \( RC \) is a finite-dimensional central simple \( C \)-algebra. Moreover, \( Z \neq 0 \) [22, Theorem 2.10] and \( C \) is the quotient field of \( Z \). Suppose that \( f : R \rightarrow RC \) is a \( Z \)-linear map. Then it is obvious that \( f \) is uniquely extended to a \( C \)-linear map from \( RC \) into \( RC \). Note that \( RC \otimes_C RC^o \cong \text{End}_C(RC) \) via a canonical map \( \phi \), defined by \( \phi(\sum_i a_i \otimes b_i^o)(x) = \sum_i a_i x b_i \) for \( x \in RC \), where \( RC^o \) denotes the ring opposite to \( RC \). Thus there exist \( a_i, b_i \in RC \) such that \( f = \phi(\sum_i a_i \otimes b_i^o) \). That is, \( f(x) = \sum_i a_i x b_i \) for all \( x \in R \), proving the lemma.

\( \square \)

**Lemma 2.3.** If \( xa - bx \in C \) for all \( x \in R \), where \( a, b \in U \), then either \( R \) is commutative or \( a = b \in C \).

**Proof.** Suppose that \( R \) is not commutative. Choose a dense right ideal \( \rho \) of \( R \) such that \( bp \subseteq R \). Let \( y \in \rho \). Then \( by \in R \) and so \((by)a -
b(by) ∈ C. That is, b(ya − by) ∈ C. Since ya − by ∈ C, either b ∈ C or ya = by. If b ∈ C, then R(a − b) ⊆ C, implying that a = b since R is not commutative. Suppose next that ya = by for all y ∈ ρ. In view of [7, Theorem 2], ya = by for all y ∈ U. In particular, set y = 1. Then a = b follows. So [a, R] ⊆ C, implying a ∈ C again. This proves the lemma.

We are now ready to the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that R ̸∼ M2(GF(2)). By assumption, we have [f(x), x^n] = 0 for all x ∈ R. Suppose first that R is not a PI-ring. Then deg(R) = ∞ in the sense of [1]. In view of [1, Theorem 4.4], there exist a, b ∈ U and maps μ, ν : R → C such that f(x) = xa + μ1(x) = bx + ν2(x) for all x ∈ R. Thus xa − bx ∈ C for all x ∈ R. It follows from Lemma 2.3 that either R is commutative or a = b ∈ C. Since R is not a PI-ring, R is not commutative. So a = b ∈ C. We are done in this case by setting λ = a ∈ C.

Suppose next that R is a PI-ring. Then Z ̸∼ 0 [22, Theorem 2.10]. By assumption, f is a Z-linear map. In view of Lemma 2.2, there exist finitely many a_i, b_i ∈ RC such that f(x) = ∑ a_i x b_i for all x ∈ R. By assumption, we see that

\[
\left[ ∑ a_i x b_i, x^n \right] = 0 \tag{2.5}
\]

for all x ∈ R and hence for all x ∈ RC ([3, Theorem 6.4.1] or [7, Theorem 2]). Define F to be the algebraic closure of C if C is infinite. Otherwise, let F = C. Then (2.5) holds for all x ∈ RC ⊗ C F. Note that x ∈ RC ⊗ C F ∼= M_m(F) for some m ≥ 1. Define g : RC ⊗ C F → RC ⊗ C F by g(x) = ∑ a_i x b_i for all x ∈ RC ⊗ C F. Then, by Lemma 2.1, there exist c ∈ F and ν : RC ⊗ C F → F such that g(x) = cx + ν(x) for all x ∈ RC ⊗ C F. Choose a basis \{β_1, β_2, \ldots\} of F over C with β_1 = 1. Write c = λ β_1 + ∑_{j=2}^s λ_j β_j for some s ≥ 1 and λ, λ_j ∈ C. Set μ(x) = g(x) − cx for x ∈ RC. Then μ(x) ∈ C for x ∈ RC. Note that f(x) = g(x) for all x ∈ R. Thus we see that f(x) = λ x + μ(x) for all x ∈ R, proving the theorem.

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