Differentiable loops on the real line

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Dedicated to Professor Adalbert Bovdi on his 70th birthday

Abstract. The paper is devoted to the study of differentiable loops $L$ on the real line such that the group $G$ topologically generated by the left translations is locally compact and hence it is isomorphic to the universal covering group of $\text{PSL}_2(\mathbb{R})$. Using the methods developed in [3] we introduce a class of natural parametrizations of the loop manifold $L$ corresponding to the Iwasawa decompositions of $G$ and find explicit expressions for the loop multiplication with respect to the given parametrizations. We characterize the differentiable curves $\mathbb{R} \to G$ consisting of the left translations of a loop $L$ in the biinvariant Lorentzian geometry of $G$.

1. Introduction

An algebra $(L, \circ, e)$ with a binary operation $(x, y) \mapsto x \circ y : L \times L \to L$ and unit element $e \in L$ is called a left loop if for any given $a, b \in L$ the equation $a \circ x = b$ is uniquely solvable for $x$, and if $e \in L$ satisfies $e \circ x = x \circ e = x$ for any $x \in L$. The left loop $(L, \circ, e)$ is called a loop if for any given $a, b \in L$ the equation $x \circ a = b$ is uniquely solvable, too. The operation $\circ : L \times L \to L$ is called multiplication, the solution of the equation $a \circ x = b$ is denoted by $a \backslash b$, the solution of the equation $x \circ a = b$ is denoted by $a/b$. The binary left division operation $(a, b) \mapsto a \backslash b$ can be defined in a left loop, the binary right division operation $(a, b) \mapsto a/b$ can be defined in a loop. The bijective maps $\lambda_x : L \to L : y \mapsto x \circ y$, $x \in L$, are called left translations of $(L, \circ, e)$. The group $G$ generated by the

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set $\Lambda = \{\lambda x, x \in L\}$ of left translations acts transitively on $L$. Hence there is a bijective correspondence $L \to G/H : x \mapsto \lambda_x H$, where $H$ is the stabilizer of $e \in L$ in the group $G$.

A loop $(L, \circ, e)$ is called differentiable of class $C^r$ if $L$ is a differentiable manifold of class $C^r$ and the multiplication $\circ$, the left and the right divisions of $(L, \circ)$ are $C^r$-differentiable maps $L \times L \to L$. In the study of the Lie theory of differentiable loops is very fruitful the investigation of the differentiable submanifold $\Lambda = \{\lambda x, x \in L\} \subset G$ of the group topologically generated by the left translations (cf. [3]), the mapping $x \mapsto \lambda_x : L \to G$ corresponds to a section with respect to the projection map $G \to G/H$.

Differentiable loops defined on the unit circle having a locally compact group $G$ as the group topologically generated by the left translations, were studied in Section 18 of the monograph [3] by P. T. Nagy and Karl Strambach. Particularly they proved the following assertion (Proposition 18.2, p. 235.) by an application of a theorem of Brouwer [1]:

Let $L$ be a topological proper loop on a connected 1-manifold, such that the group $G$ topologically generated by the left translations is locally compact. Then $G$ is a finite covering of the group $PSL_2(\mathbb{R})$ if $L$ is a circle, and $G$ is the universal covering of $PSL_2(\mathbb{R})$ if $L$ is homeomorphic to the real line $\mathbb{R}$.

Section 18 of [3] contains a detailed investigation of differentiable loops on the circle for which the group $G$ topologically generated by the left translations is isomorphic to the group $PSL_2(\mathbb{R})$. The corresponding loop multiplication were determined by a pair of real periodic differentiable functions satisfying the conditions some differential inequality. The investigation of differentiable loops on the circle is extended to the case where the group $G$ is isomorphic to $SL_2(\mathbb{R})$ in [4]. This paper is devoted to the study of differentiable proper loops on the real line $\mathbb{R}$ such that the group $G$ topologically generated by the left translations is locally compact. In this case the group $G$ is isomorphic to the universal covering group of $PSL_2(\mathbb{R})$. Modifying the description of [3] we give explicit expressions of the loop multiplications with respect to a class of natural parametrizations of the loop manifold $L$ corresponding to the Iwasawa decompositions of $G$. We characterize within the biinvariant Lorentzian geometry of the group $G$ the one-dimensional differentiable submanifolds $\Lambda = \{\lambda x, x \in \mathbb{R}\}$ of $PSL_2(\mathbb{R})$, the point of which are the left translations of a differentiable loop on the real line $\mathbb{R}$. 


2. Action of $\widetilde{PSL}_2(\mathbb{R})$ on $\mathbb{R}$

Let $\mathcal{A}$ be the algebra of real $2 \times 2$ matrices. We denote by $\widetilde{PSL}_2(\mathbb{R})$ the universal covering group of the groups

$$SL_2(\mathbb{R}) = \{ A \in \mathcal{A}; \det A = 1 \}, \quad PSL_2(\mathbb{R}) = \{ \pm A \in \mathcal{A}; \det A = 1 \}.$$

The matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. Let $\kappa : X \mapsto \frac{1}{2} \text{Trace}(X^2)$ be the normalized Cartan–Killing form on $\mathfrak{sl}_2(\mathbb{R})$ given by $\kappa(X) = -\det(X) = h^2 + t^2 - u^2$ for $X = hH + tT + uU$, which determines a pseudo-euclidean vector space structure on $\mathfrak{sl}_2(\mathbb{R})$ such that the unit vectors $\mathbf{H}$, $\mathbf{T}$, $\mathbf{U}$ form a pseudo-orthonormal basis. We denote by $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ the exponential map of $SL_2(\mathbb{R})$ and by $\tilde{\exp} : \mathfrak{sl}_2(\mathbb{R}) \to \widetilde{PSL}_2(\mathbb{R})$ the exponential map of the universal covering group $\widetilde{PSL}_2(\mathbb{R})$. Let $p : PSL_2(\mathbb{R}) \to SL_2(\mathbb{R})$ be the covering map.

The connected 1-parameter subgroups $\{ \exp(tX); t \in \mathbb{R} \}$ with $X \neq 0$ are called elliptic, hyperbolic or parabolic, if $\kappa(X) < 0$, $\kappa(X) > 0$ or $\kappa(X) = 0$, respectively. The covering map $p$ induces isomorphisms $\{ \tilde{\exp}(tX); t \in \mathbb{R} \} \to \{ \exp(tX); t \in \mathbb{R} \}$ between hyperbolic or parabolic 1-parameter subgroups, it yields proper covering homomorphisms for elliptic 1-parameter subgroups. The connected 2-dimensional subgroups of $\widetilde{PSL}_2(\mathbb{R})$ have the form $\{ \tilde{\exp}(s(Y - Z)) \cdot \tilde{\exp}(tX); s, t \in \mathbb{R} \}$, where $(X, Y, Z)$ is a pseudo-orthonormal basis of $(\mathfrak{sl}_2(\mathbb{R}), \kappa)$, which is conjugate to the basis $(\mathbf{H}, \mathbf{T}, \mathbf{U})$ of $\mathfrak{sl}_2(\mathbb{R})$. Clearly, the covering map $p$ induces isomorphisms

$$\{ \tilde{\exp}(s(Y - Z)) \cdot \tilde{\exp}(tX); s, t \in \mathbb{R} \} \to \{ \exp(s(Y - Z)) \exp(tX); s, t \in \mathbb{R} \}$$

between 2-dimensional subgroups.

The pseudo-euclidean scalar product induced by the Cartan–Killing form on $\mathfrak{sl}_2(\mathbb{R})$ is invariant with respect to the adjoint representation of $\widetilde{PSL}_2(\mathbb{R})$. Hence it can be extended by left or right translations of $\widetilde{PSL}_2(\mathbb{R})$ to a bi-invariant pseudo-riemannian metric $g(X, Y)$ on the manifold $\widetilde{PSL}_2(\mathbb{R})$ such that $g(X, X)$ at the identity element of $\widetilde{PSL}_2(\mathbb{R})$ coincides with $\kappa(X)$. A curve $t \mapsto \gamma(t) : \mathbb{R} \to \widetilde{PSL}_2(\mathbb{R})$ will be called time-like if its tangent vectors $\dot{\gamma}(t)$ satisfy $g(\dot{\gamma}(t), \dot{\gamma}(t)) < 0$ for any $t \in \mathbb{R}$.
We consider the group $G = \overline{PSL}_2(\mathbb{R})$, a 2-dimensional subgroup $H \subset G$ and the homogeneous space $G/H$ diffeomorphic to $\mathbb{R}$. Let $(X, Y, Z)$ be a pseudo-orthonormal basis of $\mathfrak{s}l_2(\mathbb{R})$, for which there exists an automorphism $\beta : \mathfrak{s}l_2(\mathbb{R}) \rightarrow \mathfrak{s}l_2(\mathbb{R})$ mapping the basis $(X, Y, Z)$ onto the canonical basis $(H, T, U)$, such that

$$H = \{ e^{\mathfrak{sp}(s(Y - Z))} \cdot e^{\mathfrak{sp}(tX)} ; s, t \in \mathbb{R} \} \subset G.$$ 

Let $K, N$ and $D$ denote the 1-parameter subgroups $K = \{ e^{\mathfrak{sp}(tZ)} ; t \in \mathbb{R} \}$, $N = \{ e^{\mathfrak{sp}(t(Y - Z))} ; t \in \mathbb{R} \}$ and $D = \{ e^{\mathfrak{sp}(tX)} ; t \in \mathbb{R} \}$. Since the vector space direct sum $\mathfrak{s}l_2(\mathbb{R}) = \mathbb{R}Z + \mathbb{R}(Y - Z) + \mathbb{R}X$ is an Iwasawa decomposition of the Lie algebra $\mathfrak{s}l_2(\mathbb{R})$, (cf. e.g. [2] p. 234), the map

$$\Delta : g \mapsto (k, n, d) : \overline{PSL}_2(\mathbb{R}) \rightarrow K \times N \times D \quad (1)$$

defined by $g = knd$ is an analytic diffeomorphism of $\overline{PSL}_2(\mathbb{R})$ onto the product manifold $K \times N \times D$. The triple $\Delta(g) = (k, n, d)$ will be called the Iwasawa decomposition of the element $g \in \overline{PSL}_2(\mathbb{R})$. It follows that any element $g \in \overline{PSL}_2(\mathbb{R})$ can be decomposed uniquely into the product $g = e^{\mathfrak{sp}(tZ)} e^{\mathfrak{sp}(u(Y - Z))} e^{\mathfrak{sp}(vX)}$ with $(t, u, v) \in \mathbb{R}^3$. Using the decomposition $G = K \cdot N \cdot D$ and $H = N \cdot D$ we obtain that any coset of the subgroup $H = N \cdot D$ can be represented uniquely by an element $e^{\mathfrak{sp}(tZ)} \in K$, $t \in \mathbb{R}$. Hence the manifold $G/H$ can be parametrized by elements of $K$. Since the mapping $t \mapsto e^{\mathfrak{sp}(tZ)} : \mathbb{R} \rightarrow K$ is bijective, we obtain that

$$\phi : t \mapsto e^{\mathfrak{sp}(tZ)}H : \mathbb{R} \rightarrow G/H \quad (2)$$

is a diffeomorphism and hence this map gives a global real parametrization of the loop manifold which will be called an Iwasawa parametrization. This parametrization $\phi$ is not canonical in the sense that it depends on the choice of the basis $(X, Y, Z)$ of $\mathfrak{s}l_2(\mathbb{R})$.

We can define the action $\alpha_g : \mathbb{R} \rightarrow \mathbb{R}$ of $g \in G$ by

$$\alpha_g t = \phi^{-1}(g \cdot e^{\mathfrak{sp}(tZ)}H) = \phi^{-1}(g \cdot \phi t),$$

where $kH \rightarrow g \cdot kH : G/H \rightarrow G/H$ denotes the natural action of $G$ on the factor space $G/H$. Hence $\alpha : G \times \mathbb{R} \rightarrow \mathbb{R}$ is a natural differentiable transformation group action of $G = \overline{PSL}_2(\mathbb{R})$ on $\mathbb{R}$.

**Proposition 1.** The mapping $\alpha_g : \mathbb{R} \rightarrow \mathbb{R}$ with $g = e^{\mathfrak{sp}(tZ)} \cdot e^{\mathfrak{sp}(u(Y - Z))} \cdot e^{\mathfrak{sp}(vX)}$ with respect to the Iwasawa parametrization $(2)$ can be expressed by

$$\alpha_g s = \begin{cases} t + \arccot \left( 2u + e^{-2v \cot s} \right) + i \pi, & \text{if } i \pi < s < (i + 1) \pi, \\ t + i \pi, & \text{if } s = i \pi, \end{cases}$$

for any $i \in \mathbb{Z}$. In particular, $\alpha_{e^{\mathfrak{sp}(tZ)}} s = t + s$. 

Differentiable loops on the real line

**Proof.** The equation \( \alpha \exp(tZ)s = t + s \) follows from the relation

\[
\tilde{\exp}(tZ) \cdot \tilde{\exp}(sZ) \cdot H = \tilde{\exp}(t + s)Z \cdot H.
\]

If \( t = 0 \) the element \( \alpha g_s \) is determined by

\[
\tilde{\exp}(u(Y - Z)) \cdot \tilde{\exp}(vX) \cdot \tilde{\exp}(sZ) \cdot H = \tilde{\exp}(\alpha g_sZ) \cdot H.
\]

Since there exists an automorphism \( \beta : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{sl}_2(\mathbb{R}) \) mapping the basis \( (X, Y, Z) \) onto the canonical basis \( (H, T, U) \) of \( \mathfrak{sl}_2(\mathbb{R}) \) this equation is equivalent to

\[
\tilde{\exp}(u(T - U)) \cdot \tilde{\exp}(vH) \cdot \tilde{\exp}(sU) \cdot \tilde{H} = \tilde{\exp}(\alpha g_sU) \cdot \tilde{H},
\]

where \( \tilde{H} = \{ \tilde{\exp}(s(T - U)) : \tilde{\exp}(tH) : s, t \in \mathbb{R} \} \subset G \). Applying the covering homomorphism \( p : \tilde{PSL}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R}) \) we obtain

\[
\begin{pmatrix}
  e^v & 0 \\
 2ue^v & e^{-v}
\end{pmatrix}
\begin{pmatrix}
  \cos s & \sin s \\
 -\sin s & \cos s
\end{pmatrix}
p(\tilde{H}) =
\begin{pmatrix}
  \cos \alpha g_s & \sin \alpha g_s \\
 -\sin \alpha g_s & \cos \alpha g_s
\end{pmatrix}
p(\tilde{H})
\]

or equivalently

\[
\begin{pmatrix}
  e^v & 0 \\
 2ue^v & e^{-v}
\end{pmatrix}
\begin{pmatrix}
  \cos s & \sin s \\
 -\sin s & \cos s
\end{pmatrix} =
\begin{pmatrix}
  \cos \alpha g_s & \sin \alpha g_s \\
 -\sin \alpha g_s & \cos \alpha g_s
\end{pmatrix}
\begin{pmatrix}
  e^x & 0 \\
 2ye^x & e^{-x}
\end{pmatrix}
\]

for some \( x, y \in \mathbb{R} \). Comparing the second columns of the product matrices we obtain

\[
\cot \alpha g_s = \frac{2u e^v \sin s + e^{-v} \cos s}{e^v \sin s} = 2u + e^{-2v} \cot s.
\]

Since the map \( \alpha_g : \mathbb{R} \rightarrow \mathbb{R} \) is a bijection and hence it is monotone. The shape of the last expression shows that \( \alpha_g \) is monotone increasing. The assertion follows from this and from \( \alpha_g 0 = 0 \).

\[\square\]

3. The left multiplication curve

We consider a differentiable left loop defined on a one-manifold and assume that the group topologically generated by the left translations is isomorphic to \( G = \widehat{PSL}_2(\mathbb{R}) \). Since \( G \) acts transitively, the left loop manifold may be identified with the factor space \( G/H \), where the 2-dimensional subgroup \( H \) of \( G \) is the identity of the left loop \((G/H, \circ, H)\). Let \( \pi : G \rightarrow G/H \) be the canonical projection map and let \( \phi \) be the parametrization \( \mathbb{R} \rightarrow G/H \) introduced in equation (2). We
The left multiplication of the associated multiplication \((\delta, (0) = 0)\) and \(x: y = \phi^{-1}\sigma_x\phi(y): \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is called the multiplication associated with the left multiplication curve \(\sigma\) with respect to the Iwasawa parametrization (2).

**Definition 2.** A differentiable curve \(\sigma: x \mapsto \sigma_x: \mathbb{R} \to G\) is called a left multiplication curve on \(G\) if \(\sigma_x\) is contained in \(\phi(x) \in G/H\) for any \(x \in \mathbb{R}\). The mapping \((x, y) \mapsto x \cdot y = \phi^{-1}\sigma_x\phi(y): \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is called the multiplication associated with the left multiplication curve \(\sigma\) on \(G/H\) as the mapping given by \(\phi\lambda_x\phi^{-1}\).

The multiplication \((x, y) \mapsto x \cdot y\) on \(\mathbb{R}\), associated with a left multiplication curve \(\sigma\) determines a differentiable loop \((\mathbb{R}, 0, 0)\) which has 0 as unit element. One can see immediately, that the left multiplication maps of this loop topologically generate a group which is isomorphic to \(G\).

Now, let \((G/H, \circ, H)\) be a differentiable loop having \(G = \widetilde{PSL}_2(\mathbb{R})\) as the group topologically generated by the left translations. The mapping \(x \mapsto \sigma_x: \mathbb{R} \to G\) satisfying \(\lambda_x = \alpha_{\sigma_x}\) determines a left multiplication curve on \(G\), and the multiplication associated with \(\sigma\) is isomorphic to the multiplication of the left loop \((G/H, \circ, H)\). According to \(G = K \cdot N \cdot D\) we have the decomposition

\[
\sigma_x = \overline{\exp}(xZ) \cdot \overline{\exp}(\nu(x)(Y - Z)) \cdot \overline{\exp}(\delta(x)X),
\]

where \(\nu(x)\) and \(\delta(x)\) are real differentiable functions having the initial value \(\nu(0) = \delta(0) = 0\). The left translations of the associated multiplication \((x, y) \mapsto x \cdot y\) have the shape \(\lambda_x = \alpha_{\sigma_x}\). If the left multiplication curve is not a one-parameter subgroup of \(G\) then the group topologically generated by the left translations is isomorphic to \(G = \widetilde{PSL}_2(\mathbb{R})\).

We obtain from Proposition 1 the following

**Proposition 3.** Let \(\sigma: x \mapsto \sigma_x: \mathbb{R} \to \widetilde{PSL}_2(\mathbb{R})\) be a left multiplication curve of the shape \(\sigma_x = \overline{\exp}(xZ) \cdot \overline{\exp}(\nu(x)(Y - Z)) \cdot \overline{\exp}(\delta(x)X)\), where \(\nu(x)\) and \(\delta(x)\) are differentiable functions having the initial value \(\nu(0) = \delta(0) = 0\). The multiplication \((x, y) \mapsto x \cdot y\) associated with the left multiplication curve \(\sigma\) with respect to the Iwasawa parametrization (2) can be expressed by

\[
x \cdot y = \begin{cases} 
x + \arccot (2\nu(x) + e^{-2\nu(x)} \cot y) + i\pi, & \text{if } i\pi < y < (i + 1)\pi, \\
x + i\pi, & \text{if } y = i\pi,
\end{cases}
\]

for any \(i \in \mathbb{Z}\).
Now, we want characterize the loop multiplications among the differentiable left loop multiplications described in the previous proposition. Clearly, if \((\mathbb{R}, \cdot, 0)\) is a loop, then for any \(y \in \mathbb{R}\) the right multiplication map \(\rho_y : \mathbb{R} \to \mathbb{R}\) is a diffeomorphism. It follows that the map \(\rho_y : \mathbb{R} \to \mathbb{R}\) and its inverse \(\rho_y^{-1} : \mathbb{R} \to \mathbb{R}\) are monotone differentiable functions and hence the function \(m(x, y) = x \cdot y\) satisfies \(\frac{\partial}{\partial x} m(x, y) = 0\) for any \(x, y \in \mathbb{R}\). Since \(m(x, 0) = x\) one has \(\frac{\partial}{\partial x} m(x, 0) = 1\) from which follows \(\frac{\partial}{\partial x} m(x, y) > 0\) for each \(x, y \in \mathbb{R}\). Now we show that this condition characterize the loop multiplications.

**Proposition 4.** The multiplication \(x \cdot y = \phi^{-1} \sigma_x \phi(y)\) associated with the left multiplication curve \(\sigma\) with respect to the Iwasawa parametrization (2) determines a differentiable loop \((\mathbb{R}, \cdot, 0)\) if and only if the function \(m(x, y) = x \cdot y\) satisfies \(\frac{\partial}{\partial x} m(x, y) > 0\).

**Proof.** If the inequality \(\frac{\partial}{\partial x} m(x, y) > 0\) is satisfied for any \(x, y \in \mathbb{R}\) then the maps \(\rho_y : \mathbb{R} \to \rho_y(\mathbb{R})\) are monotone increasing diffeomorphisms of \(\mathbb{R}\) onto the connected subset \(\rho_y(\mathbb{R}) \subset \mathbb{R}\). We have to prove that the maps \(\rho_y : \mathbb{R} \to \mathbb{R}\) are surjective. Let \(y \in \mathbb{R}\) be fixed. We assume that there exists a real number \(\alpha\) such that \(\sup_{x \in \mathbb{R}} (x \cdot y) = \lim_{x \to \infty} \rho_y(x) = \lim_{x \to -\infty} \lambda_x(y) = \alpha\). We have up to a conjugation in \(\tilde{PSL}_2(\mathbb{R})\)

\[
\exp(xU) \cdot \exp(\nu(x)(T - U)) \cdot \exp(\delta(x)H) \cdot \exp(yU) \cdot \tilde{H} = \exp((x \cdot y)U) \cdot \tilde{H}.
\]

Applying the continuous covering map \(p : \tilde{PSL}_2(\mathbb{R}) \to SL_2(\mathbb{R})\) we obtain

\[
\begin{pmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{pmatrix}
\begin{pmatrix}
e^{\delta(x)} & 0 \\
2\nu(x)e^{\delta(x)} & e^{-\delta(x)}
\end{pmatrix}
\begin{pmatrix}
\cos y & \sin y \\
-\sin y & \cos y
\end{pmatrix}
p(H)
\]

\[
= \begin{pmatrix}
\cos(x \cdot y) & \sin(x \cdot y) \\
-\sin(x \cdot y) & \cos(x \cdot y)
\end{pmatrix}
p(H).
\]

Hence we have the decomposition

\[
\begin{pmatrix}
\cos(x - x \cdot y) & \sin(x - x \cdot y) \\
-\sin(x - x \cdot y) & \cos(x - x \cdot y)
\end{pmatrix}
\begin{pmatrix}
e^{\delta(x)} & 0 \\
2\nu(x)e^{\delta(x)} & e^{-\delta(x)}
\end{pmatrix}
\begin{pmatrix}
\cos y & \sin y \\
\sin y & \cos y
\end{pmatrix}
\]

\[
= \begin{pmatrix}
e^{d(x, y)} & 0 \\
2n(x, y)e^{d(x, y)} & e^{-d(x, y)}
\end{pmatrix}
\begin{pmatrix}
\cos y & -\sin y \\
\sin y & \cos y
\end{pmatrix},
\]

where \(d(x, y)\) and \(n(x, y)\) are suitable continuous real functions. If \(\sin y = 0\) then this equation implies \(x = x \cdot y + 2k\pi\) for any \(x \in \mathbb{R}\) and hence \(y = 0\), in which case
trivially $\rho_0(\mathbb{R}) = \mathbb{R}$. Assuming $\sin y \neq 0$ and comparing the first rows of these product matrices we obtain

$$
\left( e^{\delta(x)} \cos(x - x \cdot y) + 2\nu(x)e^{\delta(x)} \sin(x - x \cdot y), e^{\delta(x)} \sin(x - x \cdot y) \right) = \left( e^{\delta(x) \cdot y}, -e^{\delta(x) \cdot y} \sin y \right).
$$

From this we have $e^{2\delta(x)}(\cot(x - x \cdot y) + 2\nu(x)) = -\cot y$. Since the function $x \mapsto x - x \cdot y$ is continuous, its range is a connected subset of $\mathbb{R}$ such that $\sup_{x \in \mathbb{R}}(x - x \cdot y) = \infty$ because of $\lim_{x \to \infty} x \cdot y = a$. Hence there exists a real value $x_0 \in \mathbb{R}$ satisfying $\sin(x_0 - x_0 \cdot y) = 0$ for which the equation

$$
e^{2\delta(x_0)} \left( \cot(x_0 - x_0 \cdot y) + 2\nu(x_0) \right) = -\cot y
$$

gives a contradiction. $\square$

We obtain the following characterization of the left multiplication curves of loop multiplications:

**Theorem 5.** Let $\sigma : x \mapsto \sigma_x : \mathbb{R} \to \tilde{PSL}_2(\mathbb{R})$ be a left multiplication curve of the shape $\sigma_x = \exp(x \mathbf{Z}) \cdot \exp(\nu(x)(\mathbf{Y} - \mathbf{Z})) \cdot \exp(\delta(x) \mathbf{X})$, where $\nu(x)$ and $\delta(x)$ are differentiable functions having the initial value $\nu(0) = \delta(0) = 0$. The multiplication $(x, y) \mapsto x \cdot y$ associated with the left multiplication curve $\sigma$ with respect to the Iwasawa parametrization (2) determines a differentiable loop on $\mathbb{R}$ if and only if the functions $\delta(t)$ and $\nu(t)$ satisfy the differential inequality

$$
g(\dot{\delta}_x, \dot{\nu}_x) = \dot{\delta}(t)^2 + 2\dot{\nu}(t) + 4\nu(t)\dot{\delta}(t) - 1 < 0,
$$

where $\dot{\delta} = \frac{d\delta}{dx}$, $\dot{\nu} = \frac{d\nu}{dx}$ and $\dot{\sigma}_x = \frac{d\sigma}{dx}$.

**Proof.** The derivation of the multiplication given in Theorem 3 gives that

$$
\frac{\partial}{\partial x} \nu(x, y) = 1 - \frac{2\dot{\nu}(x) - 2e^{-2\delta(x)}\dot{\delta}(x) \cot y}{1 + (2\nu(x) + e^{-2\delta(x)} \cot y)^2} > 0
$$

if and only if $1 + (2\nu(x) + e^{-2\delta(x)} \cot y)^2 - 2\dot{\nu}(x) + 2e^{-2\delta(x)}\dot{\delta}(x) \cot y > 0$. Equivalently,

$$
(1 + 4\nu^2(x) - 2\dot{\nu}(x)) + 2e^{-2\delta(x)}(2\nu(x) + \dot{\delta}(x)) \cot y + e^{-4\delta^2(x)} \cot^2 y > 0
$$

for all $y \in \mathbb{R}$, which gives $(2\nu(x) + \dot{\delta}(x))^2 - (1 + 4\nu^2(x) - 2\dot{\nu}(x)) < 0$ or

$$
\dot{\delta}^2(x) + 2\dot{\nu}(x) + 4\nu(x)\dot{\delta}(x) - 1 < 0.
$$

(3)
For the computation of the pseudo-Riemannian value \( g(\dot{\sigma}_x, \dot{\sigma}_x) \) we apply the covering map \( p : \widetilde{PSL}_2(\mathbb{R}) \to SL_2(\mathbb{R}) \) to \( \sigma_x \) and derivate it:

\[
\frac{d}{dx} p(\sigma_x) = \begin{pmatrix}
-\sin x & \cos x \\
-\cos x & -\sin x
\end{pmatrix}
\begin{pmatrix}
e^{\delta(x)} & 0 \\
2\nu(x)e^{\delta(x)} & e^{-\delta(x)}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{pmatrix}
\begin{pmatrix}
\delta(x)e^{\delta(x)} & 0 \\
2\nu(x) + 2\nu(x)\delta(x)e^{\delta(x)} & -\delta(x)e^{-\delta(x)}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{pmatrix}
\\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
2\nu(x) & 1
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
\delta(x) \\
2\nu(x) + 2\nu(x)\delta(x)
\end{pmatrix}
\begin{pmatrix}
0 & \delta(x) \\
0 & e^{-\delta(x)}
\end{pmatrix}
\]

Since \( g(\dot{\sigma}_x, \dot{\sigma}_x) = \gamma(\sigma_x^{-1}\dot{\sigma}_x) = -\det(p_*(\sigma_x^{-1}\dot{\sigma}_x)) = -\det(p_*(\dot{\sigma}_x)) \) we can compute

\[
g(\dot{\sigma}_x, \dot{\sigma}_x) = -\det \begin{pmatrix}
2\nu(x) + \delta(x) & 1 \\
-1 + 2\nu(x) + 2\nu(x)\delta(x) & -\delta(x)
\end{pmatrix}
\]

\[
= \delta^2(x) + 2\nu(x) + 4\nu(x)\delta(x) - 1.
\]

Hence the condition (3) is equivalent to \( g(\dot{\sigma}_x, \dot{\sigma}_x) < 0 \) which proves the assertion.

\[\square\]

**Corollary 6.** The multiplication \((x, y) \mapsto x \cdot y\) associated with the left multiplication curve \( \sigma \) is a loop multiplication if and only if the left translation curve is time-like.

**References**


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