On a new generalization of coherent rings

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Abstract. In this paper, we introduce a new generalization of coherent rings. Let \( m \) be a positive integer and \( d \) a positive integer or \( d = \infty \). A ring \( R \) is called a left \((m, d)\)-coherent ring in case every \( m \)-presented left \( R \)-module \( N \) with \( pd(N) \leq d \) is \((m + 1)\)-presented. It is shown that there are many similarities between coherent rings and \((m, d)\)-coherent rings. Some applications are also given.

1. Notation

In this section we recall some known notions and definitions needed in the sequel.

Throughout this paper, \( R \) is an associative ring with identity and all modules are unitary. For an \( R \)-module \( M \), the character module \( \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) is denoted \( M^+ \). Also \( pd(M) \) and \( fd(M) \) denote the projective and flat dimensions of \( M \) respectively.

Let \( M \) and \( N \) be \( R \)-modules. \( \text{Hom}(M, N) \) (resp. \( \text{Ext}^n(M, N) \)) means \( \text{Hom}_R(M, N) \) (resp. \( \text{Ext}^n_R(M, N) \)), and similarly \( M \otimes N \) (resp. \( \text{Tor}_n(M, N) \)) denotes \( M \otimes_R N \) (resp. \( \text{Tor}^n_R(M, N) \)) for an integer \( n \geq 1 \).

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Let $C$ be a class of $R$-modules and $M$ an $R$-module. Following [10], we say that a homomorphism $\phi : M \to C$ is a $C$-preenvelope if $C \in C$ and the abelian group homomorphism $\text{Hom}_R(\phi, C') : \text{Hom}(C, C') \to \text{Hom}(M, C')$ is surjective for each $C' \in C$. A $C$-preenvelope $\phi : M \to C$ is said to be a $C$-envelope if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism.

Given a class $L$ of $R$-modules, we will denote by $L^\perp = \{ C : \text{Ext}_1^R(L, C) = 0 \}$ for all $L \in L$, the right orthogonal class of $L$, and by $^L \! L = \{ C : \text{Ext}_1^R(C, L) = 0 \}$ for all $L \in L$, the left orthogonal class of $L$. Following [11, Definition 7.1.6], a monomorphism $\alpha : M \to C$ with $C \in C$ is said to be a special $C$-preenvelope of $M$ if $\text{coker}(\alpha) \in L^\perp$. Dually we have the definitions of a (special) $C$-precovar and a $C$-cover. Special $C$-preenvelopes (resp. special $C$-precovers) are obviously $C$-preenvelopes (resp. $C$-precovers). $C$-envelopes ($C$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

A pair $(F, C)$ of classes of $R$-modules is called a cotorsion theory [11] if $F^\perp = C$ and $C^\perp = F$. A cotorsion theory $(F, C)$ is said to be perfect (complete) if every $R$-module has a $C$-envelope and an $F$-cover (a special $C$-preenvelope and a special $F$-precovar) (see [12], [20]). A cotorsion theory $(F, C)$ is called hereditary [12] if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in F$, then $L'$ is also in $F$.

Let $M$ be a left $R$-module. $M$ is called $FP$-injective [19] if $\text{Ext}_1^R(N, M) = 0$ for all finitely presented left $R$-modules $N$. For a fixed nonnegative integer $n$, $M$ is called $n$-presented (see [2], [7]) if it has a finite $n$-presentation, that is, there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$, where each $F_i$ is finitely generated free (or projective). Clearly, an $R$-module is $0$-presented (resp. $1$-presented) if and only if it is finitely generated (resp. finitely presented).

General background materials can be found in [1], [11], [18], [21], [22].

2. Introduction

A ring for which every finitely generated left ideal is finitely presented is called a left coherent ring. Coherent rings and their generalizations have been studied extensively by many authors (see, for example, [2], [3]-[7], [10], [11], [15]-[17], [19], [21], [22]). Following Costa [7], a ring $R$ is said to be left $n$-coherent for a fixed nonnegative integer $n$ in case every $n$-presented left $R$-module is $(n + 1)$-presented. It is easy to see that $R$ is left $0$-coherent (resp. $1$-coherent) if and only if $R$ is left noetherian (resp. coherent).
On the other hand, Lee [17] introduced another concept of \( n \)-coherent rings from a different point of view. Let \( n \) be a fixed positive integer or \( \infty \). A ring \( R \) is called left \( n \)-coherent if every finitely generated submodule \( M \) of a free left \( R \)-module with \( \text{pd}(M) \leq n - 1 \) is finitely presented. Accordingly, all rings are left 1-coherent, and the left coherent rings are exactly those which are \( d \)-coherent when \( d \) is the left global dimension of \( R \), \( 0 < d \leq \infty \). Clearly the concept of \( n \)-coherent rings in [17] is different from that in [7].

In this paper, we introduce a new generalization of coherent rings, the so-called \((m, d)\)-coherent rings, which unifies the above two definitions of \( n \)-coherent rings given in [7] and [17]. To characterize \((m, d)\)-coherent rings, \((m, d)\)-injective and \((m, d)\)-flat modules are introduced. If \( \mathcal{I}_{m,d} \) denotes the class of all \((m, d)\)-injective left \( R \)-modules and \( \mathcal{F}_{m,d} \) the class of all \((m, d)\)-flat right modules, then it is shown that \((\perp \mathcal{I}_{m,d}, \mathcal{I}_{m,d})\) is a complete cotorsion theory and \((\mathcal{F}_{m,d}, \mathcal{F}_{m,d}^\perp)\) is a perfect cotorsion theory. It is also shown that there are many similarities between coherent rings and \((m, d)\)-coherent rings. For instance, we prove that a ring \( R \) is a left \((m, d)\)-coherent ring if and only if any direct product of \( R \) as a right \( R \)-module is \((m, d)\)-flat if and only if any direct product of \((m, d)\)-flat right \( R \)-modules is \((m, d)\)-injective if and only if every right \( R \)-module has an \( \mathcal{F}_{m,d} \)-preenvelope if and only if \( \text{Ext}^{m+1}(M, N) = 0 \) for any \( m \)-presented left \( R \)-module \( M \) with \( \text{pd}(M) \leq d \) and any \((m, d)\)-injective left \( R \)-module \( N \) if and only if \((\perp \mathcal{I}_{m,d}, \mathcal{I}_{m,d})\) is a hereditary cotorsion theory. Finally, some applications are given.

3. \((m, d)\)-injective and \((m, d)\)-flat modules

We begin with the following

Definition 3.1. Let \( R \) be a ring, \( m \) a positive integer, and \( d \) a positive integer or \( d = \infty \).

A left \( R \)-module \( M \) is said to be \((m, d)\)-injective if \( \text{Ext}^m(N, M) = 0 \) for any \( m \)-presented left \( R \)-module \( N \) with \( \text{pd}(N) \leq d \).

A right \( R \)-module \( F \) is said to be \((m, d)\)-flat if \( \text{Tor}_m(F, N) = 0 \) for any \( m \)-presented left \( R \)-module \( N \) with \( \text{pd}(N) \leq d \).

Remark 3.2. The concept of \((m, d)\)-injective modules unifies the concepts of \( n\)-FP-injective modules in [5] and \( n \)-absolutely pure modules in [17]. Similarly, the concept of \((m, d)\)-flat modules unifies the concepts of \( n \)-flat modules in [5] and \( n \)-flat modules in [17]. In fact, we have the following implications for any integer \( k \geq 1 \):
FP-injective modules ⇒ \( n \)-FP-injective modules in \([5]\) ⇔ \((n, \infty)\)-injective modules, where \( n \) is a positive integer.

FP-injective modules ⇒ \( n \)-absolutely pure modules in \([5]\) ⇔ \((n, \infty)\)-injective modules, where \( n \) is a positive integer or \( n = \infty \).

Flat modules ⇒ \( n \)-flat modules in \([5]\) ⇔ \((n, \infty)\)-flat modules, where \( n \) is a positive integer or \( n = \infty \).

In what follows, \( m \) is a fixed positive integer and \( d \) a fixed positive integer or \( d = \infty \). \( P_{m,d} \) stands for the class of all \( m \)-presented left \( R \)-modules \( N \) with \( \text{pd}(N) \leq d \), \( I_{m,d} \) denotes the class of all \((m, d)\)-injective left \( R \)-modules, \( F_{m,d} \) is the class of all \((m, d)\)-flat right modules.

**Theorem 3.3.** Let \( R \) be a ring. Then

1. \( (I_{m,d}, I_{m,d}) \) is a complete cotorsion theory.
2. \( (F_{m,d}, F_{m,d}) \) is a perfect cotorsion theory.

**Proof.** (1) Let \( M \) be any left \( R \)-module and \( N \in P_{m,d} \). Note that \( \text{Ext}^m(N, M) = 0 \) if and only if \( \text{Ext}^1(K_{m-1}, M) = 0 \), where \( K_{m-1} \) denotes the \((m-1)\)th syzygy of \( N \). Let \( X \) be the set of representatives of \((m-1)\)th syzygy modules of all \( n \)-presented left \( R \)-modules \( N \) with \( \text{pd}(N) \leq d \). Then \( I_{m,d} = X^1 \), and so the result follows from \([8, \text{Theorem 10}] \) and \([11, \text{Definition 7.1.5}] \).

(2) Denote by \( B \) the class of all left \( R \)-modules \( B \) with \( \text{Tor}_1(N, B) = 0 \) for all \( N \in F_{m,d} \). Then by dimension shifting one shows that \( X \in F_{m,d} \) if and only if \( \text{Tor}_1(X, B) = 0 \) for all \( B \in B \). So (2) follows from \([20, \text{Lemma 1.11 and Theorem 2.8}] \). \(\square\)

**Remark 3.4.** (1) Note that \( I_{1, \infty} (F_{1, \infty}) \) is just the class of all FP-injective left \( R \)-modules (all flat right \( R \)-modules). So \([20, \text{Theorems 3.1.1 and 3.4.2}] \) are particular cases of Theorem 3.3 where \( m = 1 \) and \( d = \infty \).

(2) Let \( M \) be an \( R \)-module over a commutative domain \( R \). Then \( M \) is \((1, 1)\)-flat if and only if \( M \) is torsion-free by \([17, \text{Lemma 1}] \), so Theorem 3.3 (2) gives the well-known result that every \( R \)-module has a torsion-free cover. On the other hand, \( M \) is \((1, 1)\)-injective if and only if \( M \) is divisible by \([17, \text{Lemma 3}] \). Since divisible envelopes may not exist (see \([13] \) and \([20, \text{Proposition 4.8}] \)), the statement of Theorem 3.3 (1) is the best possible in the sense that \((I_{m,d}, I_{m,d})\) is not a perfect cotorsion theory. However, if \( I_{m,d} \) is closed under direct limits, then \((I_{m,d}, I_{m,d})\) is a perfect cotorsion theory by Theorem 3.3 (1) and \([11, \text{Theorem 7.2.6}] \).
Lemma 3.5. A right $R$-module $M$ is $(m, d)$-flat if and only if $M^+$ is $(m, d)$-injective.

Proof. The result follows from the standard isomorphism

\[ \text{Ext}^m(N, M^+) \cong \text{Tor}_m(M, N) \]

for any left $R$-module $N$. \qed

Proposition 3.6. Let $R$ be a ring. Then

1. $I_{m,d}$ and $F_{m,d}$ are closed under pure submodules.
2. $I_{m,d}$ is closed under direct products and $F_{m,d}$ is closed under direct sums.
3. $I_{m,m}$ is closed under quotients and $F_{m,m}$ is closed under submodules.
4. $I_{m,d}$ is closed under direct sums.

Proof. (1) Let $N$ be a pure submodule of an $(m, d)$-injective left $R$-module $M$. For any $P \in P_{m,d}$ with a finite $m$-presentation $F_m \twoheadrightarrow F_{m-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow P \rightarrow 0$, let $K = \ker(F_{m-2} \rightarrow F_{m-3})$, then $K$ is finitely presented. Note that $\text{Ext}^1(K, M) \cong \text{Ext}^m(P, M) = 0$. Thus we have the exact sequence

\[ \text{Hom}(K, M) \rightarrow \text{Hom}(K, M/N) \rightarrow \text{Ext}^1(K, N) \rightarrow 0. \]

But the sequence $\text{Hom}(K, M) \rightarrow \text{Hom}(K, M/N) \rightarrow 0$ is exact since $N$ is a pure submodule of $M$, so $\text{Ext}^1(K, N) = 0$. Therefore $\text{Ext}^m(P, N) \cong \text{Ext}^1(K, N) = 0$, that is, $N$ is $(m, d)$-injective.

Let $N$ be a pure submodule of an $(m, d)$-flat right $R$-module $M$, then the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ induces the split exact sequence $0 \rightarrow (M/N)^+ \rightarrow M^+ \rightarrow N^+ \rightarrow 0$. Thus $N^+$ is $(m, d)$-injective since $M^+$ is $(m, d)$-injective by Lemma 3.5. So $N$ is $(m, d)$-flat by Lemma 3.5 again.

(2) and (3) are obvious.

(4) Let $(M_i)_{i \in I}$ be a family of $(m, d)$-injective left $R$-modules. For any $P \in P_{m,d}$ with a finite $m$-presentation $F_m \twoheadrightarrow F_{m-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow P \rightarrow 0$, we have a commutative diagram with exact rows:

\[ \text{Hom}(F_{m-1}, \oplus M_i) \rightarrow \text{Hom}(\ker(f_{m-1}), \oplus M_i) \rightarrow \text{Ext}^1(\ker(f_{m-2}), \oplus M_i) \rightarrow 0 \]

\[ \oplus \text{Hom}(F_{m-1}, M_i) \rightarrow \oplus \text{Hom}(\ker(f_{m-1}), M_i) \rightarrow \oplus \text{Ext}^1(\ker(f_{m-2}), M_i) \rightarrow 0 \]
Since \( \alpha \) and \( \beta \) are isomorphisms by [1, Exercise 16.3, p. 189] (for \( \ker(f_{m-1}) \) is finitely generated), \( \gamma \) is an isomorphism by Five Lemma. Thus

\[
\text{Ext}^m(P, \oplus M_i) \cong \text{Ext}^1(\ker(f_{m-2}), \oplus M_i) \cong \oplus \text{Ext}^1(\ker(f_{m-2}), M_i) \\
\cong \oplus \text{Ext}^m(P, M_i) = 0.
\]

So \( \oplus M_i \) is \((m, d)\)-injective. \( \square \)

In the next section, we shall discuss when \( \mathcal{F}_{m,d} \) is closed under direct products.

### 4. \((m, d)\)-coherent rings

We start with the following

**Definition 4.1.** Let \( m \) be a positive integer and \( d \) a positive integer or \( d = \infty \). A ring \( R \) is called a left \((m, d)\)-coherent ring in case every \( m \)-presented left \( R \)-module \( N \) with \( pd(N) \leq d \) is \((m + 1)\)-presented.

**Remark 4.2.**
1. Clearly, all rings are left \((m, d)\)-coherent for any pair of positive integers \( m \) and \( d \) with \( m \geq d \).
2. \( R \) is a left coherent ring if and only if \( R \) is a left \((1, \infty)\)-coherent ring if and only if \( R \) is a left \((1, d)\)-coherent ring, where \( d \) denotes the left global dimension of \( R \) and \( 0 < d \leq \infty \).
3. The concept of \((m, d)\)-coherent rings unifies two different concepts of \( n \)-coherent rings appearing in [7], [17]. In fact, we have the following implications:
   - Left coherent rings \( \Rightarrow \) left \( n \)-coherent rings in [7] \( \Leftrightarrow \) left \((n, \infty)\)-coherent rings
   - Left coherent rings \( \Rightarrow \) left \( (n, k)\)-coherent rings for every \( k \) positive or \( k = \infty \), where \( n \) is a positive integer.
   - Left coherent rings \( \Rightarrow \) left \( n \)-coherent rings in [17] \( \Leftrightarrow \) left \((1, n)\)-coherent rings
   - Left \((k, n)\)-coherent rings for every \( k \) positive, where \( n \) is a positive integer or \( n = \infty \).

Next we shall characterize \((m, d)\)-coherent rings in terms of, among others, \((m, d)\)-flat and \((m, d)\)-injective modules.

**Theorem 4.3.** The following are equivalent for a ring \( R \):
1. \( R \) is a left \((m, d)\)-coherent ring.
2. Any direct product of \( R \) as a right \( R \)-module is \((m, d)\)-flat.
Any direct product of \((m, d)\)-flat right \(R\)-modules is \((m, d)\)-flat.

Any direct limit of \((m, d)\)-injective left \(R\)-modules is \((m, d)\)-injective.

\(\lim \to \Ext^m(A, M_i) \to \Ext^m(A, \lim M_i)\) is an isomorphism for any \(A \in \mathcal{P}_{m,d}\) and any direct system \((M_i)_{i \in I}\) of left \(R\)-modules.

\(\Tor_m(\prod N_\alpha, A) \cong \prod \Tor_m(N_\alpha, A)\) for any family \(\{N_\alpha\}\) of right \(R\)-modules and any \(A \in \mathcal{P}_{m,d}\).

A left \(R\)-module \(M\) is \((m, d)\)-injective if and only if \(M^+\) is \((m, d)\)-flat.

A left \(R\)-module \(M\) is \((m, d)\)-injective if and only if \(M^{++}\) is \((m, d)\)-injective.

A right \(R\)-module \(M\) is \((m, d)\)-flat if and only if \(M^{++}\) is \((m, d)\)-flat.

**Proof.** (6) \(\Rightarrow\) (3) \(\Rightarrow\) (2) and (5) \(\Rightarrow\) (4) are obvious.

(1) \(\Rightarrow\) (6) follows from [5, Lemma 2.10 (2)].

(1) \(\Rightarrow\) (5): Let \(A \in \mathcal{P}_{m,d}\). Then \(A\) is an \((m + 1)\)-presented left \(R\)-module since \(R\) is left \((m, d)\)-coherent, and so \(\lim \to \Ext^m(A, M_i) \cong \Ext^m(A, \lim M_i)\) by [5, Lemma 2.9 (2)].

(2) \(\Rightarrow\) (1): Let \(P \in \mathcal{P}_{m,d}\) with a finite \(m\)-presentation \(F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} F_0 \to P \to 0\). Then we get an exact sequence \(0 \to K \to F_{m-1} \to L \to 0\), where \(K = \ker(f_{m-1}), L = \ker(f_{m-2})\). We shall show that \(K\) is finitely presented. Note that \(\Tor_m(\prod R, L) \cong \Tor_m(\prod R, P) = 0\). Thus we have a commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
0 & \to & (\prod R) \otimes K & \to & (\prod R) \otimes F_{m-1} & \to & (\prod R) \otimes L & \to & 0 \\
& & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & \\
0 & \to & \prod K & \to & \prod F_{m-1} & \to & \prod L & \to & 0
\end{array}
\]

Since \(\beta\) and \(\gamma\) are isomorphisms by [11, Theorem 3.2.22], \(\alpha\) is an isomorphism by Five Lemma. So \(K\) is finitely presented by [11, Theorem 3.2.22] again. Thus \(P\) is \((m + 1)\)-presented.

(4) \(\Rightarrow\) (1): Let \(P \in \mathcal{P}_{m,d}\) with a finite \(m\)-presentation \(F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} F_0 \to P \to 0\). Then we get an exact sequence \(0 \to K \to F_{m-1} \to L \to 0\), where \(K = \ker(f_{m-1}), L = \ker(f_{m-2})\). We claim that \(K\) is finitely presented. In fact, let \((M_i)_{i \in I}\) be a family of injective left \(R\)-modules, where \(I\) is a directed set. Then \(\lim M_i\) is \((m, d)\)-injective by (4). Note that \(\Ext^1(L, \lim M_i) \cong\)
Ext\(^m\)(P, lim \(M_i\)) = 0. Thus we have a commutative diagram with exact rows:

\[
\begin{array}{ccc}
\text{Hom}(L, \lim \rightarrow M_i) & \rightarrow & \text{Hom}(L, \lim \rightarrow M_i) \\
\downarrow \alpha & & \downarrow \beta \\
\lim \text{Hom}(L, M_i) & \rightarrow & \lim \text{Hom}(F_{m-1}, M_i) \\
\downarrow \gamma & & \downarrow \gamma \\
\lim \text{Hom}(K, M_i) & \rightarrow & \lim \text{Hom}(K, M_i) \\
\end{array}
\]

Since \(\alpha\) and \(\beta\) are isomorphisms by [16, Proposition 2.5], \(\gamma\) is an isomorphism by Five Lemma. So \(K\) is finitely presented by [16, Proposition 2.5] again. Therefore \(P\) is \((m + 1)\)-presented.

(1) \(\Rightarrow\) (7): Let \(A \in \mathcal{P}_{m,d}\). Since \(R\) is left \((m, d)\)-coherent, \(A\) has a projective resolution \(\cdots \rightarrow F_m \rightarrow F_{m-1} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0\) with each \(F_i\) finitely generated. Thus Tor\(_1\)(\(M^+, A\)) \(\cong\) Ext\(_1\)(\(A, M^+)\) by [18, Theorem 9.51] and the remark following it. So \(M\) is \((m, d)\)-injective if and only if \(M^+\) is \((m, d)\)-flat.

(7) \(\Rightarrow\) (8) is obvious since \(M^+\) is \((m, d)\)-flat if and only if \(M^{++}\) is \((m, d)\)-injective by Lemma 3.5.

(8) \(\Rightarrow\) (9): If \(M\) is an \((m, d)\)-flat right \(R\)-module, then \(M^+\) is \((m, d)\)-injective by Lemma 3.5. So \(M^{++}\) is \((m, d)\)-injective by (8), and hence \(M^{++}\) is \((m, d)\)-flat by Lemma 3.5. Conversely, if \(M^{++}\) is \((m, d)\)-flat, then \(M\) is \((m, d)\)-flat by Proposition 3.6 (1) since \(M\) is a pure submodule of \(M^{++}\).

(9) \(\Rightarrow\) (3): Let \((M_i)_{i \in I}\) be a family of \((m, d)\)-flat right \(R\)-modules. Then \(\oplus M_i\) is \((m, d)\)-flat by Proposition 3.6 (2). So \((\oplus M_i)^{++} \cong (\oplus M_i^+)^+\) is \((m, d)\)-flat by (9). But \(\oplus M_i^+\) is a pure submodule of \(\oplus M_i^+\) by [3, Lemma 1 (1)]. Thus \((\oplus M_i^+)^+ \rightarrow (\oplus M_i^+)^+ \rightarrow 0\) is split. Hence \(\oplus M_i^{++} \cong (\oplus M_i^+)^+\) is \((m, d)\)-flat. Since \(\oplus M_i\) is a pure submodule of \(\oplus M_i^{++}\) by [3, Lemma 1 (2)], \(\oplus M_i\) is \((m, d)\)-flat by Proposition 3.6 (1).

It is well known that \(R\) is a left coherent ring if and only if every right \(R\)-module has a flat preenvelope (see [10, Proposition 5.1]) if and only if every factor module of an \(FP\)-injective left \(R\)-module by a pure submodule is \(FP\)-injective (see [21, 35.9]). Now we have similar characterizations of \((m, d)\)-coherent rings as shown in the following theorem.

**Theorem 4.4.** The following are equivalent for a ring \(R\):

1. \(R\) is a left \((m, d)\)-coherent ring.
2. Every right \(R\)-module has an \(\mathcal{F}_{m,d}\)-preenvelope.
3. \(\text{Ext}^{m+1}(M, N) = 0\) for any \(M \in \mathcal{P}_{m,d}\) and any \(N \in \mathcal{I}_{m,d}\).
(4) \( \text{Ext}^{n+1}(M, N) = 0 \) for any \( M \in \mathcal{P}_{m,d} \) and any \( N \in \mathcal{I}_{1,\infty} \).

(5) \( \text{Ext}^{n+j}(M, N) = 0 \) for any \( j \geq 1 \), any \( M \in \mathcal{P}_{m,d} \) and any \( N \in \mathcal{I}_{m,d} \).

(6) \( \text{Ext}^{n+j}(M, N) = 0 \) for any \( j \geq 1 \), any \( M \in \mathcal{P}_{m,d} \) and any \( N \in \mathcal{I}_{1,\infty} \).

(7) If \( 0 \to A \to B \to C \to 0 \) is exact with \( B \in \mathcal{P}_{m,d} \) and \( C \in \mathcal{P}_{m,d} \), then \( A \in \mathcal{P}_{m,d} \).

(8) If \( 0 \to N \to M \to L \to 0 \) is exact with \( N \in \mathcal{I}_{m,d} \) and \( M \in \mathcal{I}_{m,d} \), then \( L \in \mathcal{I}_{m,d} \).

(9) If \( 0 \to N \to M \to L \to 0 \) is exact with \( N \in \mathcal{I}_{1,\infty} \) and \( M \in \mathcal{I}_{m,d} \), then \( L \in \mathcal{I}_{m,d} \).

(10) \((\mathcal{I}_{m,d}, \mathcal{I}_{m,d})\) is a hereditary cotorsion theory.

Proof. (1) \(\Rightarrow\) (2): Let \( N \) be any right \( R \)-module. By [11, Lemma 5.3.12], there is an infinite cardinal number \( \aleph_\alpha \) such that for any \( R \)-homomorphism \( f : N \to L \) with \( L \) \((m, d)\)-flat, there is a pure submodule \( Q \) of \( L \) such that \( \text{Card}(Q) \leq \aleph_\alpha \) and \( f(N) \subseteq Q \). Note that \( Q \) is \((m, d)\)-flat by Proposition 3.6 (1), and so \( N \) has an \( F_{m,d} \)-preenvelope by Theorem 4.3 and [11, Proposition 6.2.1].

(2) \(\Rightarrow\) (1): Note that \( F_{m,d} \) is closed under direct products by [4, Lemma 1] and so (1) follows from Theorem 4.3.

(1) \(\Rightarrow\) (7): Since \( C \) is \((m + 1)\)-presented by (1), \( A \) is \( m \)-presented by [2, Exercise 6, p. 61]. In addition, \( pd(A) \leq d \) is obvious.

(7) \(\Rightarrow\) (3): Let \( M \in \mathcal{P}_{m,d} \). There is an exact sequence \( 0 \to K \to F \to M \to 0 \) with \( F \) finitely generated projective. Then \( K \in \mathcal{P}_{m,d} \) by (7). Thus for any \( N \in \mathcal{I}_{m,d} \), we have the exact sequence

\[
0 = \text{Ext}^m(K, N) \to \text{Ext}^{m+1}(M, N) \to \text{Ext}^{m+1}(F, N) = 0.
\]

So \( \text{Ext}^{m+1}(M, N) = 0 \).

(3) \(\Rightarrow\) (4), (5) \(\Rightarrow\) (6) \(\Rightarrow\) (4) and (8) \(\Rightarrow\) (9) are trivial.

(3) \(\Rightarrow\) (8): Let \( P \in \mathcal{P}_{m,d} \). The exact sequence \( 0 \to N \to M \to M/N \to 0 \)

induces the exactness of \( 0 = \text{Ext}^m(P, M) \to \text{Ext}^m(P, M/N) \to \text{Ext}^{m+1}(P, N) = 0 \).

Thus \( \text{Ext}^m(P, M/N) = 0 \), that is, \( M/N \in \mathcal{I}_{m,d} \).

(4) \(\Rightarrow\) (9): The proof is similar to that of (3) \(\Rightarrow\) (8).

(9) \(\Rightarrow\) (1): Let \( P \in \mathcal{P}_{m,d} \) with a finite \( m \)-presentation \( F_m \overset{f_m}{\to} F_{m-1} \overset{f_{m-1}}{\to} \cdots \overset{f_1}{\to} F_0 \to P \to 0 \). We claim that \( K = \ker(f_{m-1}) \) is finitely presented. In fact, for any \( FP \)-injective left \( R \)-module \( N \), there is an exact sequence \( 0 \to N \to E \to E/N \to 0 \) with \( E \) injective. Note that \( E/N \) is \((m, d)\)-injective by (9). Hence we get the exact sequence

\[
0 = \text{Ext}^m(P, E/N) \to \text{Ext}^{m+1}(P, N) \to \text{Ext}^{m+1}(P, E) = 0.
\]
Thus $\text{Ext}^{m+1}(P, N) = 0$, and so $\text{Ext}^1(K, N) \cong \text{Ext}^{m+1}(P, N) = 0$. It follows that $K$ is finitely presented by [9] since $K$ is finitely generated. Therefore $R$ is left $(m, d)$-coherent.

(3) ⇒ (5) holds by induction and the equivalence of (3) and (7).
(8) ⇔ (10) follows from Theorem 3.3 and [12, Proposition 1.2]. □

5. Applications

Some applications are given in this section. We start by considering when every right $R$-module has a monic $F_{m,d}$-preenvelope.

**Proposition 5.1.** The following are equivalent for a ring $R$:

1. Every right $R$-module has a monic $F_{m,d}$-preenvelope.
2. $R$ is a left $(m, d)$-coherent ring and $(m, d)$-injective as a left $R$-module.
3. $R$ is left $(m, d)$-coherent and every injective right $R$-module is $(m, d)$-flat.
4. $R$ is left $(m, d)$-coherent and every flat left $R$-module is $(m, d)$-injective.

**Proof.** (2) ⇒ (1): Let $M$ be any right $R$-module. Then $M$ has an $F_{m,d}$-preenvelope $f : M \to F$ by Theorem 4.4. Since $(R^+R)^+$ is a cogenerator in the category of right $R$-modules, there is an exact sequence $0 \to M \to \Pi(R^+R)^+$. Note that $(R^+R)^+$ is $(m, d)$-flat by Theorem 4.3 (7), and so $\Pi(R^+R)^+$ is $(m, d)$-flat by Theorem 4.3 (3). Thus $f$ is monic, and hence (1) follows.

(1) ⇒ (3) follows from Theorem 4.4.

(3) ⇒ (4): Let $M$ be a flat left $R$-module. Then $M^+$ is injective, and so $M^+$ is $(m, d)$-flat by (3). Thus $M$ is $(m, d)$-injective by Theorem 4.3.

(4) ⇒ (2) is clear. □

Next we discuss when every right $R$-module has an epic $F_{m,d}$-(pre)envelope.

**Theorem 5.2.** The following are equivalent for a ring $R$:

1. Every right $R$-module has an epic $F_{m,d}$-envelope.
2. Every left $R$-module has a monic $I_{m,d}$-cover.
3. $R$ is a left $(m, d)$-coherent ring and submodules of $(m, d)$-flat right $R$-modules are $(m, d)$-flat.
4. $R$ is a left $(m, d)$-coherent ring and every left $R$-module in $\perp I_{m,d}$ has a monic $I_{m,d}$-cover.
5. Every quotient of any $(m, d)$-injective left $R$-module is $(m, d)$-injective.
Proof. (1) $\Leftrightarrow$ (3) follows from [4, Theorem 2].

(3) $\Rightarrow$ (5): Let $X$ be any $(m, d)$-injective left $R$-module and $N$ any submodule of $X$. Then the exact sequence $0 \to N \to X \to X/N \to 0$ induces the exactness of $0 \to (X/N)^+ \to X^+ \to N^+ \to 0$. Since $X^+$ is $(m, d)$-flat by (3) and Theorem 4.3, so is $(X/N)^+$ by (3). Thus $X/N$ is $(m, d)$-injective by Theorem 4.3 again.

(5) $\Rightarrow$ (3): $R$ is a left $(m, d)$-coherent ring by Theorem 4.4.

Now let $A$ be any submodule of an $(m, d)$-flat right $R$-module $B$. Then the exactness of $0 \to A \to B \to B/A \to 0$ induces an exact sequence $0 \to (B/A)^+ \to B^+ \to A^+ \to 0$. Note that $B^+$ is $(m, d)$-injective by Lemma 3.5, so $A^+$ is $(m, d)$-injective by (5), and hence $A$ is $(m, d)$-flat by Lemma 3.5 again.

(2) $\Leftrightarrow$ (5) holds by [14, Proposition 4] since the class of $(m, d)$-injective left modules is closed under direct sums by Proposition 3.6 (4).

(4) $\Rightarrow$ (5): Let $M$ be any $(m, d)$-injective left $R$-module and $N$ any submodule of $M$. We have to prove that $M/N$ is $(m, d)$-injective. In fact, note that $N$ has a special $I_{m,d}$-preenvelope by Theorem 3.3, that is, there exists an exact sequence $0 \to N \to E \xrightarrow{\beta} L \to 0$ with $E \in I_{m,d}$ and $L \in \perp I_{m,d}$. Since $L$ has a monic $I_{m,d}$-cover $\phi: F \to L$ by (4), there is $\alpha : E \to F$ such that $\beta = \phi \alpha$. Thus $\phi$ is epic, and hence it is an isomorphism. So $L$ is $(m, d)$-injective. For any $K \in P_{m,d}$, we have the exact sequence

$$0 = \Ext^m(K, L) \to \Ext^{m+1}(K, N) \to \Ext^{m+1}(K, E).$$

Note that $\Ext^{m+1}(K, E) = 0$ by Theorem 4.4 since $R$ is left $(m, d)$-coherent. So $\Ext^{m+1}(K, N) = 0$. On the other hand, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces the exactness of the sequence

$$0 = \Ext^m(K, M) \to \Ext^m(K, M/N) \to \Ext^{m+1}(K, N) = 0.$$

Therefore $\Ext^m(K, M/N) = 0$, as desired.

(5) $\Rightarrow$ (4) follows from Theorem 4.4 (8) and the equivalence of (2) and (5). □

Note that all rings are left $(m, m)$-coherent by Remark 4.2 (1). So we have the following

Corollary 5.3. The following are true for any ring $R$:

1. $F_{m,m}$ is closed under direct products and $I_{m,m}$ is closed under direct limits.
2. Every right $R$-module has an epic $F_{m,m}$-envelope and every left $R$-module has a monic $I_{m,m}$-cover.
3. $R$ is $(m, m)$-injective as a left $R$-module if and only if every (injective) right $R$-module is $(m, m)$-flat if and only if every (flat) left $R$-module is $(m, m)$-injective.
Proof. (1) follows from Theorem 4.3. (2) holds by Proposition 3.6 (3) and Theorem 5.2. (3) comes from (2) and Proposition 5.1.

Corollary 5.4. The following are equivalent for a ring $R$:

1. Every $m$-presented left $R$-module has projective dimension at most $m$.
2. Every quotient of any $(m, \infty)$-injective left $R$-module is $(m, \infty)$-injective.
3. $R$ is a left $(m, \infty)$-coherent ring and submodules of $(m, \infty)$-flat right $R$-modules are $(m, \infty)$-flat.
4. $R$ is a left $(m, \infty)$-coherent ring and every $m$-presented left $R$-module has flat dimension at most $m$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear. (2) $\Rightarrow$ (1): Let $M$ be an $m$-presented left $R$-module and $N$ any left $R$-module, then there is a short exact sequence $0 \to N \to E \to L \to 0$ with $E$ injective. Note that $L$ is $(m, \infty)$-injective by (2). Thus we have an exact sequence $0 = \text{Ext}^m(M, L) \to \text{Ext}^{m+1}(M, N) \to \text{Ext}^{m+1}(M, E) = 0$, and so $\text{Ext}^{m+1}(M, N) = 0$. It follows that $M$ has projective dimension at most $m$.

(2) $\Leftrightarrow$ (3) follows from Theorem 5.2.

(4) $\Rightarrow$ (1): Let $M$ be any $m$-presented left $R$-module. Then $M$ is $(m+1)$-presented since $R$ is left $(m, \infty)$-coherent. So there is an exact sequence $F_{m+1} \xrightarrow{f_{m+1}} \cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i$ finitely generated projective. Note that $\ker(f_{m-1})$ is finitely presented. Since $fd(M) \leq m$ by (4), $\ker(f_{m-1})$ is flat. Thus $\ker(f_{m-1})$ is projective, and hence $pd(M) \leq m$.

Proposition 5.5. The following are equivalent for a ring $R$:

1. All finitely presented left $R$-modules are of projective dimension $\leq d$.
2. Every $(1, d)$-injective left $R$-module is FP-injective.

Moreover, if $R$ is a left $(1, d)$-coherent ring, then the above conditions are equivalent to:

3. Every $(1, d)$-flat right $R$-module is flat.

In this case, $R$ is a left coherent ring.

Proof. (1) $\Rightarrow$ (2) is obvious by definition. (2) $\Rightarrow$ (1): Let $M$ be a finitely presented left $R$-module. Then $M \in \mathcal{I}_{1,d}$ by (2). So by Theorem 3.3 (1) and the proof of [20, Theorem 3.4], $M$ is a direct summand in a left $R$-module $N$ such that $N$ is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal $\lambda$, $N_0 = 0$, and $N_{\alpha+1}/N_\alpha$ is a finitely presented left
R-module of projective dimension \( \leq d \) for all \( \alpha < \lambda \). It follows that \( pd(M) \leq d \) by [11, Exercise 7.3.2, p. 162].

(2) \( \Rightarrow \) (3): Let \( M \) be any \((1, d)\)-flat right \( R \)-module. Then \( M^+ \) is \((1, d)\)-injective by Lemma 3.5, and so \( M^+ \) is \( FP \)-injective by (2). Hence \( M \) is flat.

(3) \( \Rightarrow \) (2): Let \( M \) be any \((1, d)\)-injective left \( R \)-module. Then \( M^+ \) is \((1, d)\)-flat by Theorem 4.3 since \( R \) is left \((1, d)\)-coherent, and so \( M^+ \) is flat by (3). On the other hand, by [5, Lemma 2.7 (1)], for any finitely presented left \( R \)-module \( N \), there is an exact sequence

\[
\text{Tor}_1(M^+, N) \to (\text{Ext}^1(N, M))^+ \to 0.
\]

Thus \( M \) is \( FP \)-injective.

In this case, note that every direct product \( \Pi M_i \) of any family \( \{ M_i \} \) of flat right \( R \)-modules is \((1, d)\)-flat by Theorem 4.3 since \( R \) is left \((1, d)\)-coherent, and hence \( \Pi M_i \) is flat by (3). Thus \( R \) is left coherent. \( \square \)

Let \( d_1 \) and \( d_2 \) be positive integers such that \( d_1 < d_2 \). If \( M \) is \((m, d_2)\)-injective (resp. \((m, d_2)\)-flat), then \( M \) is \((m, d_1)\)-injective (resp. \((m, d_1)\)-flat). However, the converse is not true in general as shown by the following example.

Example 5.6. Take \( R \) to be a commutative coherent ring with \( wD(R) = d_2 \), for example, let \( R = S[X_1, X_2, \ldots, X_{d_2}] \), the ring of polynomials in \( d_2 \) indeterminates over a commutative von Neumann regular ring \( S \) (see [15, Theorem 1.3.17]). Then all finitely presented \( R \)-modules are of projective dimension \( \leq d_2 \) by [19, Theorem 3.3]. Thus there exists a \((1, d_1)\)-injective \( R \)-module (resp. \((1, d_1)\)-flat \( R \)-module) which is not \((1, d_2)\)-injective (resp. \((1, d_2)\)-flat) by Proposition 5.5.

Recall that a ring \( R \) is called left semihereditary if every finitely generated left ideal of \( R \) is projective. Specializing Proposition 5.5 to the case \( d = 1 \), we have

Corollary 5.7 ([17, Corollary 1]). The following are equivalent for a ring \( R \):

1. \( R \) is a left semihereditary ring.
2. Every \((1, 1)\)-injective \( R \)-module is \( FP \)-injective.
3. Every \((1, 1)\)-flat right \( R \)-module is flat.

Theorem 5.8. The following are equivalent for a ring \( R \):

1. Every left \( R \)-module in \( P_{m,d} \) is of projective dimension \( \leq m - 1 \).
2. Every left \( R \)-module is \((m, d)\)-injective.
3. Every right \( R \)-module is \((m, d)\)-flat.
Every left $R$-module in $\perp I_{m,d}$ is projective.

Every right $R$-module in $\mathcal{F}_{m,d}^\perp$ is injective.

$R$ is a left $(m, d)$-coherent ring, and every left $R$-module in $\perp I_{m,d}$ is $(m, d)$-injective.

$R$ is a left $(m, d)$-coherent ring, and every right $R$-module in $\mathcal{F}_{m,d}^\perp$ is $(m, d)$-flat.

**Proof.** (1) $\Rightarrow$ (2): Let $M$ be any left $R$-module and $N \in P_{m,d}$. Then $pd(N) \leq m - 1$ by (1), and so $\text{Ext}^m(N, M) = 0$. Thus $M$ is $(m, d)$-injective.

(2) $\Rightarrow$ (3) follows from Lemma 3.5.

(3) $\Rightarrow$ (1): Let $P \in P_{m,d}$. Then there is an exact sequence $F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{F_1} F_0 \xrightarrow{P} 0$ with each $F_i$ finitely generated projective. Since $\text{T}or^m(Q, P) = 0$ for any right $R$-module $Q$ by (3), $fd(P) \leq m - 1$. Hence $K = \ker(f_{m-2})$ is flat. Note that $K$ is finitely presented, and so $K$ is projective. Thus $pd(P) \leq m - 1$.

(2) $\Leftrightarrow$ (4) and (3) $\Leftrightarrow$ (5) follow from Theorem 3.3.

(2) $\Rightarrow$ (6) and (3) $\Rightarrow$ (7) are obvious by Theorem 4.3.

(6) $\Rightarrow$ (2): Let $M$ be a left $R$-module. By Theorem 3.3, $M$ has a special $\perp I_{m,d}$-precover, that is, there is a short exact sequence $0 \rightarrow K \rightarrow N \rightarrow M \rightarrow 0$, where $K \in I_{m,d}$ and $N \in \perp I_{m,d}$. Since $N \in I_{m,d}$ by (6), $M \in I_{m,d}$ by Theorem 4.4.

(7) $\Rightarrow$ (3): Let $M$ be a right $R$-module. By Theorem 3.3 and Wakamatsu’s Lemma (see [22, Section 2.1]), there is a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$, where $N \in \mathcal{F}_{m,d}^\perp$ and $L \in \mathcal{F}_{m,d}$. Then $0 \rightarrow L^+ \rightarrow N^+ \rightarrow M^+ \rightarrow 0$ is exact. Note that $N \in F_{m,d}$ by (7). So $L^+ \in I_{m,d}$ and $N^+ \in I_{m,d}$ by Lemma 3.5. Thus $M^+ \in I_{m,d}$ by Theorem 4.4, and hence $M \in F_{m,d}$, as required. □

**Corollary 5.9.** The following are equivalent for a ring $R$:

(1) For any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of left $R$-modules, if $A$ and $B$ are finitely generated and projective, so is $C$.

(2) For any exact sequence $0 \rightarrow A_n \rightarrow A_{n-1} \rightarrow \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow 0$ with $n \geq 2$, if each $A_i (1 \leq i \leq n)$ is finitely generated and projective, then so is $A_0$.

(3) Every left $R$-module is $(1, 1)$-injective.

(4) Every right $R$-module is $(1, 1)$-flat.

(5) $R$ is $(1, 1)$-injective as a left $R$-module.

**Proof.** The equivalence of (1) through (4) follows from Theorem 5.8 by letting $m = d = 1$. (4) $\Leftrightarrow$ (5) comes from Corollary 5.3 (3). □
We end this paper with the following

Remark 5.10. (1) Any von Neumann regular ring satisfies the equivalent conditions in Corollary 5.9, but the converse is not true. For example, let $R$ be an algebra over a field $F$ with basis $\{1\} \cup \{e_i : i = 0, 1, 2, \ldots\} \cup \{x_i : i = 1, 2, \ldots\}$ such that $1$ is the unity of $R$ and, for all $i$ and $j$, $e_i e_j = \delta_{ij} x_j$, $x_i e_j = \delta_{i,j+1} x_j$, $e_i x_j = \delta_{ij} x_j$, and $x_i x_j = 0$. Then $R$ is $FP$-injective as a left $R$-module but not $FP$-injective as a right $R$-module (see [6, Example 2]). So $R$ satisfies the equivalent conditions in Corollary 5.9 but it is not von Neumann regular.

(2) Let $0 \to A \to B \to C \to 0$ be an exact sequence of left $R$-modules with $A$ and $B$ finitely generated projective. In general, $C$ is not projective. For example, let $R = \mathbb{Z}$, the ring of integers. In the exact sequence $0 \to 2R \to R \to R/2R \to 0$, $2R$ and $R$ are finitely generated projective, but $R/2R$ is not projective.

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