Prime ideals and complex ring homomorphisms
on a commutative algebra

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Abstract. We give a characterization of prime ideals $P$ of a commutative complex
algebra $A$ in order that $P$ be the kernel of some complex ring homomorphism on $A$. If, in
addition, $A$ is a uniform algebra on an infinite compact metric space, then we show that
there are exactly $2^c$ complex ring homomorphisms on $A$, whose kernels are non-maximal
prime ideals. Moreover, it turns out that ring homomorphisms on a commutative Banach
algebra are deeply connected with the existence of discontinuous homomorphisms.

1. Introduction and the statement of results

Let $A$ and $B$ be algebras over the complex number field $\mathbb{C}$. We say that a
mapping $\rho : A \to B$ is a ring homomorphism, provided that
\[
\rho(f + g) = \rho(f) + \rho(g)
\]
and
\[
\rho(fg) = \rho(f)\rho(g)
\]
for every $f, g \in A$. Moreover if $\rho$ is homogeneous, that is $\rho(\lambda f) = \lambda \rho(f)$ for every
$f \in A$ and $\lambda \in \mathbb{C}$, then $\rho$ is an ordinary homomorphism. It is obvious that if
$\rho : A \to B$ is a ring homomorphism, then $\rho(rf) = r\rho(f)$ for every rational number $r$ and $f \in A$. If, in addition, $\rho$ is assumed to be continuous, then we see that $\rho$ is real linear, that is, $\rho(tf) = t\rho(f)$ for every real number $t$ and $f \in A$. So, we consider ring homomorphisms which need not be continuous. The study of
ring homomorphisms between two Banach algebras has a long history. In 1944,
Arnold [1] proved that a ring isomorphism between the two Banach algebras of all bounded linear operators on two infinite dimensional Banach spaces is linear or conjugate linear. Kaplansky [6] generalized this result as follows: If $\rho$ is a ring isomorphism from one semisimple Banach algebra $A$ onto another, then $A$ is the direct sum of closed ideals $A_1$, $A_2$ and $A_3$ such that $\rho|_{A_1}$ is linear, $\rho|_{A_2}$ is conjugate linear and that $A_3$ is finite dimensional. The finite dimensional part is not trivial in general. In fact, Kestelman [7] proved that there exists a ring homomorphism $\rho : C \to C$ such that $\rho$ is neither linear nor conjugate linear. Moreover, Charnow [2, Theorem 3] proved that there exist $2^{|S|}$ ring automorphisms for every algebraically closed field $k$. Here and after, $|S|$ denotes the cardinal number of a set $S$. In particular, there are $2^{|C|}$ ring automorphisms on $C$. Molnár [10, Theorem 1] essentially gave a representation of a ring homomorphism between two commutative $C^*$-algebras.

Suppose $\rho : A \to B$ is a ring homomorphism between two commutative Banach algebras $A$ and $B$ with the maximal ideal spaces $M_A$ and $M_B$, respectively. When studying such mappings, a natural approach would be to consider ring homomorphisms $\varphi \circ \rho : A \to C$ for every $\varphi \in M_B$, and patch them by a continuous mapping from a suitable subset of $M_B$ into $M_A$. Indeed, some representations of ring homomorphisms, with additional conditions, are proved in this way (cf. [5, Theorem 2.3], [9, Theorem 2.6], [11, Theorem 5.1, 5.2]). Unfortunately this approach does not work in general because the kernel $\ker(\varphi \circ \rho)$ need not be a maximal ideal of $A$. On the other hand, Molnár [10, Theorem 2] essentially gave a representation of ring homomorphisms between two commutative $C^*$-algebras by another approach. Although we are concerned with ring homomorphism, the term ideal will mean an algebra ideal. Let $C(X)$ denote the commutative Banach algebra of all complex valued continuous functions on a compact Hausdorff space $X$. Suppose $\tau : C \to C$ is a ring homomorphism, and suppose $x_0 \in X$. Šemrl [11, Example 5.4] considered a complex ring homomorphism $\rho : C(X) \to C$ of the form

$$\rho(f) = \tau(f(x_0)) \quad (f \in C(X))$$

and gave the following example: If $\mathbb{N}$ is the set of all natural numbers and if $K$ is the closure of $\{1/n : n \in \mathbb{N}\}$ with its usual topology, then there is a non-zero ring homomorphism $\phi : C(K) \to C$ such that $\phi$ is not of the form $(*)$. In particular, $\ker \phi$ is a non-maximal prime ideal of $C(K)$. The first author [9, Lemma 2.1] gave a characterization of a ring homomorphism $\rho$ between two commutative Banach algebras in order that $\ker \rho$ be a maximal ideal (cf. [5, Lemma 2.2]).

In this note, we are concerned with complex ring homomorphisms $\rho$ on a commutative complex algebra $A$. If $\rho$ is non-zero, then it is easy to see that
ker $\rho$ is a prime ideal of $\mathcal{A}$. Recall that if $\mathcal{A}$ is a unital commutative Banach algebra, then there is a one-to-one correspondence between non-zero complex homomorphisms on $\mathcal{A}$ and maximal ideals of $\mathcal{A}$. With this in mind, one might expect that there is also a correspondence between complex ring homomorphisms and prime ideals of a complex commutative algebra $\mathcal{A}$. In this note, we give a characterization of prime ideals that can be represented as the kernels of some complex ring homomorphisms. Before we state our main result, we need some terminology. If $\mathcal{A}$ is unital, then we define $\mathcal{A}_e \overset{\text{def}}{=} \mathcal{A}$; otherwise, $\mathcal{A}_e$ denotes the commutative complex algebra obtained by adjunction of a unit element $e$ to $\mathcal{A}$. As usual, we may identify $f \in \mathcal{A}$ with $(f, 0) \in \mathcal{A}_e$.

Let $K$ be the closure of $\{1/n : n \in \mathbb{N}\}$ with its usual topology. As stated above, Šemrl gave a complex ring homomorphism on $C(K)$, whose kernel is a non-maximal prime ideal. In the following corollary we see that there exist $2^\mathfrak{c}$ such mappings. Moreover, the following is true.

**Corollary 1.2.** If $\mathcal{A}$ is a uniform algebra on an infinite compact metric space, then there are exactly $2^\mathfrak{c}$ complex ring homomorphisms on $\mathcal{A}$, whose kernels are non-maximal prime ideals.

### 2. A proof of results

Recall that an ideal $\mathcal{P}$ of a commutative algebra is prime if $\mathcal{P}$ is proper and $fg \notin \mathcal{P}$ whenever $f \notin \mathcal{P}$ and $g \notin \mathcal{P}$.

It is advisable to note that the quotient field of an integral domain $R$ is well defined even if $R$ has no unit: If $a \in R \setminus \{0\}$, then the “fraction” $a/a$ is a unit, and we may identify $b \in R$ with $ab/a$. 

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**Theorem 1.1.** Suppose $\mathcal{A}$ is a commutative complex algebra and $\mathcal{P}$ is a prime ideal of $\mathcal{A}$. Put $\mathfrak{c} = \sharp \mathbb{C}$. Then each of the following four properties implies the other three:

(a) There exists a non-zero ring homomorphism $\rho : \mathcal{A} \to \mathbb{C}$ such that $\ker \rho = \mathcal{P}$.

(b) The quotient algebra $\mathcal{A}/\mathcal{P}$ has the cardinal number $\mathfrak{c}$.

(c) There exists a prime ideal $\bar{\mathcal{P}}$ of $\mathcal{A}_e$ such that $\mathcal{P} = \bar{\mathcal{P}} \cap \mathcal{A}$ and that $\mathcal{A}_e/\bar{\mathcal{P}}$ has the cardinal number $\mathfrak{c}$.

(d) There exists a non-zero ring homomorphism $\bar{\rho} : \mathcal{A}_e \to \mathbb{C}$ such that $\mathcal{A} \cap \ker \bar{\rho} = \mathcal{P}$. 

**Proof.** The proof follows from the fundamental properties of complex ring homomorphisms and the construction of $\mathcal{A}_e$. The details are left to the reader.
Lemma 2.1. Suppose $\mathcal{A}$ is a commutative complex algebra and $\rho : \mathcal{A} \to \mathbb{C}$ is a non-zero ring homomorphism. Then
(a) the kernel $\ker \rho$ is a prime ideal of $\mathcal{A}$, and
(b) $\rho$ is of the form $\rho = \tau \circ \pi$, where $\tau$ is a non-zero field homomorphism on the quotient field $\mathcal{F}$ of $\mathcal{A}/\ker \rho$ into $\mathbb{C}$, and $\pi : \mathcal{A} \to \mathcal{A}/\ker \rho$ is the quotient mapping.

Proof. Choose $a \in \mathcal{A}$ such that $\rho(a) \neq 0$: This is possible since $\rho$ is assumed to be non-zero. (a) Pick $f \in \ker \rho$ and $\lambda \in \mathbb{C}$ arbitrarily. It follows that

$$
\rho(\lambda f) \rho(a) = \rho(f) \rho(\lambda a) = 0,
$$

and hence $\lambda f \in \ker \rho$. We thus obtain that $\ker \rho$ is an (algebra) ideal. Now it is obvious that $\ker \rho$ is a prime ideal.

(b) Let $\mathcal{F}$ be the quotient field of $\mathcal{A}/\ker \rho$. $\mathcal{F}$ is well defined since $\mathcal{A}/\ker \rho$ is an integral domain by (a). We define the mapping $\tau : \mathcal{F} \to \mathbb{C}$ by

$$
\tau(\pi(f)/\pi(g)) = \frac{\rho(f)}{\rho(g)} (\pi(f)/\pi(g) \in \mathcal{F}).
$$

A simple calculation shows that $\tau$ is a well defined non-zero field homomorphism. As usual we may identify $\pi(f) \in \mathcal{A}/\ker \rho$ with $\pi(fa)/\pi(a) \in \mathcal{F}$. We get

$$
\tau(\pi(f)) = \tau(\pi(fa)/\pi(a)) = \frac{\rho(fa)}{\rho(a)} = \rho(f) \quad (f \in \mathcal{A}),
$$

and hence $\rho = \tau \circ \pi$. \hfill \Box

Lemma 2.2. Suppose $\mathcal{A}$ is a commutative complex algebra and $\mathcal{P}$ is a prime ideal of $\mathcal{A}$. Then
(a) $\epsilon = \sharp \mathbb{C} \leq \sharp(\mathcal{A}/\mathcal{P})$, and
(b) if $a \in \mathcal{A} \setminus \mathcal{P}$, the set $\mathcal{P}_e \overset{\text{def}}{=} \{(f, \lambda) \in \mathcal{A}_e : fa + \lambda a \in \mathcal{P}\}$ is a prime ideal of $\mathcal{A}_e$ such that $\mathcal{P} = \mathcal{P}_e \cap \mathcal{A}$.

Proof. Pick $a \in \mathcal{A} \setminus \mathcal{P}$ arbitrarily.

(a) Let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{P}$ be the quotient mapping. Since $a \notin \mathcal{P}$ and since $\mathcal{P}$ is an ideal, $\pi(\lambda a) = \pi(\mu a)$ implies $\lambda = \mu$ for $\lambda, \mu \in \mathbb{C}$. This shows that the mapping $\lambda \mapsto \pi(\lambda a)$ is an injection, so that

$$
\epsilon = \sharp \mathbb{C} = \sharp\{\pi(\lambda a) : \lambda \in \mathbb{C}\} \leq \sharp(\mathcal{A}/\mathcal{P}).
$$
(b) It is easy to see that \( \mathcal{P}_e \) is a proper ideal of \( \mathcal{A}_e \) such that \( \mathcal{P} = \mathcal{P}_e \cap \mathcal{A} \). To show that \( \mathcal{P}_e \) is prime, suppose \((f_1, \lambda_1)(f_2, \lambda_2) \in \mathcal{P}_e \). By definition, this implies that \((f_1 f_2 + \lambda_2 f_1 + \lambda_1 f_2) a + (\lambda_1 \lambda_2) a \in \mathcal{P} \), and so we obtain \((f_1 a + \lambda_1 a)(f_2 a + \lambda_2 a) \in \mathcal{P} \). Since \( \mathcal{P} \) is a prime ideal, \((f_1 a + \lambda_1 a) \) or \((f_2 a + \lambda_2 a) \) belongs to \( \mathcal{P} \). This implies \((f_1, \lambda_1) \in \mathcal{P}_e \) or \((f_2, \lambda_2) \in \mathcal{P}_e \), and hence \( \mathcal{P}_e \) is prime.

Let \( \mathcal{K} \) be a transcendental extension field of a commutative field \( k \), and \( S \) a subset of \( \mathcal{K} \). We recall that \( S \) is said to be algebraically independent over \( k \), if the set of all finite products of elements of \( S \) is linearly independent over \( k \). A subset \( T \) of \( \mathcal{K} \) is called a transcendence base of \( \mathcal{K} \) over \( k \), if \( T \) is algebraically independent over \( k \) which is maximal with respect to the inclusion ordering. The existence of a transcendence base of \( \mathcal{K} \) over \( k \) is well known (cf. [8, Theorem 1.1 of Chapter X]). The maximality of \( T \) shows that \( \mathcal{K} \) is algebraic over \( k(T) \), the field generated by \( T \) over \( k \).

**Lemma 2.3.** Let \( Q \) be the rational number field and \( k \) a transcendental extension field of \( Q \) such that \( \sharp k = \mathfrak{c} \). If \( T \) is a transcendence base of \( k \) over \( Q \), then \( \sharp T = \mathfrak{c} \).

**Proof.** Suppose \( T \) is a transcendence base of \( k \) over \( Q \). Let \( Q(T) \) be the field generated by \( T \) over \( Q \). Since \( T \subset Q(T) \subset k \), we obtain \( \sharp T \leq \sharp \{Q(T)\} \leq \mathfrak{c} \).

So, we show that \( \mathfrak{c} \leq \sharp T \). Since \( k \) is algebraic over \( Q(T) \), each element of \( k \) is a zero point of some function in \( \varphi \), the set of all monic polynomials over \( Q(T) \). Note that for each monic polynomial, its zero points in \( k \) is at most finite. Put \( \mathfrak{a} = \sharp Q \), then we have

\[
\mathfrak{c} = \sharp k \leq (\sharp Q) \times \mathfrak{a} \leq (\sharp \{Q(T)\} \times \mathfrak{a}) \times \mathfrak{a} = \sharp \{Q(T)\},
\]

and hence \( \mathfrak{c} \leq \sharp \{Q(T)\} \).

We thus obtain

\[
\mathfrak{c} = \sharp \{Q(T)\} \leq \mathfrak{a} \times \sharp T.
\]

If \( T \) were finite, then we would have \( \sharp \{Q(T)\} = \mathfrak{a} \), in contradiction to \( \sharp \{Q(T)\} = \mathfrak{c} \).

It follows that \( \sharp T \geq \mathfrak{a} \), and so \( \mathfrak{a} \times \sharp T = \sharp T \). By \((**)\) we get \( \mathfrak{c} \leq \sharp T \), proving \( \sharp T = \mathfrak{c} \).

**Proof of Theorem 1.1.** Let \( \pi : \mathcal{A} \to \mathcal{A}/\mathcal{P} \) be the quotient mapping and fix \( a \in \mathcal{A} \setminus \mathcal{P} \).

(a) \( \Rightarrow \) (b) By (a) of Lemma 2.2, we obtain \( \mathfrak{c} \leq \sharp (\mathcal{A}/\mathcal{P}) \). To prove the opposite inequality, let \( \mathcal{F} \) be the quotient field of \( \mathcal{A}/\mathcal{P} \). By (b) of Lemma 2.1, we can write \( \rho = \tau \circ \pi \), where \( \tau : \mathcal{F} \to \mathbb{C} \) is a field homomorphism, and hence injective. It
follows that $\sharp F \leq \sharp C = \mathfrak{c}$. If we regard $A/P$ as a subset of $F$, it follows that $\sharp(A/P) \leq \sharp F \leq \mathfrak{c}$, proving $\sharp(A/P) = \mathfrak{c}$.

(b) $\Rightarrow$ (c) Let $P_e$ be the prime ideal of $A_e$ as in (b) of Lemma 2.2. Let $	ilde{\pi} : A_e \to A_e/P_e$ be the quotient mapping. Identification of $f$ and $(f, 0)$ shows that $\pi(f) = \pi(g)$ if and only if $\tilde{\pi}(f, 0) = \tilde{\pi}(g, 0)$ for $f, g \in A$, and so $\mathfrak{c} = \sharp(A/P) \leq \sharp(A_e/P_e)$. To show the opposite inequality, we define the mapping $\psi : A_e/P_e \to A/P$ by

$$\psi(\tilde{\pi}(f, \lambda)) = \pi(f + \lambda e) \quad (\tilde{\pi}(f, \lambda) \in A_e/P_e).$$

A simple calculation shows that $\psi$ is a well defined injection. Hence $\sharp(A_e/P_e) \leq \sharp(A/P) = \mathfrak{c}$, proving $\sharp(A_e/P_e) = \mathfrak{c}$.

(c) $\Rightarrow$ (d) Let $\tilde{F}$ be the quotient field of $A_e/\tilde{P}$. Then

$$\mathfrak{c} = \sharp(A_e/\tilde{P}) \leq \sharp \tilde{F} \leq \sharp(A_e/\tilde{P}) \times \sharp(A_e/\tilde{P}) = \mathfrak{c},$$

so that $\tilde{F}$ also has the cardinal number $\mathfrak{c}$. Note that $\tilde{F}$ is a transcendental extension of $\mathbb{Q}$ since $\tilde{F}$ contains a unital algebra $A_e/\tilde{P} \supset \mathbb{C}$.

Let $T$ and $\tilde{T}$ be transcendence bases of $\mathbb{C}$ and $\tilde{F}$ over $\mathbb{Q}$, respectively. By Lemma 2.3, we see that $\sharp T = \mathfrak{c} = \sharp \tilde{T}$. Thus we can find a bijection $\theta : \tilde{T} \to T$. Since $T$ is algebraically independent over $\mathbb{Q}$, the mapping $\theta$ is naturally extended to a field homomorphism $\hat{\theta} : \mathbb{Q}(\tilde{T}) \to \mathbb{Q}(T)$ so that $\hat{\theta}(r) = r$ for every $r \in \mathbb{Q}$. Since $\tilde{F}$ is an algebraic extension of $\mathbb{Q}(\tilde{T})$ and since $\mathbb{C}$ is algebraically closed, $\hat{\theta}$ can be extended to a field homomorphism on $\tilde{F}$ into $\mathbb{C}$ (cf. [8, Theorem 2.8 of Chapter VIII]), which is also denoted by $\hat{\theta}$. Define $\tilde{\rho} = \hat{\theta} \circ \tilde{\pi}$, where $\tilde{\pi} : A_e \to A_e/\tilde{P}$ is the quotient mapping. Then $\tilde{\rho} : A_e \to \mathbb{C}$ is a ring homomorphism whose kernel is equal to $\tilde{P}$, proving $A \cap \ker \tilde{\rho} = A \cap \tilde{P} = P$.

(d) $\Rightarrow$ (a) Put $\rho = \tilde{\rho}|_A$. Then $\rho : A \to \mathbb{C}$ is a non-zero ring homomorphism such that $\ker \rho = P$. \square

**Proof of Corollary 1.2.** Suppose $A$ is a uniform algebra on an infinite compact metric space $X$. Let $\Delta$ be the set of all non-maximal prime ideals of $A$. It is well known [4, Corollary 1] that there exist exactly $2^2$ non-maximal prime ideals of $A$, and hence $\sharp \Delta = 2^\mathfrak{c}$. Since $X$ is separable, $\sharp A = \mathfrak{c}$. For if $X_0$ is a countable dense subset of $X$, then the restriction map $f \mapsto f|_{X_0}$ ($f \in A$) is injective since each element of $A$ is continuous, and hence $\mathfrak{c} \leq \sharp A \leq \mathfrak{a} \times \mathfrak{c} = \mathfrak{c}$. So, there exist exactly $2^\mathfrak{c}$ functions on $A$ into $\mathbb{C}$, which need not be continuous nor ring homomorphic. This implies that there are at most $2^\mathfrak{c}$ complex ring homomorphisms on $A$. 
Conversely pick $P \in \Delta$ arbitrarily. By (a) of Lemma 2.2, we have
\[ c \leq \sharp \{ A/P \} \leq \sharp A = c, \]
and hence $\sharp \{ A/P \} = c$. So, by Theorem 1.1, to each $P \in \Delta$ there corresponds a complex ring homomorphism $\rho_P : A \to \mathbb{C}$ such that $\ker \rho_P = P$. Suppose $\rho_{P_1} = \rho_{P_2}$ for $P_1, P_2 \in \Delta$. It follows from $\ker \rho_{P_j} = P_j$ for $j = 1, 2$ that $P_1 = P_2$, and hence the mapping $P \mapsto \rho_P$ is an injection. We conclude that
\[ 2^c = \sharp \Delta \leq \sharp \{ \rho_P : P \in \Delta \}, \]
and the proof is complete. \[ \square \]

**Remark.** Let $A$ be a commutative Banach algebra. It is well-known (cf. [3, Theorem 5.7.32]) that under the continuum hypothesis there is a discontinuous homomorphism on $A$ into some Banach algebra whenever there is a non-maximal prime ideal $P$ of $A$ with $\sharp \{ A/P \} = c$. It follows from Theorem 1.1 that under the continuum hypothesis a discontinuous homomorphisms on $A$ exists whenever there is a non-zero ring homomorphism $\rho : A \to \mathbb{C}$ such that $\ker \rho$ is non-maximal.

**Example 2.1.** Let $\overline{D}$ denote the closure of the open unit disk $D$ in $\mathbb{C}$. The disk algebra $A(\overline{D})$ is a typical example of uniform algebras. HATORI, ISHI with the first and second author ([5, Corollary 5.3]) proved that if $\rho : A(\overline{D}) \to A(\overline{D})$ is a ring homomorphism whose range contains a non-constant function, then $\rho$ is linear or conjugate linear.

Here, let us consider complex ring homomorphisms on $A(\overline{D})$, that is, the range contains only constant functions. It is well known that the set of all non-zero complex homomorphisms on $A(\overline{D})$ can be identified with $\overline{D}$. So, there are $c$ complex homomorphisms on $A(\overline{D})$. On the other hand, by Corollary 1.2, we see that there are $2^c$ ring homomorphisms whose kernels are non-maximal prime ideals.

Finally, we give a pathological feature of complex ring homomorphisms (cf. [5, Corollary 5.2]).

**Example 2.2.** If $H(\Omega)$ is the algebra of all analytic functions on a region $\Omega \subset \mathbb{C}$, then, as we shall show, $H(\Omega)$ is a subring of $\mathbb{C}$. In particular, it will follow that every subalgebra $A$ of $H(\Omega)$, which is with or without unit, is a subring of $\mathbb{C}$. In fact, the ideal $(0)$ containing only zero is a prime ideal of $H(\Omega)$. Moreover $\sharp H(\Omega) = c$ since $c = \sharp \mathbb{C} \leq \sharp H(\Omega) \leq \sharp C(\Omega) = c$, and so by Theorem 1.1 there exists a non-zero complex ring homomorphism $\rho$ on $H(\Omega)$ such that $\ker \rho = (0)$. Therefore, $\rho$ is an injective complex ring homomorphism on $H(\Omega)$. 

\[ \square \]
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