A Radon–Nikodym theorem for completely $n$-positive linear maps on pro-$C^*$-algebras and its applications

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Abstract. The order relation on the set of completely $n$-positive linear maps from a pro-$C^*$-algebra $A$ to $L(H)$, the $C^*$-algebra of bounded linear operators on a Hilbert space $H$, is characterized in terms of the representation associated with each completely $n$-positive linear map. Also, the pure elements in the set of all completely $n$-positive linear maps from $A$ to $L(H)$ and the extreme points in the set of completely $n$-positive linear maps from a unital $C^*$-algebra $A$ to $L(H)$ are characterized in terms of the representation induced by each completely $n$-positive linear map.

1. Introduction and preliminaries

STINESPRING [12] showed that any completely positive linear map from a $C^*$-algebra $A$ to $L(H)$, the $C^*$-algebra of bounded linear operators on a Hilbert space $H$, induces a representation of $A$ on another Hilbert space that generalizes the GNS construction for positive linear functionals on $C^*$-algebras. In [1], ARVESON proved a Radon–Nikodym type theorem which gives a description of the order relation on the set $CP_\infty(A, L(H))$ of all completely positive linear maps from a $C^*$-algebra $A$ to $L(H)$ in terms of the Stinespring representation associated with each completely positive linear map, and using this theorem, he established characterizations of pure elements in the set of completely positive linear maps from $A$ to $L(H)$ and extreme elements in the set of completely $n$-positive linear maps from $A$ to $L(H)$.

Mathematics Subject Classification: 46L05.

Key words and phrases: completely $n$-positive linear maps, pure completely $n$-positive linear maps; extreme completely $n$-positive linear maps, pro-$C^*$-algebras.

This research was supported by grant CNCSIS (Romanian National Council for Research in High Education)-code A 1065/2006.
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maps $[p_{ij}]_{i,j=1}^n$ from a unital $C^*$-algebra $A$ to $L(H)$ such that $p_{ij}(1) = I_H$ for all $i \in \{1, 2, \ldots, n\}$ and $p_{ij}(1) = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ in terms of the Stinespring representation associated with each completely positive linear map.

Kaplan [10] introduced the notion of multi-positive linear functional on a $C^*$-algebra $A$ (that is, an $n \times n$ matrix of linear functionals on $A$ which verifies the positivity condition) and showed that any multi-positive linear functional on $A$ induces a representation of $A$ on a Hilbert space. Also, he proved a Radon–Nikodym type theorem for multi-positive linear functionals on a $C^*$-algebra $A$. In [13], [14] Suen considered the $n \times n$ matrices of continuous linear maps from a $C^*$-algebra $A$ to $L(H)$ and showed that a unital completely $n$-positive linear map from a unital $C^*$-algebra $A$ to $L(H)$ induces a representation of $A$ on a Hilbert space in terms of the Stinespring construction.

In this paper we generalize various earlier results by Arveson and Kaplan. The paper is organized as follows. In Section 2, we establish a relationship between the comparability of nondegenerate representations of $A$ and the matricial order structure of $B(A, L(H))$, the vector space of continuous linear maps from $A$ to
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Let $A$ be a pro-$C^*$-algebra and let $H$ be a Hilbert space. An $n \times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ of continuous linear maps from $A$ to $L(H)$ can be regarded as a linear map $\rho$ from $M_n(A)$ to $M_n(L(H))$ defined by

$$\rho ([a_{ij}]_{i,j=1}^n) = [\rho_{ij}(a_{ij})]_{i,j=1}^n.$$  

It is not difficult to check that $\rho$ is continuous. We say that $[\rho_{ij}]_{i,j=1}^n$ is an $n$-positive (respectively completely $n$-positive) linear map from $A$ to $L(H)$ if the linear map $\rho$ from $M_n(A)$ to $M_n(L(H))$ is positive (respectively completely positive). The set of all completely positive linear maps from $A$ to $L(H)$ is denoted by $CP^n_{\infty}(A, L(H))$ and the set of all completely $n$-positive linear maps from $A$ to $L(H)$ is denoted by $CP^n_{\infty}(A, L(H))$. If $\rho$ and $\theta$ are two elements in $CP^n_{\infty}(A, L(H))$, we say that $\theta \leq \rho$ if $\rho - \theta$ is an element in $CP^n_{\infty}(A, L(H))$.

Remark 2.1. In the same manner as in the proof of Theorem 1.4 in [5], we can show that the map $S$ from $CP^n_{\infty}(A, L(H))$ to $CP^n_{\infty}(A, M_n(L(H)))$ defined by

$$S ([\rho_{ij}]_{i,j=1}^n) (a) = [\rho_{ij}(a)]_{i,j=1}^n$$

is an affine order isomorphism.

The following theorem is a particular case of Theorem 3.4 in [9].
Theorem 2.2. Let $A$ be a pro-$C^*$-algebra, let $H$ be a Hilbert space and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely $n$-positive linear map from $A$ to $L(H)$. Then there is a representation $\Phi_\rho$ of $A$ on a Hilbert space $H_\rho$ and there are $n$ elements $V_{\rho,1}, \ldots, V_{\rho,n}$ in $L(H_\rho)$ such that
(a) $\rho_{ij}(a) = V_{\rho,i}^* \Phi_\rho(a) V_{\rho,j}$ for all $a \in A$ and for all $i, j \in \{1, \ldots, n\};$
(b) $\{\Phi_\rho(a) V_{\rho,i} \xi; a \in A, \xi \in H, 1 \leq i \leq n\}$ spans a dense subspace of $H_\rho$.

The $n+2$ tuple $(\Phi_\rho, H_\rho, V_{\rho,1}, \ldots, V_{\rho,n})$ constructed in Theorem 2.2 is said to be the Stinespring representation of $A$ associated with the completely $n$-positive linear map $\rho$.

Remark 2.3. The representation associated with a completely $n$-positive linear map is unique up to unitary equivalence [9, Theorem 3.4].

Remark 2.4. Let $(\Phi_\rho, H_\rho, V_{\rho,1}, \ldots, V_{\rho,n})$ be the Stinespring representation associated with a completely $n$-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$. For each $i \in \{1, \ldots, n\}$, we denote by $H_i$ the Hilbert subspace of $H_\rho$ generated by $\{\Phi_\rho(a) V_{\rho,i} \xi; a \in A, \xi \in H\}$ and by $P_i$ the projection in $L(H_\rho)$ whose range is $H_i$. Then $P_i \in \Phi_\rho(A)'$, the commutant of $\Phi_\rho(A)$ in $L(H_\rho)$, and $a \mapsto \Phi_\rho(a) P_i$ is a representation of $A$ on $H_i$ which is unitarily equivalent with the Stinespring representation associated with $\rho_{ii}$ for all $i \in \{1, \ldots, n\}$. Since $\Phi_\rho$ is a nondegenerate representation of $A,$
$$\lim_{\lambda} \Phi_\rho(e_\lambda) \xi = \xi$$
for some approximate unit $\{e_\lambda\}_{\lambda \in \Lambda}$ for $A$ and for all $\xi \in H$ [7, Proposition 4.2], and then
$$P_i V_{\rho,i} \xi = \lim_{\lambda} P_i \Phi_\rho(e_\lambda) V_{\rho,i} \xi = V_{\rho,i} \xi$$
for all $\xi \in H$ and for all $i \in \{1, \ldots, n\}$. Therefore $P_i V_{\rho,i} = V_{\rho,i}$ for all $i \in \{1, \ldots, n\}$.

Remark 2.5. If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a diagonal completely $n$-positive linear map from $A$ to $L(H)$ (that is, $\rho_{ij} = 0$ if $i \neq j$), then the Stinespring representation associated with $\rho$ is unitarily equivalent with the direct sum of the Stinespring representations associated with $\rho_{ii}, i \in \{1, \ldots, n\}$.

Two completely $n$-positive linear maps $\rho$ and $\theta$ from $A$ to $L(H)$ are called disjoint (respectively unitarily equivalent) if the representations of $A$ induced by $\rho$ respectively $\theta$ are disjoint (respectively unitarily equivalent).

The following proposition is an analogue of Proposition 2.2 in [10] for completely $n$-positive linear maps.
Proposition 2.6. Let \( \rho_{11} \) and \( \rho_{22} \) be two completely positive linear maps from \( A \) to \( L(H) \). Then \( \rho_{11} \) and \( \rho_{22} \) are disjoint if and only if there are no nonzero continuous linear maps \( \rho_{12} \) and \( \rho_{21} \) such that \( \rho = [\rho_{ij}]_{i,j=1}^{2} \) is a completely 2-positive linear map from \( A \) to \( L(H) \).

Proof. Suppose that \( \rho_{11} \) and \( \rho_{22} \) are disjoint and \( \rho = [\rho_{ij}]_{i,j=1}^{2} \) is a completely 2-positive linear map from \( A \) to \( L(H) \). Let \( (\Phi_{\rho}, H_{\rho}, V_{\rho,1}, V_{\rho,2}) \) be the Stinespring representation associated with \( \rho \). Since \( \rho_{11} \) and \( \rho_{22} \) are disjoint, the central carriers of projections \( P_{1} \) and \( P_{2} \) (see Remark 2.4) are orthogonal and so \( P_{1}P_{2} = 0 \). Then

\[
\rho_{12}(a) = V_{\rho,1}^{*}\Phi_{\rho}(a)V_{\rho,2} = V_{\rho,1}^{*}P_{1}\Phi_{\rho}(a)P_{2}V_{\rho,2} = V_{\rho,1}^{*}\Phi_{\rho}(a)P_{2}V_{\rho,2} = 0
\]

for all \( a \in A \), and since \( \rho_{21}(a) = (\rho_{12}(a^{*}))^{*} \) for all \( a \in A \), we conclude that \( \rho_{12} = \rho_{21} = 0 \).

Conversely, suppose that there are no nonzero continuous linear maps \( \rho_{12} \) and \( \rho_{21} \) such that \( \rho = [\rho_{ij}]_{i,j=1}^{2} \) is a completely 2-positive linear map from \( A \) to \( L(H) \). Let \( (\Phi_{\rho_{i}}, H_{\rho_{i}}, V_{\rho_{i}}) \) be the Stinespring representation associated with \( \rho_{i} \), \( i \in \{1,2\} \). The direct sum \( \Phi = \Phi_{\rho_{1}} \oplus \Phi_{\rho_{2}} \) of the representations \( \Phi_{\rho_{1}} \) and \( \Phi_{\rho_{2}} \) is a representation of \( A \) on the Hilbert space \( H_0 = H_{\rho_{1}} \oplus H_{\rho_{2}} \). We consider the linear operators \( V_{1}, V_{2} \in L(H,H_0) \) defined by \( V_{1}(\xi) = V_{\rho_{1}}\xi \) respectively \( V_{2}(\xi) = 0 \). For each \( i \in \{1,2\} \), the orthogonal projection of \( H_{0} \) on \( H_{\rho_{i}} \) is denoted by \( E_{i} \). It is not difficult to check that the range of \( E_{i} \) is generated by \( \{\Phi_{\rho}(a)V_{i}\xi; a \in A, \xi \in H_{i}\} \), \( i \in \{1,2\} \). Moreover, \( \rho_{i}(a) = V_{i}^{*}\Phi_{\rho}(a)V_{i} \), \( i \in \{1,2\} \).

Suppose that \( \rho_{11} \) and \( \rho_{22} \) are not disjoint. Then there are two nonzero projections \( F_{1} \) and \( F_{2} \) in \( \Phi(A)^{\prime} \) majorized by \( E_{1} \) respectively \( E_{2} \), and a partial isometry \( V \) in \( \Phi(A)^{\prime} \) such that \( V^{*}V = F_{1} \) and \( VV^{*} = F_{2} \). It is not difficult to check that the range of \( F_{1} \) is generated by \( \{\Phi_{\rho}(a)V_{i}\xi; a \in A, \xi \in H_{i}\} \), \( i \in \{1,2\} \). We consider the linear map \( \theta \) from \( A \) to \( M_{2}(L(H)) \) defined by

\[
\theta(a) = \begin{bmatrix}
V_{1}^{*}F_{1}\Phi_{\rho}(a)V_{1} & V_{1}^{*}\Phi_{\rho}(a)V_{1}^{*}V_{2} \\
V_{2}^{*}F_{2}\Phi_{\rho}(a)V_{2} & V_{1}^{*}F_{2}\Phi_{\rho}(a)V_{2}
\end{bmatrix}.
\]

It is not difficult to check that \( \theta \) is continuous and

\[
\theta(a^{*}b) = (M(a)W)^{*}M(b)W,
\]

where

\[
M(a) = \begin{bmatrix}
\Phi(a) & \Phi(a) \\
0 & 0
\end{bmatrix}
\]

and \( W = \begin{bmatrix} F_{1}V_{1} & 0 \\
0 & V^{*}V_{2} \end{bmatrix} \).
for all $a$ and $b$ in $A$. Then
\[
\sum_{k, l=1}^{m} T_k^\varepsilon \theta(a_k^* a_l) T_l = \left( \sum_{k=1}^{m} M(a_k) W T_k \right)^* \left( \sum_{l=1}^{m} M(a_l) W T_l \right) \geq 0
\]
for all $T_1, \ldots, T_m \in M_2(L(H))$ and for all $a_1, \ldots, a_m \in A$. This implies that $\theta \in CP_{\infty}(A, M_2(L(H)))$ and so $S^{-1}(\theta) \in CP_{\infty}^2(A, L(H))$ (see Remark 2.1). Let $\rho = [\rho_{ij}]_{i,j=1}^{2}$, where $\rho_{12} = (S^{-1}(\theta))_{12}$ and $\rho_{21} = (S^{-1}(\theta))_{21}$. Then
\[
(S(\rho) - \theta)(a) = \begin{bmatrix}
V_1^* (E_1 - F_1) \Phi(a) V_1 & 0 \\
0 & V_2^* (E_2 - F_2) \Phi(a) V_2
\end{bmatrix}
\]
for all $a \in A$. Therefore $S(\rho) - \theta \in CP_{\infty}(A, M_2(L(H)))$. From this fact and taking into account that $\theta \in CP_{\infty}(A, M_2(L(H)))$, we conclude that $S(\rho) \in CP_{\infty}(A, M_2(L(H)))$ and then by Remark 2.1, $\rho \in CP_{\infty}^2(A, L(H))$. If $\rho_{12} = 0$, then $V_1^* \Phi(a) V_2 = 0$ for all $a \in A$. This implies that
\[
(V^* \Phi(a) V_2 \xi, \Phi(a) F_1 V_1 \eta) = (V_1^* \Phi(a^* a) V^* V_2 \xi, \eta) = 0
\]
for all $a \in A$ and for all $\xi, \eta \in H$. Since the range of $F_1$ is generated by $\{\Phi(a) F_1 V_2 \xi ; a \in A, \xi \in H\}$ and for all $\xi \in H$ and $a \in A$, $V^* \Phi(a) V_2 \xi$ is an element in the range of $F_1$, the preceding relation yields that $V^* \Phi(a) V_2 \xi = 0$ for all $a \in A$ and for all $\xi \in H$. Then $\Phi(a) F_2 V_2 \xi = V V^* \Phi(a) V_2 \xi = 0$ for all $a \in A$ and for all $\xi \in H$ and so $F_2 = 0$. This is a contradiction, since $F_2$ is a nonzero projection in $\Phi(A)'$. Therefore $\rho_{12} \neq 0$. Thus we have found a completely 2-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^{2}$ such that $\rho_{12} \neq 0$, a contradiction. Therefore $\rho_{11}$ and $\rho_{22}$ are disjoint and the proposition is proved.

\[
\square
\]

3. The Radon–Nikodym theorem for completely $n$-positive linear maps

Let $A$ be a pro-$C^*$-algebra and let $H$ be a Hilbert space.

**Lemma 3.1.** Let $\rho = [\rho_{ij}]_{i,j=1}^{n}$ be an element in $CP_{\infty}^n(A, L(H))$. If $T$ is a positive element in $\Phi_\rho(A)'$, the commutant of $\Phi_\rho(A)$ in $L(H_\rho)$, then the map $\rho_T$ from $M_n(A)$ to $M_n(L(H))$ defined by
\[
\rho_T \left([a_{ij}]_{i,j=1}^{n}\right) = \left[V_1^* T \Phi_\rho(a_{ij}) V_1\right]_{i,j=1}^{n}
\]
is a completely $n$-positive linear map from $A$ to $L(H)$.
PROOF. It is not difficult to check that $\rho_T$ is a matrix of continuous linear maps from $A$ to $L(H)$, the $(i,j)$-entry of the matrix $\rho_T$ being the continuous linear map $(\rho_T)_{ij}$ from $A$ to $L(H)$ defined by

$$(\rho_T)_{ij}(a) = V_{\rho, i}^* (T \Phi_{\rho}(a)) V_{\rho, j}.$$ 

Also it is not difficult to check that

$$S(\rho_T)(a^* b) = \left(M_{T^\frac{1}{2}}(a)V\right)^* \left(M_{T^\frac{1}{2}}(b)V\right),$$

for all $a, b \in A$, where $T^\frac{1}{2}$ is the square root of $T$,

$$M_{T^\frac{1}{2}}(a) = \begin{bmatrix} T^\frac{1}{2} \Phi_{\rho}(a) & \cdots & T^\frac{1}{2} \Phi_{\rho}(a) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} V_{\rho, 1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & V_{\rho, n} \end{bmatrix}.$$ 

Then

$$\sum_{k, l=1}^m T^\frac{1}{2} S(\rho_T)(a^*_k a_k) T_k = \sum_{k, l=1}^m T^\frac{1}{2} \left(M_{T^\frac{1}{2}}(a_l)V\right)^* \left(M_{T^\frac{1}{2}}(a_k)V\right) T_k$$

$$= \left(\sum_{k=1}^m M_{T^\frac{1}{2}}(a_k) VT_k\right) \left(\sum_{k=1}^m M_{T^\frac{1}{2}}(a_k) VT_k\right)^* \geq 0$$

for all $T_1, \ldots, T_m \in M_n(L(H))$ and for all $a_1, \ldots, a_m \in A$. This shows that $S(\rho_T) \in CP_{\infty}(A, M_n(L(H)))$ and by Remark 2.1, $\rho_T \in CP_{\infty}(A, L(H))$. \hfill $\square$

Remark 3.2. Let $\rho = [\rho_{ij}]_{i,j=1}^n \in CP_{\infty}(A, L(H))$.

1. If $I_{H_{\rho}}$ is the identity map on $H_{\rho}$, then $\rho_{I_{H_{\rho}}} = \rho$.
2. If $T_1$ and $T_2$ are two positive elements in $\Phi_{\rho}(A)'$, then $\rho_{T_1 + T_2} = \rho_{T_1} + \rho_{T_2}$.
3. If $T$ is a positive element in $\Phi_{\rho}(A)'$ and $\alpha$ is a positive number, then $\rho_{\alpha T} = \alpha \rho_T$.

Lemma 3.3. Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be an element in $CP_{\infty}(A, L(H))$ and let $T_1$ and $T_2$ be two positive elements in $\Phi_{\rho}(A)'$. Then $T_1 \leq T_2$ if and only if $\rho_{T_1} \leq \rho_{T_2}$.

PROOF. First, we suppose that $T_1 \leq T_2$. Then $\rho_{T_2 - T_1} \in CP_{\infty}(A, L(H))$, and since

$$(\rho_{T_2} - \rho_{T_1})([a_{ij}]_{i,j=1}^n) = [V_{\rho, i}^* (T_2 - T_1) \Phi_{\rho}(a_{ij}) V_{\rho, j}]_{i,j=1}^n = \rho_{T_2 - T_1}([a_{ij}]_{i,j=1}^n)$$

for all $[a_{ij}]_{i,j=1}^n \in M_n(A)$, we conclude that $\rho_{T_1} \leq \rho_{T_2}$. 

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Conversely, suppose that $\rho_{T_1} \leq \rho_{T_2}$. Let $\sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\rho}(a_{ki})V_{\rho,i}\xi_{ki} \in H_{\rho}$. Then

$$\left\langle (T_2 - T_1) \left( \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\rho}(a_{ki})V_{\rho,i}\xi_{ki} \right), \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\rho}(a_{ki})V_{\rho,i}\xi_{ki} \right\rangle$$

$$= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle V_{\rho,i}^*(T_2 - T_1)\Phi_{\rho}(a_{ki}^*a_{lj})V_{\rho,j}\xi_{lj}, \xi_{ki} \rangle$$

$$= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \left( \mathcal{S}(\rho_{T_2}) - \mathcal{S}(\rho_{T_1}) \right) (a_{ki}^*a_{lj}) \left( \tilde{\xi}_{ljp} \right)_{p=1}^{n}, \left( \tilde{\xi}_{kip} \right)_{p=1}^{n} \geq 0,$$

where $\tilde{\xi}_{kip} = \begin{cases} \xi_{ki} & \text{if } p = i \text{ and } 1 \leq k \leq m \\ 0 & \text{if } p \neq i \text{ and } 1 \leq k \leq m \end{cases}$. From this fact and taking into account that $\{\Phi_{\rho}(a)V_{\rho,i}\xi; a \in A, \xi \in H, 1 \leq i \leq n\}$ spans a dense subspace of $H_{\rho}$, we conclude that $T_2 - T_1 \geq 0$ and the lemma is proved. \[\square\]

**Lemma 3.4.** Let $\rho = [\rho_{ij}]_{i=1}^{n}$ and $\theta = [\theta_{ij}]_{i=1}^{n}$ be two elements in $CP_{\infty}^{n}(A, L(H))$. If $\theta \leq \rho$, then there is an element $W \in L(H_{\rho}, H_{\theta})$ such that

(a) $||W|| \leq 1$;
(b) $WV_{\rho,i} = V_{\theta,i}$ for all $i \in \{1, \ldots, n\}$;
(c) $W\Phi_{\rho}(a) = \Phi_{\theta}(a)W$ for all $a \in A$.

**Proof.** Since

$$\left\langle \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\theta}(a_{ki})V_{\theta,i}\xi_{ki}, \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\theta}(a_{ki})V_{\theta,i}\xi_{ki} \right\rangle$$

$$= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle V_{\theta,j}^*\Phi_{\theta}(a_{kj}^*a_{ki})V_{\theta,i}\xi_{ki}, \xi_{lj} \rangle = \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \langle \theta_{ji} (a_{kj}^*a_{ki}) \xi_{ki}, \xi_{lj} \rangle$$

$$= \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \left( \mathcal{S}(\theta) (a_{kj}^*a_{ki}) \left( \tilde{\xi}_{kip} \right)_{p=1}^{n}, \left( \tilde{\xi}_{ljp} \right)_{p=1}^{n} \right)$$

$$\leq \sum_{k,l=1}^{m} \sum_{i,j=1}^{n} \left( \mathcal{S}(\rho) (a_{kj}^*a_{ki}) \left( \tilde{\xi}_{kip} \right)_{p=1}^{n}, \left( \tilde{\xi}_{ljp} \right)_{p=1}^{n} \right)$$

$$= \left\langle \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\rho}(a_{ki})V_{\rho,i}\xi_{ki}, \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_{\rho}(a_{ki})V_{\rho,i}\xi_{ki} \right\rangle$$

for all $a_{ki} \in A$ and $\xi_{ki} \in H$, $1 \leq i \leq n$, $1 \leq k \leq m$, and since $\{\Phi_{\rho}(a)V_{\rho,i}\xi; a \in A, \xi \in H, 1 \leq i \leq n\}$ spans a dense subspace of $H_{\rho}$, there is a unique bounded linear
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Let $W$ be a linear map from $H_\rho$ to $H_\theta$ such that

$$W \left( \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_\rho(a_{ki})V_{\rho,i} \xi_{ki} \right) = \sum_{k=1}^{m} \sum_{i=1}^{n} \Phi_\theta(a_{ki})V_{\theta,i} \xi_{ki}.$$

Moreover, $\|W\| \leq 1$. Let \{e_\lambda\}_{\lambda \in \Lambda} be an approximate unit for $A$, $\xi \in H$ and $i \in \{1, \ldots, n\}$. Then

$$WV_{\rho,i} \xi = \lim_{\lambda} W\Phi_\rho(e_\lambda)V_{\rho,i} \xi = \lim_{\lambda} \Phi_\theta(e_\lambda)V_{\theta,i} \xi = V_{\theta,i} \xi.$$

Therefore $WV_{\rho,i} = V_{\theta,i}$ for all $i \in \{1, \ldots, n\}$. It is not difficult to check that $W \Phi_\rho(a) = \Phi_\theta(a)W$ for all $a \in A$.

Let $\rho = [\rho_{ij}]_{i,j=1}^{n} \in CP^\infty_n(A, L(H))$. We consider the following sets:

$$[0, \rho] = \{ \theta \in CP^\infty_n(A, L(H)); \theta \leq \rho \}$$

and

$$[0, I_\rho] = \{ T \in \Phi_\rho(A)'; 0 \leq T \leq I_{H_{\rho}} \}.$$

The following theorem can be regarded as a Radon–Nikodym type theorem for completely multi-positive linear maps.

**Theorem 3.5.** The map $T \mapsto \rho_T$ from $[0, I_\rho]$ to $[0, \rho]$ is an affine order isomorphism.

**Proof.** According to Lemmas 3.1 and 3.3 and Remark 3.2, it remains to show that the map $T \mapsto \rho_T$ from $[0, I_\rho]$ to $[0, \rho]$ is bijective.

Let $T \in [0, I_\rho]$ such that $\rho_T = 0$. Then $V_{\rho,i}^{*}T \Phi_\rho(a^*a)V_{\rho,i} = 0$ and so $T^{\frac{1}{2}} \Phi_\rho(a)V_{\rho,i} = 0$ for all $a \in A$ and for all $i \in \{1, \ldots, n\}$. From this fact and taking into account that $\{\Phi_\rho(a)V_{\rho,i} \xi; a \in A, \xi \in H, 1 \leq i \leq n\}$ spans a dense subspace of $H_\rho$, we conclude that $T = 0$ and so the map $T \mapsto \rho_T$ from $[0, I_\rho]$ to $[0, \rho]$ is injective.

Let $\theta \in [0, \rho]$. If $W$ is the bounded linear operator from $H_\rho$ to $H_\theta$ constructed in Lemma 3.4, then $W^*W \in [0, I_\rho]$. Let $T = W^*W$. From

$$\theta ([a_{ij}]_{i,j=1}^{n}) = [V_{\rho,i}^{*}W^* \Phi_\rho(a_{ij})V_{\rho,j}]_{i,j=1}^{n} = [(WV_{\rho,i})^* \Phi_\rho(a_{ij})WV_{\rho,j}]_{i,j=1}^{n} = [V_{\theta,i}^{*}W^* \Phi_\theta(a_{ij})V_{\theta,j}]_{i,j=1}^{n} = \rho_T ([a_{ij}]_{i,j=1}^{n})$$

for all $[a_{ij}]_{i,j=1}^{n} \in M_n(A)$, we deduce that $\theta = \rho_T$ and the theorem is proved. \qed
We say that a completely \( n \)-positive linear map \( \rho \) from a pro-\( C^* \)-algebra \( A \) to \( L(H) \) is pure if for every \( \theta \in CP_{n}^{\infty}(A, L(H)) \) with \( \theta \leq \rho \) there is \( \alpha \in [0, 1] \) such that \( \rho = \alpha \theta \).

A representation \( \Phi \) of a pro-\( C^* \)-algebra \( A \) on a Hilbert space \( H \) is irreducible if and only if the commutant \( \Phi(A)' \) of \( \Phi(A) \) in \( L(H) \) consists of the scalar multiplies of the identity map on \( H \) \cite{4}.

**Corollary 3.6.** Let \( \rho \in CP_{n}^{\infty}(A, L(H)) \). Then the following statements are equivalent:

1. \( \rho \) is pure;
2. The representation \( \Phi_{\rho} \) of \( A \) associated with \( \rho \) is irreducible.

**Remark 3.7.** If \( \rho \) is a continuous positive functional on \( A \), then \( \rho \in CP_{1}^{\infty}(A, L(C)) \). Moreover, \( \Phi_{\rho} \) is the GNS representation of \( A \) associated with \( \rho \), and Corollary 3.6 states that \( \rho \) is pure if and only if the representation \( \Phi_{\rho} \) is irreducible. This is a particular case of the known result which states that a continuous positive linear functional \( \rho \) on a lmc\( * \)-algebra \( A \) with bounded approximate unit is pure if and only if the GNS representation of \( A \) associated with \( \rho \) is topological irreducible (see, for example, Corollary 3.7 of \cite{4}).

**4. Applications of the Radon–Nikodym theorem**

Let \( A \) be a unital pro-\( C^* \)-algebra and let \( H \) be a Hilbert space.

**Lemma 4.1.** Let \( \rho = [\rho_{ij}]_{i,j=1}^{n} \in CP_{n}^{\infty}(A, L(H)) \). If \( \rho_{ii}, i \in \{1,2,\ldots,n\} \) are unitarily equivalent pure unital completely positive linear maps from \( A \) to \( L(H) \) and for all \( i,j \in \{1,\ldots,n\} \) with \( i \neq j \) there is a unitary element \( u_{ij} \) in \( A \) such that \( \rho_{ij}(u_{ij}) = 1_{H} \), then \( \rho \) is pure.

**Proof.** Let \( (\Phi_{\rho}, H_{\rho}, V_{\rho,1}, \ldots, V_{\rho,n}) \) be the Stinespring representation associated with \( \rho \). Let \( i, j \in \{1,\ldots,n\} \) with \( i \neq j \). Since

\[
\|\Phi_{\rho}(u_{ij})V_{\rho,j} - V_{\rho,i}\|^2 = \| (V_{\rho,j}^{*} \Phi_{\rho}(u_{ij}^{*}) - V_{\rho,i}^{*}) (\Phi_{\rho}(u_{ij})V_{\rho,j} - V_{\rho,i}) \|
= \| \rho_{jj}(1) - \rho_{ij}(u_{ij}) - (\rho_{ij}(u_{ij}))^{*} + \rho_{ii}(1) \| = 0
\]

we conclude that \( \Phi_{\rho}(u_{ij})V_{\rho,j} = V_{\rho,i} \). This implies that the vector subspaces \( H_{i} \) and \( H_{j} \) of \( H_{\rho} \) coincide. Therefore \( H_{\rho} = H_{i} \) for all \( i \in \{1,\ldots,n\} \), and since \( \Phi_{\rho}(\cdot)P_{i} \) acts irreducibly on \( H_{i}, i \in \{1,\ldots,n\} \), the representation \( \Phi_{\rho} \) of \( A \) is irreducible and, by Corollary 3.6, \( \rho \) is pure. \( \square \)
The following proposition is a generalization of Propositions 2.5 in [10] and 4.5 in [8].

**Proposition 4.2.** Let \( \rho = [\rho_{ij}]_{i,j=1}^n \in CP_n^\infty(A, L(H)) \) such that \( \rho_{ii}, i \in \{1, 2, \ldots, n\} \) are pure unital completely positive linear maps from \( A \) to \( L(H) \). If whenever \( \rho_{ii} \) is unitarily equivalent with \( \rho_{jj} \) for some \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), there is a unitary element \( u_{ij} \in A \) such that \( \rho_{ij}(u_{ij}) = I_H \), then \( \rho \) is an extreme point in the set of all \( \theta = [\theta_{ij}]_{i,j=1}^n \in CP_n^\infty(A, L(H)) \) such that \( \theta_{ii}(1) = I_H \) for all \( i \in \{1, \ldots, n\} \).

**Proof.** By Proposition 2.6 and Lemma 4.1, \( \rho \) is a block-diagonal sum of disjoint pure completely \( n_r \)-positive linear maps \( \rho_r = [\rho_{(k)(j)}]_{k=1}^{n_r}, 1 \leq r \leq m \), where \( m \) is the number of equivalence classes of the pure completely positive linear maps \( \rho_{ii} \) and \( n_1 + \cdots + n_m = n \). By Corollary 3.6, the representation \( \Phi_\rho \) of \( A \) induced by \( \rho \) is irreducible and, moreover, it is unitarily equivalent to a subrepresentation \( \Phi_r \) of \( \Phi_\rho \). Therefore \( \Phi_\rho \) is a direct sum of disjoint irreducible representations \( \Phi_r, r \in \{1, \ldots, m\} \). For each \( r \in \{1, \ldots, m\} \), we denote by \( E_r \) the central support of \( \Phi_r \) (this is, \( \Phi_r(a) = \Phi_r(a)E_r \) for all \( a \in A \)).

Let \( \theta = [\theta_{ij}]_{i,j=1}^n, \sigma = [\sigma_{ij}]_{i,j=1}^n \in CP_n^\infty(A, L(H)) \) with \( \theta_{ii}(1) = \sigma_{ii}(1) = I_H \) for all \( i \in \{1, \ldots, n\} \) and \( \lambda \in (0, 1) \) such that

\[
\lambda \theta + (1 - \lambda) \sigma = \rho.
\]

Then, by Theorem 3.5, there is \( T \) in \( \Phi_\rho(A)' \), \( 0 \leq T \leq I_{H_\rho} \) such that \( \lambda \theta = \rho_T \).

From this fact and taking into account that \( \Phi_r \) is irreducible and \( E_rTE_r \) is an element in \( \Phi_r(A)' \) for all \( r \in \{1, \ldots, m\} \) and \( \theta_{ii}(1) = I_H \) for all \( i \in \{1, \ldots, n\} \), we conclude that \( E_rTE_r = \lambda E_r \) for all \( r \in \{1, \ldots, m\} \). Consequently, \( T = \lambda I_{H_\rho} \) and so \( \theta = \rho \). In the same manner we obtain \( \sigma = \rho \) and so \( \rho \) is an extreme point in the set of all \( \theta = [\theta_{ij}]_{i,j=1}^n \in CP_n^\infty(A, L(H)) \) such that \( \theta_{ii}(1) = I_H \) for all \( i \in \{1, \ldots, n\} \).

Let \( A \) be a unital pro-C*-algebra, \( H \) a Hilbert space and \( CP_n^\infty(A, L(H), I) = \{\theta \in CP_n^\infty(A, L(H)); \theta_{ii}(1) = I_H, 1 \leq i \leq n \) and \( \theta_{ij}(1) = 0, 1 \leq i < j \leq n\} \). From Remark 2.1 and Proposition 4.2 we obtain the following corollary that generalizes Corollaries 2.7 in [10] and 4.7 in [8].

**Corollary 4.3.** Let \( \rho = [\rho_{ij}]_{i,j=1}^n \in CP_n^\infty(A, L(H), I) \) such that \( \rho_{ii}, i \in \{1, 2, \ldots, n\} \) are pure completely positive linear maps from \( A \) to \( L(H) \). If whenever \( \rho_{ii} \) is unitarily equivalent with \( \rho_{jj} \) for some \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), there is a unitary element \( u_{ij} \in A \) such that \( \rho_{ij}(u_{ij}) = I_H \), then the map \( \varphi \) from \( A \) to \( M_n(L(H)) \) defined by \( \varphi(a) = [\rho_{ij}(a)]_{i,j=1}^n \) is an extreme point in the set of all unital completely positive linear maps from \( A \) to \( M_n(L(H)) \).
The following theorem gives a characterization of the extreme points in $CP^n_\infty(A, L(H), I)$.

**Theorem 4.4.** Let $\rho \in CP^n_\infty(A, L(H), I)$ and $P_{H_0}$ the projection of $H_0$ on the Hilbert subspace $H_0$ generated by $\{V_{\rho,i}; \xi \in H, 1 \leq i \leq n\}$. Then $\rho$ is an extreme point in $CP^n_\infty(A, L(H), I)$ if and only if the map $T \mapsto P_{H_0}TP_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_0)$ is injective.

**Proof.** First we suppose that $\rho$ is an extreme point in $CP^n_\infty(A, L(H), I)$. Let $T \in \Phi_\rho(A)'$ such that $P_{H_0}TP_{H_0} = 0$. Since $P_{H_0}T^*P_{H_0} = (P_{H_0}TP_{H_0})^* = 0$, we can suppose that $T = T^*$. From

$$\langle V^*_{\rho,i}TV_{\rho,j, \xi, \eta} \rangle = \langle TV_{\rho,j, \xi, \eta} \rangle = \langle PH_0TP_{H_0}V_{\rho,j, \xi, \eta} \rangle = 0$$

for all $i, j \in \{1, \ldots, n\}$ and for all $\xi, \eta \in H$, it follows that $V^*_{\rho,i}TV_{\rho,j} = 0$ for all $i, j \in \{1, \ldots, n\}$. It is not difficult to check that there are two positive numbers $\alpha$ and $\beta$ such that $\frac{1}{2}I_{H_\rho} \leq \alpha T + \beta I_{H_\rho} \leq \frac{3}{4}I_{H_\rho}$. Moreover, $\beta \in (0, 1)$. Let $T_1 = \frac{\alpha}{\beta}T + I_{H_\rho}$ and $T_2 = I_{H_\rho} - \frac{\alpha}{1-\beta}T$. Clearly, $T_1$ and $T_2$ are positive elements in $\Phi_\rho(A)'$. Therefore $\rho_{T_1}$ and $\rho_{T_2}$ are completely $n$-positive linear maps from $A$ to $L(H)$. Moreover, since

$$(\rho_{T_1})_{ij}(1) = V^*_{\rho,i} \left( \frac{\alpha}{\beta}T + I_{H_\rho} \right) V_{\rho,j} = V^*_{\rho,i}V_{\rho,j} = \rho_{ij}(1)$$

and

$$(\rho_{T_2})_{ij}(1) = V^*_{\rho,i} \left( I_{H_\rho} - \frac{\alpha}{1-\beta}T \right) V_{\rho,j} = V^*_{\rho,i}V_{\rho,j} = \rho_{ij}(1)$$

for all $i, j \in \{1, \ldots, n\}$, $\rho_{T_1}, \rho_{T_2} \in CP^n_\infty(A, L(H), I)$. A simple calculation shows that

$$\beta\rho_{T_1} + (1-\beta)\rho_{T_2} = \rho.$$

From this fact, since $\rho$ is an extreme point, we conclude that $\rho_{T_1} = \rho_{T_2} = \rho$ and then by Theorem 3.5, $T_1 = T_2 = I_{H_\rho}$, whence $T = 0$.

Conversely, we suppose that the map $T \mapsto P_{H_0}TP_{H_0}$ from $\Phi_\rho(A)'$ to $L(H_0)$ is injective. Let $\theta, \sigma \in CP^n_\infty(A, L(H), I)$ and $\alpha \in (0, 1)$ such that $\alpha\theta + (1-\alpha)\sigma = \rho$. Then by Theorem 3.5, there is $T \in \Phi_\rho(A)'$, $0 \leq T \leq I_{H_\rho}$ such that $\alpha\theta = \rho_T$ and so

$$V^*_{\rho,i}TV_{\rho,j} = \begin{cases} \alpha I_H & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

From

$$\langle P_{H_0}(T - \alpha I_{H_\rho})P_{H_0}V_{\rho,j, \xi, \eta} \rangle = \langle V^*_{\rho,i}TV_{\rho,j, \xi, \eta} \rangle - \alpha \langle V^*_{\rho,i}V_{\rho,j, \xi, \eta} \rangle = 0.$$
for all \(i, j \in \{1, \ldots, n\}\) and for all \(\xi, \eta \in H\), we conclude that \(P_{H_0}(T - \alpha I_{H_0})P_{H_0} = 0\) and so \(T = \alpha I_{H_0}\). Consequently \(\theta = \rho\). In the same way we obtain \(\sigma = \rho\). Therefore \(\rho\) is an extreme point in \(CP_\infty(A, L(H), I)\).

\[\square\]

References


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(Received January 9, 2006; revised December 22, 2006)