The $\mathcal{L}$-dual of a Matsumoto space

By IOANA MONICA MASCA (Brasov), VASILE SORIN SABAU (Sapporo)
and HIDEO SHIMADA (Sapporo)

Abstract. In [HS1], [MHSS] the $\mathcal{L}$-duals of a Randers and Kropina space were studied. In this paper we shall discuss the $\mathcal{L}$-dual of a Matsumoto space. The metric of this $\mathcal{L}$-dual space is completely new and it brings a new idea about $\mathcal{L}$-duality because the $\mathcal{L}$-dual of Matsumoto metric can be given by means of four quadratic forms and 1-forms on $T^*M$ constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

1. Introduction

The study of $\mathcal{L}$-duality of Lagrange and Finsler space was initiated by R. Miron [Mi2] around 1980. Since then, many Finsler geometers studied this topic.

One of the remarkable results obtained are the concrete $\mathcal{L}$-duals of Randers and Kropina metrics [HS2]. However, the importance of $\mathcal{L}$-duality is by far limited to computing the dual of some Finsler fundamental functions.

Recently, in [BRS], the complicated problem of classifying Randers metrics of constant flag curvature was solved by means of duality. Other geometrical problems of $(\alpha, \beta)$-metrics might be solved on future by considering not the metric itself, but its $\mathcal{L}$-dual.

The concrete examples of $\mathcal{L}$-dual metrics are quite few [HS1], [HS2]. In the present paper we succeeded to compute the dual of another well known $(\alpha, \beta)$-metric, the Matsumoto metric. Surprisingly, despite of the quite complicated computations involved, we obtain the Hamiltonian function by means of four

Mathematics Subject Classification: 53B40, 53C60.

Key words and phrases: Matsumoto space, Finsler space, Cartan space, the duality between Finsler and Cartan spaces.
quadratic forms and a 1-form on $T^*M$. This metric is completely new and it brings a new idea about $L$-duality. The dual of an $(\alpha, \beta)$-metric can be given by means of several quadratic forms and 1-forms on $T^*M$ constructed only with the Riemannian metric coefficients, $a_{ij}(x)$ and the 1-form coefficients $b_i(x)$.

2. The Legendre transformation

2.1. Definitions. Let $F^n = (M, F)$ be a $n$-dimensional Finsler space. The fundamental function $F(x, y)$ is called an $(\alpha, \beta)$-metric if $F$ is homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a(y, y) = a_{ij}y^i y^j$, $y = y^i \frac{\partial}{\partial x^i}|_{x} \in T_x M$ is Riemannian metric, and $\beta = b_i(x) y^i$ is a 1-form on $T^*M = TM \setminus \{0\}$.

A Finsler space with the fundamental function:

$$F(x, y) = \alpha(x, y) + \beta(x, y),$$

(2.1)

is called a Randers space.

A Finsler space having the fundamental function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)},$$

(2.2)

is called a Kropina space, and one with

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)},$$

(2.3)

is called a Matsumoto space.

Let $C^n = (M, K)$ be an $n$-dimensional Cartan space having the fundamental function $K(x, p)$. We also consider Cartan spaces having the metric function of the following form:

$$K(x, p) = \sqrt{a^{ij}(x)p_ip_j + b^i(x)p_i},$$

(2.4)

or

$$K(x, p) = \frac{a^{ij}(x)p_ip_j}{b^i(x)p_i},$$

(2.5)

with $a_{ij}a^{jk} = \delta^k_i$ and we will again call these spaces Randers and Kropina spaces on the cotangent bundle $T^*M$, respectively.

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. 
The $\mathcal{L}$-dual of a Matsumoto space

It is known [MHSS] that if $L$ is a differentiable function, we can consider the fiber derivative of $L$, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$:

$$\varphi(x, y) = (x^i, \dot{\partial}_a L(x, y))$$

which is called the Legendre transformation. We can define, in this case, the function $H : U^* \to \mathbb{R}$:

$$H(x, p) = p_a y^a - L(x, y),$$

where $y = (y^a)$ is the solution of the equations:

$$p_a = \dot{\partial}_a L(x, y).$$

In the same manner, the fiber derivative is locally given by:

$$\psi(x, p) = (x^i, \dot{\partial}_a H(x, p)).$$

The function $\psi$ is a diffeomorphism between the same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \to \mathbb{R}$:

$$L(x, y) = p_a y^a - H(x, p),$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}_a H(x, p).$$

The Hamiltonian given by (2.7) is called the Legendre transformation of the Lagrangian $L$ and the Lagrangian given by (2.10) is called the Legendre transformation of the Hamiltonian $H$.

If $(M, K)$ is a Cartan space, then $(M, H)$ is a Hamilton manifold [MHSS], where $H(x, p) = \frac{1}{2} K^2(x, p)$ is 2-homogeneous on a domain of $T^* M$. So, we get the following transformation of $H$ on $U$:

$$L(x, y) = p_a y^a - H(x, p) = H(x, p).$$

**Proposition 1** ([MHSS]). The scalar field $L(x, y)$ defined by (2.12) is a positively 2-homogeneous regular Lagrangian on $U$.

Therefore, we get the Finsler metric $F$ of $U$, such that

$$L = \frac{1}{2} F^2$$

Thus, for the Cartan space $(M, K)$ one always can locally associate a Finsler space $(M, F)$ which will be called the $\mathcal{L}$-dual of a Cartan space $(M, K|_{U^*})$.

Conversely, we can associate, locally, a Cartan space to every Finsler space which will be called the $\mathcal{L}$-dual of a Finsler space $(M, F|_U)$.
3. The $(\alpha, \beta)$ Finsler – $(\alpha^*, \beta^*)$ Cartan $\mathcal{L}$-duality

Let us recall some known results.

**Theorem 3.1** ([HS1], [MHSS]). Let $(M, F)$ be a Randers space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of $b_i$. Then:

(1) If $b^2 = 1$, the $\mathcal{L}$-dual of $(M, F)$ is a Kropina space on $T^*M$ with:

$$H(x, p) = \frac{1}{2} \left( \frac{a^{ij} p_i p_j}{2b/p_i} \right)^2.$$  \hspace{1cm} (3.1)

(2) If $b^2 \neq 1$, the $\mathcal{L}$-dual of $(M, F)$ is a Randers space on $T^*M$ with:

$$H(x, p) = \frac{1}{2} \left( \sqrt{\tilde{\alpha}^{ij} p_i p_j \pm \tilde{b}^i p_i} \right)^2,$$ \hspace{1cm} (3.2)

where

$$\tilde{\alpha}^{ij} = \frac{1}{1 - b^2} a^{ij} + \frac{1}{(1 - b^2)^2} b^i b^j; \quad \tilde{b}^i = \frac{1}{1 - b^2} b^i,$$

(in (3.2) \(\prime - \prime\) corresponds to $b^2 < 1$ and \(\prime + \prime\) corresponds to $b^2 > 1$).

**Theorem 3.2** ([HS1], [MHSS]). The $\mathcal{L}$-dual of a Kropina space is a Randers space on $T^*M$ with the Hamiltonian:

$$H(x, p) = \frac{1}{2} \left( \sqrt{\tilde{\alpha}^{ij} p_i p_j \pm \tilde{b}^i p_i} \right)^2,$$ \hspace{1cm} (3.3)

where

$$\tilde{\alpha}^{ij} = \frac{b^2}{4} a^{ij}; \quad \tilde{b}^i = \frac{1}{2} b^i,$$

(in (3.3) \(\prime - \prime\) corresponds to $\beta < 0$ and \(\prime + \prime\) corresponds to $\beta > 0$).

In [HS1] the notation $\alpha^* = (a^{ij}(x)p_ip_j)^{\frac{1}{2}}, \beta^* = b^i(x)p_i$ are used, where $a^{ij}(x)$ are the reciprocal components of $a_{ij}(x)$ and $b^i(x)$ are the components of the vector field on $M$, $b^i(x) = a^{ij}(x)b_j(x)$. We can consider the metric functions $K = \alpha^* + \beta^*$ (Randers metric on $T^*M$) or $K = \frac{\alpha^2}{\beta^2}$ (Kropina metric on $T^*M$) defined on a domain $D^* \subset T^*M$. So, one can easily rewrite the previous theorems:

**Theorem 3.3.** Let $(M, F)$ be a Randers space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of $b_i$. Then:
The $L$-dual of a Matsumoto space

(1) If $b^2 = 1$, the $L$-dual of $(M, F)$ is a Kropina space on $T^*M$ with:

$$H(x, p) = \frac{1}{2} \left( \frac{\alpha^*}{\beta^*} \right)^2.$$

(3.4)

(2) If $b^2 \neq 1$, the $L$-dual of $(M, F)$ is a Randers space on $T^*M$ with:

$$H(x, p) = \frac{1}{2} \left( \alpha^* \pm \beta^* \right)^2,$$

(3.5)

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_i p_j}$ and $\beta^* = \tilde{b}^i p_i$ where

$$\tilde{a}^{ij} = \frac{1}{1 - b^2} a^{ij} + \frac{1}{(1 - b^2)^2} b^i b^j; \quad \tilde{b}^i = \frac{1}{1 - b^2} b^i,$$

(in (3.5) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

**Theorem 3.4.** The $L$-dual of a Kropina space is a Randers space on $T^*M$ with the Hamiltonian:

$$H(x, p) = \frac{1}{2} \left( \alpha^* \pm \beta^* \right)^2,$$

(3.6)

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_i p_j}$ and $\beta^* = \tilde{b}^i p_i$ where

$$\tilde{a}^{ij} = \frac{b^2}{4} a^{ij}; \quad \tilde{b}^i = \frac{1}{2} b^i,$$

(in (3.6) $'-'$ corresponds to $\beta < 0$ and $'+$' corresponds to $\beta > 0$).

We are going to compute now the dual of a Matsumoto metric. We obtain:

**Theorem 3.5.** Let $(M, F)$ be a Matsumoto space and $b = (a_{ij}b^i b^j)^{\frac{1}{2}}$ the Riemannian length of $b_i$. Then

(1) If $b^2 = 1$, the $L$-dual of $(M, F)$ is the space having the fundamental function:

$$H(x, p) = \frac{1}{2} \left( -\frac{b^i p_i}{2} \left( \sqrt{a^{ij} p_i p_j} + \sqrt{\frac{b^i p_i + \sqrt{a^{ij} p_i p_j}}{a^{ij} p_i p_j}} \right)^2 \right)^2.$$

(3.7)

where

$$\tilde{a}^{ij} = b^i b^j - a^{ij}.$$
If \( b^2 \neq 1 \), the \( \mathcal{L} \)-dual of \((M, F)\) is the space on \( T^*M \) having the fundamental function:

\[
H(x, p) = \frac{1}{2} \left( -\frac{b^2 p_i}{200} \frac{25}{\sqrt{d_{ij}^2 p_ip_j + d_{ij}^2 p_ip_j + d_{ij}^2 p_ip_j}} \right)^2,
\]

where

\[
\begin{align*}
\hat{a}_{ij} & = (b^2 - 2) a_{ij} + 2 \varepsilon_{1} a_{ij}, \\
\hat{c}_{ij} & = a_{ij} (\theta^2_1 b^2 b^j + a_{ij} a_{ij}), \\
\hat{c}_{ij} & = (2a_{ij})^2 \theta^3, \\
\sqrt{\hat{a}_{ij}^2} & = \sqrt{c_{ij}^1} - 2 \sqrt{c_{ij}^2} + \sqrt{c_{ij}^3}, \\
\hat{d}_{ij} & = d_{ij}^3 + 4m(a_{ij} b^2 - b^2 b^i), \\
\sqrt{\hat{d}_{ij}^3} & = \sqrt{d_{ij}^3} + 3\sqrt{a_{ij}}, \\
\sqrt{\hat{d}_{ij}^5} & = \sqrt{d_{ij}^3} a_{ij}, \\
\hat{d}_{ij} & = d_{ij}^3 a_{ij}, \\
\sqrt{\hat{d}_{ij}^7} & = 2 \sqrt{d_{ij}^3 + d_{ij}^4}, \\
\hat{d}_{ij} & = 200 \left( \sqrt{d_{ij}^6} + 2m a_{ij} \right) - 5 \left( 4\sqrt{d_{ij}^3} + \sqrt{d_{ij}^4} \right), \\
\hat{d}_{8} & = 4 \sqrt{d_{ij}^6} + 4a_{ij} p + 9 \sqrt{d_{ij}^3},
\end{align*}
\]

and

\[
\begin{align*}
m & = 1 - b^2, \\
n & = \frac{20b^2 - 29}{29}, \\
p & = \frac{1 - 2b^2}{2}, \\
\theta_1 & = -\frac{712b^6 - 452b^4 + 24b^2 + 1}{1728}, \\
\theta_2 & = \frac{576b^4 - 2232b^2 + 2628}{1728},
\end{align*}
\]
\[ \theta_3 = -\left( \frac{8b^2 + 1}{12} \right)^2, \]
\[ \theta_4 = \frac{2\theta^2 + 1}{6}, \]
\[ \theta_5 = \frac{11b^2 + 1}{12}, \]
\[ \varepsilon_1 = 2(\theta_2^2 - \theta_4), \]
\[ \varepsilon_2 = 3\theta_3\theta_4^2 + \theta_2^2, \]
\[ \varepsilon_3 = 4\varepsilon_2 - 2\theta_1 - \varepsilon_1. \]

**Proof.** By putting:
\[ \alpha_2 = y_i y_j, \quad b_i = a_{ij} b_j, \quad \beta = b_i y_i, \quad \beta^* = b_i p_i, \quad p^i = a_{ij} p_j, \]
\[ \alpha'^2 = p_i p^i = a_{ij} p_i p_j, \]
and
\[ F = \frac{\alpha^2}{a - p}, \]
with
\[ p_i = \frac{1}{2} \frac{\partial F^2}{\partial y_i} = \frac{y_i}{\alpha - \beta} + \frac{\alpha^2 b_i - y_i \beta}{(\alpha - \beta)^2}. \quad (3.9) \]

Contracting in (3.9) by \( p^i \) and \( b^i \) we get:
\[ \alpha'^2 = \frac{F}{(\alpha - \beta)^2} [F^2(\alpha - 2\beta) + \alpha^2 \beta^*], \]
\[ \beta^* = \frac{F}{(\alpha - \beta)^2} [\beta(\alpha - 2\beta) + \alpha^2 b^2]. \quad (3.10) \]

In [Sh], for a Finsler \((\alpha, \beta)\)-metric \( F \) on a manifold \( M \), one constructs a positive function \( \phi = \phi(s) \) on \((-b_0; b_0)\) with \( \phi(0) = 1 \) and \( F = \alpha \phi(s) \), \( s = \frac{\beta}{\alpha} \), where
\[ \alpha = \sqrt{a_{ij} y^i y^j} \quad \text{and} \quad \beta = b_i y_i \quad \text{with} \quad ||\beta||_x < b_0, \forall x \in M. \]

The function \( \phi \) satisfies:
\[ \phi(s) - s \phi'(s) + (b^2 - s^2) \phi''(s) > 0, \quad (|s| \leq b_0). \]

A Matsumoto metric is a special \((\alpha, \beta)\)-metric with \( \phi = \frac{1}{1-s^2} \).

Using Shen’s [Sh] notation \( s = \frac{b}{a} \), the formula (3.10) become:
\[ \alpha'^2 = F^2 \frac{1 - 2s}{(1-s)^2} + F \frac{1}{(1-s)^2} \beta^*, \]
\[ \beta^* = F s \frac{1 - 2s}{(1-s)^2} + F \frac{1}{(1-s)^2} b^2. \quad (3.11) \]

Now we put \( 1 - s = t \), i.e. \( s = 1 - t \) and both equations become:
\[ \alpha'^2 = F^2 \frac{2t - 1}{t^3} + F \frac{1}{t^2} \beta^*, \quad (3.12) \]
\[ \beta^* = F(1 - t) \frac{2t - 1}{t^2} + F \frac{1}{t^2} b^2. \]  
(3.13)

We get
\[ \beta^* t^2 = M(-2t^2 + 3t + b^2 - 1). \]  
(3.14)

For \( b^2 = 1 \) from (3.13) we obtain:
\[ F = -\frac{\beta^* t^2}{2t - 3}. \]  
(3.15)

and by substitution of \( F \) in (3.12), after some computations we get a cubic equation:
\[ t^3 - 3t^2 + \frac{9}{4}t - \frac{\beta^*}{2\alpha^2} = 0. \]  
(3.16)

Using Cardano’s method for solving cubic equation [Wi], we get:
\[ F = -\frac{\beta^*}{2\alpha^2} \left( \frac{(2P - 1)^2}{3P^2 + (P - 1)^2} \right), \]  
(3.17)

where for \( P \) we have:
\[ P = \frac{1}{2} \left[ \sqrt[3]{\left( \frac{\beta^* + \sqrt{\beta^*^2 - \alpha^*^2}}{\alpha^*} \right)^2} \right]. \]  
(3.18)

After some computations, for \( F \) we get:
\[ F = -\frac{\beta^*}{2\alpha^2} \left( \sqrt[3]{\beta^*^2 + \alpha^*^2} \right)^3. \]  
(3.19)

Substituting now \( \beta^* = b^i p_i \) and \( \alpha^*^2 = p_ip^i = a^ijp_ip_j \) we can easily get (3.7).

If \( b^2 \neq 1 \), the formula (3.15) is more complicated because:
\[ F = \frac{\beta^* t^2}{2t^2 + 3t + b^2 - 1}, \]  
(3.20)

and by substituting this in (3.12) we obtain the quadric equation:
\[ t^4 - 3t^3 + t^2 \frac{13 - 4b^2}{4} + t \frac{6\alpha^*^2(b^2 - 1)}{4\alpha^*^2} + \frac{\alpha^*^2(b^2 - 1)^2 + \beta^*^2(1 - b^2)}{4\alpha^*^2} = 0. \]  
(3.21)
After a quite long computation, formula (3.21) becomes a cubic equation (different from (3.16)) and solving it, we get:

\[
F = -\frac{\beta^*}{2} \left( \left( \sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^*2}{\alpha^2}\right)}} + \frac{A}{2} + \frac{3}{4} \right)^2 
+ \sqrt{A^2 + m\left(b^2 - \frac{\beta^*2}{\alpha^2}\right) - \frac{5}{4}(A + \frac{3}{10})^2 + n} \right)
/ \left( \left( \frac{3}{2} + 2A \right) \left( \sqrt{-A^2 + 3A + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^*2}{\alpha^2}\right)}} \right)^2 + 2\sqrt{A^2 + m\left(b^2 - \frac{\beta^*2}{\alpha^2}\right)} + \frac{9}{2}A + p \right),
\]

(3.22)

where

\[A^2 = \sqrt{\left( \frac{1}{2} \frac{\beta^*2}{\alpha^2} + \varepsilon_1 \right)^2 + \varepsilon_3 + \sqrt{4\left( \theta_1 \frac{\beta^*2}{\alpha^2} + \varepsilon_2 \right) + \theta_5}.\]

(3.23)

By substituting now \(\beta^* = b^i p_i\) and \(\alpha^* = p^i = a^i j p_i p_j\), after some computations, from (3.23) we obtain (3.8).

\[\square\]

3.1. Remarks.

(1) It is easy to see that both relations, (3.7) and (3.8), are coming from (3.14). Indeed, substituting \(b^2 = 1\) in (3.14) we get the cubic equation (3.16). As solution, we find (3.7). For \(b^2 \neq 1\), from (3.14) we get the complicated quadric equation (3.21) with (3.8) as solution. If in (3.21) we would replace \(b^2 = 1\) we would get \(t^4 - 3t^3 + \frac{9}{4} = 0\) with \(t_1 = t_2 = 0\) and \(t_3 = t_4 = \frac{3}{2}\). It is impossible for these four solutions to exist in our proof. So, we can easily see that (3.7) and (3.8) are two different relations and we can’t get (3.7) as a particular case of (3.8).

(2) Using \(\alpha^*\) and \(\beta^*\) we can get, for the \(\mathcal{L}\)-dual of \((M,F)\), in the case \(b^2 = 1\), the fundamental function:

\[H(x,p) = \frac{1}{2} \left( -\frac{\beta^*}{2} \left( \sqrt{\frac{\alpha^*2}{a^*2} + \varepsilon_1} + \sqrt{\frac{\beta^*2}{a^*2} - \frac{\alpha^*2}{a^*2}} \right)^3 \right)^2.\]

(3.24)

(3) In (3.7) \(\hat{a}^{ij}\) is positive-definite and the Randers metric on \(T^*M\)

\[p_i b^i + \sqrt{p_i p_j \hat{a}^{ij}}\] is positive-valued for any \(p\).
4. Conclusions

Let’s take a second look at formula (3.8). If we introduce the following quadratic forms:
\[
\begin{align*}
\alpha^*_2 &= \sqrt{d^i_j p_i p_j}, \\
\alpha^*_4 &= \sqrt{d^i_j p_i p_j}, \\
\alpha^*_8 &= \sqrt{d^i_j p_i p_j}, \\
\alpha^*_9 &= \sqrt{d^i_j p_i p_j},
\end{align*}
\]
defined on \( T^*M \) by the corresponding matrices, then (3.8) becomes:
\[
H(x, p) = \frac{1}{2} \left( -\frac{\beta^*}{200} \left( 2\alpha^*_2 + \alpha^*_4 \right)^2 + \left( \alpha^*_8 \right)^2 \right)^2,
\]
(4.1)
for \( b^2 \neq 1 \).

In other words, the \( L \)-duals of a Randers and Kropina metrics are expressed only with the duals \( \alpha^* \), \( \beta^* \) of \( \alpha \), \( \beta \), respectively. However, the \( L \)-dual of a Matsumoto metric is given by means of four distinct quadratic forms on \( T^*M \). Remark that the coefficients of the quadratic forms are constructed only from the Riemannian metric matrix element, \( a_{ij} \) and the 1-forms \( \beta \)'s coefficients \( b_i(x) \).

Inevitably, the following question occurs: if \( d^i_j, d^i_j, d^i_j, d^i_j \) are positively defined and therefore making sure that \( \alpha^*_2, \alpha^*_4, \alpha^*_8, \alpha^*_9 \) exist.

The answer is not quite immediate and depends both on the value of \( b^2 \) and on \( a^{ij}, b^i, b^j \). For example, if we take \( b^2 < \frac{1}{2} \) and \( a^{ij} > 2b^i b^j \) then, not only \( d^i_j, d^i_j, d^i_j, d^i_j \) are positively defined but also the four quadric forms are defined.

Certainly, there are many other values for \( b^2, a^{ij}, b^i, b^j \) which give a certain positive answer, but the above values justify the existence of (4.1).

4.1. Remarks, examples.

Remark 4.1. For the \( L \)-dual of (4.1) we obtain the Matsumoto space with the fundamental function:
\[
F = \frac{\tilde{a}_{ij} y^i y^j}{\sqrt{b^2a_{ij} y^i y^j - b_i y^i}},
\]
(4.2)
where
\[
\begin{align*}
\tilde{b}_i &= 4b^2 a_i, \\
\tilde{a}_{ij} &= a_{ij} b_i b_j (7 + 8b^2) - \sqrt{a_{ij}} b_i \left[ a_{ij} (1 + 2b^2) - 12b_i b_j \right] \\
&\pm m \left[ a_{ij}^2 b_i (7 + 8b^2) - \sqrt{a_{ij}} (a_{ij} - 12b_i b_j) \right],
\end{align*}
\]
and
\[
m = \sqrt{b_i b_j - b^2 a_{ij}}.
\]
The other properties like curvature and the relation between geometrical properties of the $\mathcal{L}$-dual metric (4.1) and the initial Matsumoto metric will be studied elsewhere.

**Example 1.** Let us consider a particular example and find its $\mathcal{L}$-dual. For this, let us consider a surface $S$ embedded in the usual Euclidean space $R^3$, i.e. $S \hookrightarrow R^3$, $(x, y) \in S \rightarrow (x, y, z = f(x, y)) \in R^3$.

It is known that the induced Riemannian metric on the surface $S$ is given by:

$$(a_{ij}) = \begin{pmatrix} 1 + (f_x)^2 & f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix},$$

where $f_x$ and $f_y$ means partial derivative with respect to $x$ and $y$, respectively.

If we consider now a coordinate system $(x, y, u, v) \in TM$ in the tangent bundle $TM$, then for $\alpha$ and $\beta$ one can choose:

$$\alpha^2 = (1 + f_x^2) u^2 + 2 f_x f_y uv + (1 + f_y^2) v^2,$$

and

$$\beta = f_x u + f_y v.$$

Now, for the induced Riemannian metric, we have:

$$\det ||a_{ij}|| = 1 + f_x^2 + f_y^2,$$

$$(a^{ij}) = \begin{pmatrix} 1 + (f_x)^2 & -f_x f_y \\ f_x f_y & 1 + (f_y)^2 \end{pmatrix},$$

$$\bar{b}^1 = \frac{f_x}{1 + f_x^2 + f_y^2}, \quad \bar{b}^2 = \frac{f_y}{1 + f_x^2 + f_y^2},$$

and for the Riemannian length of $\bar{b}_i$:

$$b^2 = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}, \quad 0 < b^2 < 1.$$

Using these and following step by step the second case of Theorem 3.5, we find:

$$d_{11}^2 = M(A + 4B) - A^2,$$
where

\[ M = \sqrt{\frac{1 + (f_x)^2}{1 + f_x^2 + f_y^2}}, \quad N = \sqrt{\frac{1 + (f_x)^2}{\frac{1}{1 + f_x^2 + f_y^2}}}, \quad P = \sqrt{-\frac{f_x f_y}{\frac{1}{1 + f_x^2 + f_y^2}}}, \]

and

\[ A = \sqrt{R_1} - R_2 + 2M^2\theta_3, \quad B = \sqrt{R_1} - R_2 + M^2\theta_6, \]
\[ C = \sqrt{R_3} - R_4 + 2N^2\theta_5, \quad D = \sqrt{R_3} - R_4 + N^2\theta_6, \]

and

\[ E = \sqrt{R_5}, \quad F = \sqrt{R_5 + \frac{4}{c}}, \]

where

\[ R_1 = 2\sqrt{\frac{2}{\frac{f_x^2(1 + f_y^2)}{(1 + f_x^2 + f_y^2)^5} + 8\epsilon_1 \frac{f_x^2(1 + f_y^2)^2}{(1 + f_x^2 + f_y^2)^4} + 8\epsilon_4 \frac{(1 + f_y^2)^3}{(1 + f_x^2 + f_y^2)^3}}}, \]
\[ R_2 = 4\sqrt{\frac{2\epsilon_2}{\frac{(1 + f_y^2)^3}{(1 + f_x^2 + f_y^2)^3} + \theta^2 \frac{f_x^2(1 + f_y^2)^2}{(1 + f_x^2 + f_y^2)^4}}}, \]
\[ R_3 = 2\sqrt{\frac{2}{\frac{f_y^2(1 + f_x^2)}{(1 + f_x^2 + f_y^2)^5} + 8\epsilon_1 \frac{f_y^2(1 + f_x^2)^2}{(1 + f_x^2 + f_y^2)^4} + 8\epsilon_4 \frac{(1 + f_x^2)^3}{(1 + f_x^2 + f_y^2)^3}}}. \]
The $\mathcal{L}$-dual of a Matsumoto space

\[ R_4 = 4 \sqrt{2 \varepsilon_2 \left( \frac{(1 + f_x^2)^3}{(1 + f_x^2 + f_y^2)} \right) + \theta_2^2 \frac{f_y^2(1 + f_x^2)^2}{(1 + f_x^2 + f_y^2)^3}}, \]

\[ R_5 = 2 \left( \sqrt{(1 - 2 \varepsilon_1)^2 + 8 \varepsilon_2 + 2 \theta_5 - 2 \sqrt{2 \varepsilon_2 - \frac{2}{c} \theta_4^2}} \right), \]

and

\[ e = 1 + f_x^2 + f_y^2, \]

\[ m = \frac{1}{1 + f_x^2 + f_y^2}, \]

\[ n = -\frac{29 + 9 f_x^2 + 9 f_y^2}{29(1 + f_x^2 + f_y^2)}, \]

\[ p = \frac{1 - f_x^2 - f_y^2}{2(1 + f_x^2 + f_y^2)}, \]

\[ \theta_1 = -\frac{258 c^3 - 1256 c^2 + 1684 c - 712}{124 c^3}, \]

\[ \theta_2 = \frac{81 c^2 + 90 c + 48}{124 c^3}, \]

\[ \theta_3 = -\left( \frac{9c - 8}{12c} \right)^2, \]

\[ \theta_4 = \frac{3c - 2}{6c}, \]

\[ \theta_5 = \frac{12c - 11}{12c}, \]

\[ \theta_6 = \frac{12c^2 + 13c - 24}{6c^3}, \]

\[ \varepsilon_1 = -\frac{45c^2 - 138c - 32}{124 c^2}, \]

\[ \varepsilon_2 = \frac{-2187 c^4 + 41796 c^3 - 15660 c^2 + 24768 c - 768}{124 c^4}, \]

\[ \varepsilon_3 = \frac{921 c^4 + 14732 c^3 - 1084 c^2 + 6832 c - 256}{124 c^4}, \]

\[ \varepsilon_4 = \frac{13077 c^4 + 189204 c^3 + 8916 c^2 + 90816 c - 2048}{124 c^4}, \]

getting in this way all the four quadric form which allow us to find, in $T^*M$, using (4.1), the $\mathcal{L}$-dual of our particular Matsumoto space from above.
For the above construction, we need to analyze the existence of the expressions under the radicals. $M, N$ always exist.

First of all, because of the radical in the expression of $P$ we must have $f_x f_y \leq 0$. If $f_x f_y = 0$ we get $d_{22}^{12} = d_{22}^{21} = 0$ and $d_{41}^{12} = d_{41}^{21} = 0$, $d_{32}^{12} = d_{32}^{21} = 0$.

Let us put $\Delta = (\varepsilon_1 - \theta_2^2)^2 - 4(\varepsilon_4 - 2\varepsilon_2)$ and $S = 4(\varepsilon_4 - 2\varepsilon_2)$. Therefore, we have:

If $\Delta < 0$ then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ for any value of $c$. This allows us to conclude that $A, B, C, D$ always exist. If $\Delta \geq 0$ and $c \in [1, \frac{4}{3}]$ or $\Delta \geq 0$ and $S \geq 0$, then $R_1 - R_2 \geq 0$ and $R_3 - R_4 \geq 0$ proving the existence of $A, B, C, D$.

We also need to have $R_5 \geq 0$. But this depends on the value of $c \geq 1$. For example, if $c \in [1, \frac{4}{3}]$ we have $R_5 \in [-0.0701; 2.1898]$.

To complete our discussion, we mention here the following result [SS1]: if $f_x^2 + f_y^2 \leq \frac{1}{4}$ i.e. $1 \leq c \leq \frac{4}{3}$, then $\frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2} \leq \frac{1}{4}$ and the fundamental tensor $g^{ij}$ of Matsomoto space $F = \frac{\alpha^2}{\alpha - \beta}$ with $\alpha$ and $\beta$ defined above is positively defined, or equivalently, the indicatrix is convex.

**Example 2.** Let us consider the surface $S$ to be a plane, $z = f(x, y) = \frac{1}{2}x$.

The convexity condition for the indicatrix is satisfied, i.e.: $f_x^2 + f_y^2 = \frac{1}{4} < \frac{1}{4}$.

Now, $f_x = \frac{1}{2}, f_y = 0$,

$$\begin{pmatrix} a_{ij} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \det \|a_{ij}\| = \frac{5}{4}, \quad \left( a^{ij} \right)^{\frac{1}{4}} = \begin{pmatrix} 4 & 0 \\ 5 & 0 \\ 0 & 1 \end{pmatrix},$$

and $\tilde{b}^1 = \frac{2}{5}$, $\tilde{b}^2 = 0$ and $b^2 = \frac{1}{5}$.

Following the calculus from above, we get:

$$d_{22}^{11} = 10.7621695,$$

$$d_{22}^{12} = d_{22}^{21} = 0,$$

$$d_{22}^{41} = 18.5916118,$$

$$d_{41}^{11} = 4.1619406,$$

$$d_{41}^{12} = d_{41}^{21} = 0,$$

$$d_{41}^{8} = 3.3692342,$$

$$d_{41}^{11} = 255.0575035.$$
The $\mathcal{L}$-dual of a Matsumoto space

\begin{align*}
    d_{12}^1 &= d_{12}^2 = 0, \\
    d_{22}^2 &= 185.6868118, \\
    d_{11}^1 &= 24.6023378, \\
    d_{12}^1 &= d_{12}^1 = 0, \\
    d_{0}^2 &= 23.1147203, \\
    \end{align*}

and for the four quadratic forms and $\beta^*$ we get:

\begin{align*}
    \alpha_2^* &= 10.7621695t^2 + 18.5916118s^2, \\
    \alpha_4^* &= 4.1619406t^2 + 3.3692342s^2, \\
    \alpha_8^* &= 255.0575035t^2 + 185.6868118s^2, \\
    \alpha_9^* &= 24.6023378t^2 + 23.1147203s^2, \\
    \beta^* &= 0.4t.
\end{align*}

References


I. M. Masca et al. : The \( \mathcal{L} \)-dual of a Matsumoto space


